CHAPTER III

FINITE ELEMENT FORMULATIONS OF BEAM AND SHELL ELEMENTS

I. Equation of motion of the system.

The equation of motion of linear multi degree of freedom system resulting from finite element method of a structure can be taken in the form [1,2,3,7,13,19]:

\[ M\ddot{U}(t) + C\dot{U}(t) + Ku(t) = P(t) \]  

(3.1)

where:
- \( M \) - global mass matrix of the structure: \( M = \sum_{i=1}^{E} M_{e} \)
- \( K \) - global stiffness matrix of the structure: \( K = \sum_{E} K_{e} \)
- \( C \) - global damping matrix of the structure: \( C = \sum_{E} C_{e} \)
- \( P \) - total load vector:

\[ P = \sum_{E} P_{e} + P_{n} \]

\( U, \dot{U}, \ddot{U} \) - global displacement, velocity and acceleration vector of the structure

In the static analysis eq. (3.1) become:

\[ KU = P \]

(3.2)

and eigenfrequency analysis equation (3.1) become:

\[ \{ K - \omega^{2}M \} \phi = \{ 0 \} \]

(3.3)

where:
- \( \omega^{2}, \phi \) - set of eigenvalue and eigenvector respectively

For analysis of very large structural systems, the method of superelement for static analysis and reduction algorithms for eigenfrequency analysis are discussed in the chapter four.

II. Formulation of element stiffness and mass matrices.

II.1. Finite element beam without shear deformation.

The stiffness and mass matrices of the frame element in the local coordinate system as [1,2,3,4,818,37...]:

where:
- \( E, G \) - Modulus of elasticity, and shear modulus respectively
- \( A \) - cross section area, \( I_{y}, I_{z} \) - Moment of inertia, \( l \) - length of element.
\[
\begin{bmatrix}
\frac{12E_l}{l^3} & 0 & 0 & \frac{12E_y}{l^3} & \frac{GJ_x}{l} & \frac{4E_l}{l} \\
0 & 0 & 0 & \frac{-6E_y}{l^3} & 0 & \frac{4E_l}{l} \\
0 & 0 & \frac{6E_y}{l^2} & 0 & 0 & 0 \\
\frac{-12E_l}{l^3} & 0 & 0 & 0 & 0 & \frac{12E_y}{l^3} \\
0 & 0 & \frac{-12E_y}{l^3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-6E_l}{l^2} & 0 & 0 \\
0 & 0 & \frac{6E_y}{l^2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2E_y}{l} & 0 & \frac{4E_l}{l} \\
0 & 0 & \frac{6E_y}{l^2} & 0 & 0 & 0
\end{bmatrix}
\]

\[K_e = \begin{bmatrix}
\frac{1}{3} & 0 & 0 & 13 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{35} & 0 & 0 \\
0 & 0 & 0 & \frac{J_x}{3F} & 0 & 0 \\
0 & 0 & \frac{-111}{210} & 0 & \frac{1}{105} & 105 \\
0 & \frac{111}{210} & 0 & 0 & 0 & \frac{1}{105} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 9 & 0 & 0 & \frac{13}{35} \\
0 & 0 & \frac{9}{70} & 0 & -131 & 0 \\
0 & 0 & \frac{131}{420} & 0 & 0 & \frac{13}{35} \\
0 & 0 & 0 & 0 & 0 & \frac{J_x}{3F} \\
0 & 0 & \frac{131}{420} & 0 & -\frac{1}{140} & 0 \\
0 & 0 & \frac{-131}{420} & 0 & \frac{11.1}{210} & 0 \\
0 & -\frac{131}{420} & 0 & 0 & -\frac{1}{140} & \frac{11.1}{210} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{105}
\end{bmatrix}
\]

\[M_e = \rho_A.1 \]

\[(3.4)\]

\[(3.5)\]
While assemblage element stiffness or mass matrices into global stiffness or mass matrices must be transformed from local coordinate system of the element into global coordinate system. The element stiffness and mass matrices become:

\[ \mathbf{K}_e = \mathbf{T}^T \mathbf{\hat{K}}_e \mathbf{T} \quad \text{and} \quad \mathbf{M}_e = \mathbf{T}^T \mathbf{\hat{M}}_e \mathbf{T} \]

where: \( \mathbf{\hat{K}}_e \), \( \mathbf{\hat{M}}_e \) - element stiffness and mass matrices in local coordinate

\( \mathbf{T} \) : transformation matrix.

II.2. General shell elements

Up to now, there are many finite element shell theories have been suggested. History, trend of development and formulations of some shell theories are introduced in most text book about finite element method [3,13,22,26,27,41]. In this study, Bilinear Degenerated Shell (BDS) element which is presented by K.N Worsak[26] and Z.H. Zhong[49] is used. Here present brief formulation of the BDS, for fully, can be referred to [19,26,49].

The BDS element (fig 3.2) evolves from an eight-node three dimensional brick. The mid surface encloses by four straight side forms a hyperbolic paraboloid, the concept of degenerated is used.

*In the bilinear degenerated shell two assumption are made:*

- Normally to the midsurface remain straight after deformation. Thus, the formulation include transverse shear deformation. Kirchhoff-Love hypothesis is not assumed.
- Stress normal to the midsurface are zero.

II.2.1. Shape functions for geometry and displacement fields.

The shell element fig. 3.2 is defined by the natural, curvilinear coordinate \{r,s,t\} such that a bi-unit cube is uniquely mapped into the shell element. The shape functions to describe the midsurface in terms of natural coordinate is bilinear. Thus,

\[ N_i = \frac{1}{4} (1 + r_i r) (1 + s_i s) \quad i = 1..4 \quad (3.6) \]

where \( r_i \) and \( s_i \) are the nature coordinate of the node \( i \).

Using shape functions, the coordinate of any point in the element can be uniquely given in terms of nodal coordinates and thickness as,

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \frac{1}{2} \mathbf{t} \begin{bmatrix}
m_{1i} \\
m_{2i} \\
m_{3i}
\end{bmatrix} \quad (3.7)
\]
where:  \( x_i, y_i, z_i \) - the global coordinates of midsurface node \( i \).
\( h_i \) - thickness of element at node \( i \).
\( l_{3i}, m_{3i}, n_{3i} \) - the normal unit vector at node \( i \).

At any point \((r,s)\) on the midsurface \((t=0)\) an orthogonal set of local coordinate axes \(x', y', z'\) are constructed, \(e_3'\) is the normal unit vector and \(e_1', e_2'\) are tangent to the midsurface. Thus,

\[
\left\{ e_3' \right\} = \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix}_{(R,s)} = \left( \begin{bmatrix} \frac{\partial x}{\partial S} \\ \frac{\partial x}{\partial T} \\ \frac{\partial x}{\partial R} \end{bmatrix}_{(R,s)} \right) \times \left( \begin{bmatrix} \frac{\partial x}{\partial S} \\ \frac{\partial x}{\partial T} \\ \frac{\partial x}{\partial R} \end{bmatrix}_{(R,s)} \right)
\]

(3.8)
The partial derivatives can be obtained from (3.6), the direction cosine of the new axes $x'$, $y'$, $z'$ with respect to $x$, $y$, $z$ are defined by $[D]$ matrix as:

$$[D] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$ (3.11)

The displacement variation in the element may be expressed as:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \begin{bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{bmatrix}$$ (3.12)

where $u_i$, $v_i$, $w_i$ - displacement of the node $i$ on the midsurface $x,y,z$ direction.

$u_i^*$, $v_i^*$, $w_i^*$ - relative nodal displacement along $x$, $y$, $x$ direction.

The displacements $u_i^*$, $v_i^*$, $w_i^*$ are to be expressed explicitly in terms of the rotation $\theta_{x_i}$, $\theta_{y_i}$, $\theta_{z_i}$ at each node $i$, about global axes. Using shell assumptions, the displacement produced by the normal rotation $\alpha_{1i}$, $\alpha_{2i}$ (fig. 3.3) can be calculated as:

$$\begin{bmatrix} u_i' \\ v_i' \\ w_i' \end{bmatrix} = \frac{1}{2} \begin{pmatrix} \alpha_{2i} \\ -\alpha_{1i} \\ 0 \end{pmatrix}$$ (3.13)

where $u_i'$, $v_i'$, $w_i'$ displacement components along $x'$, $y'$, $z'$ at node $i$.

$\alpha_{1i}$, $\alpha_{2i}$ rotation about $x'$, $y'$.

The components of these displacement along global direction $u_i^*$, $v_i^*$, $w_i^*$, can now be got by knowing the direction cosines of $x'$, $y'$, $z'$ with respect to $x$, $y$, $z$. 

(3.9)
Substituting (3.13) into (3.14) and arranging the terms in matrix form

\[
\begin{bmatrix}
    u_i^* \\
    v_i^* \\
    w_i^*
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
    l_{1i} & -l_{2i} \\
    m_{1i} & -m_{2i} \\
    n_{1i} & -n_{2i}
\end{bmatrix} \begin{bmatrix}
    \alpha_{1i}' \\
    \alpha_{2i}'
\end{bmatrix}
\]

(3.15)

and express \( \alpha_{1i}' \) and \( \alpha_{2i}' \) in terms of global rotation \( \theta_{xi}, \theta_{yi}, \theta_{zi} \) as:

\[
\alpha_{1i}' = l_{1i}\theta_{xi} + m_{1i}\theta_{yi} + n_{1i}\theta_{zi} \quad \alpha_{2i}' = l_{2i}\theta_{xi} + m_{2i}\theta_{yi} + n_{2i}\theta_{zi}
\]

(3.16)

Rewriting (3.16) in matrix form

\[
\begin{bmatrix}
    \alpha_{1i}' \\
    \alpha_{2i}'
\end{bmatrix} = \begin{bmatrix}
    l_{1i} & m_{1i} & n_{1i} \\
    l_{2i} & m_{2i} & n_{2i}
\end{bmatrix} \begin{bmatrix}
    \theta_{xi} \\
    \theta_{yi} \\
    \theta_{zi}
\end{bmatrix}
\]

(3.17)

![Diagram](image)

Fig. 3.3a Rotation about local axis 3.3b Rotation of normal due \( \alpha_{2i}', \alpha_{1}' \)

Substituting (3.12) into (3.10) obtained

\[
\begin{bmatrix}
    u_i^* \\
    v_i^* \\
    w_i^*
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
    \theta_{xi} \\
    \theta_{yi} \\
    \theta_{zi}
\end{bmatrix}
\]

(3.18)
where

\[
[D_i] = \begin{bmatrix}
0 & n_{3i} & -m_{3i} \\
-n_{3i} & 0 & l_{3i} \\
-m_{3i} & -l_{3i} & 0
\end{bmatrix}
\] (3.19)

The displacement variations are expressed in term of nodal value as,

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \frac{1}{2} t_i \theta_i \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix}
\] (3.20)

Finally:

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
u_i \\
v_i \\
w_i
\end{bmatrix} + \frac{1}{2} t_i \theta_i \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix}
\] (3.21)

II.2.2. Strain displacement matrix

Assuming \( \varepsilon_z = 0 \), the strain component along the local axes of the shell are given by

\[
\varepsilon' = \begin{bmatrix}
\varepsilon'_x \\
\varepsilon'_y \\
\gamma'_{xy} \\
\gamma'_{xz} \\
\gamma'_{yz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial x} + \frac{\partial \varepsilon'_y}{\partial t} \\
\frac{\partial v'}{\partial y} + \frac{\partial \varepsilon'_x}{\partial t} \\
\frac{\partial \varepsilon'_x}{\partial x} + \frac{\partial \varepsilon'_y}{\partial y} \\
\frac{\partial \varepsilon'_x}{\partial y} + \frac{\partial \varepsilon'_y}{\partial x} \\
\frac{\partial \varepsilon'_y}{\partial z} + \frac{\partial \varepsilon'_z}{\partial y}
\end{bmatrix}
\] (3.22)

For this purpose the derivatives of \( u, v, w \) with respect to \( r, s, t \) are required and they are obtained by differentiating (3.20) thus,

\[
\frac{\partial}{\partial t} \begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial t} \\
\frac{\partial w}{\partial t}
\end{bmatrix}
\]
The strain displacement matrix \([B]\) may be split up conventionally into two matrices \([B_m]\) and \([B_s]\) such as:

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial x}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & \frac{\partial N_i}{\partial y}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & \frac{\partial N_i}{\partial z}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) \\
\frac{\partial N_i}{\partial x}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & \frac{\partial N_i}{\partial y}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) & \frac{\partial N_i}{\partial z}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) \\
\frac{\partial N_i}{\partial x}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) & \frac{\partial N_i}{\partial y}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & \frac{\partial N_i}{\partial z}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi})
\end{bmatrix}
\]

\[+ \sum_{i=1}^{4} \frac{h_i}{2} \left( \begin{array}{ccc}
\frac{1}{N_i}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & \frac{1}{N_i}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & \frac{1}{N_i}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) \\
\frac{1}{N_i}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi}) & \frac{1}{N_i}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) & \frac{1}{N_i}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) \\
\frac{1}{N_i}(m_{3i}\theta_{xi} - l_{3i}\theta_{yi}) & \frac{1}{N_i}(n_{3i}\theta_{yi} - m_{3i}\theta_{zi}) & \frac{1}{N_i}(l_{3i}\theta_{zi} - n_{3i}\theta_{xi})
\end{array} \right) \] (3.23)

The strain displacement matrix \([B]\) may be split up conventionally into two matrices \([B_m]\) and \([B_s]\) such as:

\[
\begin{bmatrix}
\varepsilon_{x'}^i \\
\varepsilon_{y'}^i \\
\varepsilon_{z'}^i
\end{bmatrix} = \sum_{i=1}^{4} [B_m] \{d_i\}
\]

\[
\begin{bmatrix}
\varepsilon_{x'}^i \\
\varepsilon_{y'}^i
\end{bmatrix} = \sum_{i=1}^{4} [B_s] \{d_i\}
\]

(3.24)

where: \(\{d_i\}\) - global displacements and rotations at each node.

The strain matrix-displacement matrices \([B_{mi}]\) is further split as:

\([B_{1mi}], [B_{2mi}], [B_{3mi}]\).

where: \([B_{1mi}]\) is formed considering only in the plane displacement \(u_i, v_i, w_i\).

\([B_{2mi}]\) and \([B_{3mi}]\) are formed considering rotation \(\theta_{xi}, \theta_{yi}, \theta_{zi}\).

Similarly, the strain displacement matrix \([B_{si}]\) is split into \([B_{1si}], [B_{2si}], [B_{3si}]\).

\([B_{1si}]\) is formed considering only in the plane displacement \(u_i, v_i, w_i\), \([B_{2si}]\) and \([B_{3si}]\) are formed considering rotation.

Formulation of \([B_{1mi}]\): Making use of (3.23), the derivative of \(u'\) and \(v'\) with respect to \(x'\) and \(y'\) are computed. Arranging the term with respect to inplane displacement \(u_i, v_i\) and \(w_i\), \([B_{1mi}]\) matrix is constructed. A typical term is given below

\[
\frac{\partial u'}{\partial x'} = \sum_{i=1}^{4} \left[ \left( \frac{\partial N_i}{\partial x} l_{1i} u_i + \frac{\partial N_i}{\partial y} m_{1i} v_i + \frac{\partial N_i}{\partial z} n_{1i} w_i \right) + m_i \left( \frac{\partial N_i}{\partial y} l_{1i} v_i + \frac{\partial N_i}{\partial z} m_{1i} v_i + \frac{\partial N_i}{\partial z} n_{1i} w_i \right) + \right. \\
+ \left. n_i \left( \frac{\partial N_i}{\partial z} l_{1i} v_i + \frac{\partial N_i}{\partial z} m_{1i} v_i + \frac{\partial N_i}{\partial z} n_{1i} w_i \right) \right]
\]

(3.25)

Similarly \(\frac{\partial v'}{\partial x'}\) and \(\frac{\partial w'}{\partial x'} + \frac{\partial v'}{\partial y'}\) can be computed. The inplane strain due to \(v_i, v_i\) and \(w_i\) can be expressed as:
\begin{equation}
\begin{pmatrix}
\varepsilon_x' \\
\varepsilon_y' \\
\varepsilon_z'
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u'}{\partial x'} \\
\frac{\partial u'}{\partial y'} \\
\frac{\partial u'}{\partial z'} + \frac{\partial \gamma'}{\partial z'}
\end{pmatrix} = \sum_{i=1}^{4} [B_{1mi}] \begin{pmatrix}
u_i \\
v_i \\
\omega_i
\end{pmatrix}
\end{equation}

where:

\begin{equation}
[B_{1mi}] = \begin{bmatrix}
l_{1,1}'B'(1,i) & m_{1,1}'B'(1,i) & n_{1,1}'B'(1,i) \\
l_{1,2}'B'(2,i) & m_{1,2}'B'(2,i) & n_{1,2}'B'(2,i) \\
l_{1,3}'B'(2,i) + l_{2,3}'B'(1,i) & m_{1,3}'B'(2,i) + m_{2,3}'B'(1,i) & n_{1,3}'B'(2,i) + n_{2,3}'B'(1,i)
\end{bmatrix}
\end{equation}

(3.27)

- **Formulation of \([B_{2mi}]\) and \([B_{3mi}]\)** the same procedure as in the case of \([B_{1mi}]\) is followed and here contributions due to rotations are considered. A typical term is given below:

\begin{equation}
[B_{2mi}] = \begin{bmatrix}
l_{2,1}' & m_{2,1} & n_{2,1} \\
l_{2,2} & m_{2,2} & n_{2,2} \\
l_{2,3} + l_{3,2}' & m_{2,3} + m_{3,2} & n_{2,3} + n_{3,2}
\end{bmatrix}
\end{equation}

\begin{equation}
(3.28)
\end{equation}

Arranging the relation between the inplane strains and rotations in matrix form we get:

\begin{equation}
\begin{pmatrix}
u_1 \\
v_2 \\
\omega_1
\end{pmatrix} = \begin{pmatrix}
l_{1,1}' & m_{1,1} & n_{1,1} \\
l_{1,2} & m_{1,2} & n_{1,2} \\
l_{1,3} + l_{3,2}' & m_{1,3} + m_{3,2} & n_{1,3} + n_{3,2}
\end{pmatrix} \begin{pmatrix}
u_1 \\
v_2 \\
\omega_1
\end{pmatrix}
\end{equation}

(3.30)

In \([B_{2mi}]\), terms such as \(J_{13}^* = 13; \ J_{23}^* = m_3; \ J_{33}^* = n_3\); and by orthogonality condition:

\begin{itemize}
  \item It can be shown that
\end{itemize}
\[ \begin{align*}
&l_1l_3 + m_1m_3 + n_1n_3 = 0 \\
&l_2l_3 + m_2m_3 + n_2n_3 = 0 \quad (3.31)
\end{align*} \]

Hence, it can be observed that: \[ [B_{2mi}] = [0] \quad (3.32) \]

Therefore, (3.30) reduced to,

\[ \begin{bmatrix}
\varepsilon_x' \\
\varepsilon_y' \\
\gamma_{x'y'} \\
\gamma_{y'x'}
\end{bmatrix} = \sum_{i=1}^{4} t[B_{3mi}] \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix} \quad (3.33) \]

where:

\[ [B_{3mi}] = \frac{h_i}{2} \begin{bmatrix}
B'(1, i)(m_{3i}n_1 - n_{3i}m_1) & B'(1, i)(n_{3i}l_1 - l_{3i}n_1) & B'(1, i)(l_{3i}m_1 - m_{3i}l_1) \\
B'(2, i)(m_{3i}n_2 - n_{3i}m_2) & B'(2, i)(n_{3i}l_2 - l_{3i}n_2) & B'(2, i)(l_{3i}m_2 - m_{3i}l_2) \\
B'(2, i)(m_{3i}n_1 - n_{3i}m_1) + B'(1, i)(m_{3i}n_2 - n_{3i}m_2) & B'(2, i)(n_{3i}l_1 - l_{3i}n_1) + B'(1, i)(n_{3i}l_2 - l_{3i}n_2) & B'(2, i)(l_{3i}m_1 - m_{3i}l_1) + B'(1, i)(l_{3i}m_2 - m_{3i}l_2)
\end{bmatrix} \quad (3.34) \]

- **Formulation of \([B_{1si}]\):** Similarly the derivatives are worked out for computing \(\gamma_{x'z'}\) and \(\gamma_{y'z'}\) due to displacement \(u', v', w'\) thus,

\[ \begin{bmatrix}
\gamma_{x'z'} \\
\gamma_{y'z'}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial y'} \\
\frac{\partial v'}{\partial x'} + \frac{\partial w'}{\partial y'}
\end{bmatrix} = \sum_{i=1}^{4} [B_{1si}] \begin{bmatrix}
u_i \\
w_i
\end{bmatrix} \quad (3.35) \]

where:

\[ [B_{1si}] = \begin{bmatrix}
l_1B'(3, i) + l_1B'(1, i) & m_1B'(3, i) + m_1B'(1, i) & n_1B'(3, i) + n_1B'(1, i) \\
l_2B'(3, i) + l_2B'(2, i) & m_2B'(3, i) + m_2B'(2, i) & n_2B'(3, i) + n_2B'(2, i)
\end{bmatrix} \quad (3.36) \]

and

\[ B'(3, i) = \frac{\partial N_i}{\partial x} l_3 + \frac{\partial N_i}{\partial y} m_3 + \frac{\partial N_i}{\partial z} n_3 \quad (3.37) \]

- **Formulation of \([B_{2mi}]\) and \([B_{3si}]\):** the strain \(\gamma_{x'z'}\) and \(\gamma_{y'z'}\) due to rotations \(\theta_{xi}, \theta_{yi}, \theta_{zi}\) are expressed as

\[ \begin{bmatrix}
\gamma_{x'z'} \\
\gamma_{y'z'}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial y'} \\
\frac{\partial v'}{\partial x'} + \frac{\partial w'}{\partial y'}
\end{bmatrix} = \sum_{i=1}^{4} ([B_{2si}] + [B_{3si}]) \begin{bmatrix}
\theta_{xi} \\
\theta_{yi} \\
\theta_{zi}
\end{bmatrix} \quad (3.38) \]

The matrices \([B_{2si}]\) and \([B_{3si}]\) are constructed in similar manner as before and given as follow:

\[ [B_{2si}] = \frac{h_i}{2} N_i B' \begin{bmatrix}
m_{3i}n_1 - n_{3i}m_1 & n_{3i}l_1 - l_{3i}n_1 & l_{3i}m_1 - m_{3i}l_1 \\
m_{3i}n_2 - n_{3i}m_2 & n_{3i}l_2 - l_{3i}n_2 & l_{3i}m_2 - m_{3i}l_2
\end{bmatrix} \quad (3.39) \]
where: \[ B' = l_1 J_1^* + m_1 J_2^* + n_1 J_3^* \] (3.40) and
\[
[B_{3i}] = \frac{h_1}{2} \begin{bmatrix}
B'(3, 1)(m_3 n_1 - n_3 m_1) + & B'(3, 1)(n_3 l_1 - l_3 n_1) + & B'(3, 1)(l_3 m_1 - m_3 l_1) + \\
B'(1, 1)(m_3 n_3 - n_3 m_3) + & B'(1, 1)(n_3 l_3 - l_3 n_3) + & B'(1, 1)(l_3 m_3 - m_3 l_3) + \\
B'(2, 1)(m_3 n_3 - n_3 m_3) + & B'(2, 1)(n_3 l_3 - l_3 n_3) + & B'(2, 1)(l_3 m_3 - m_3 l_3) + \\
\end{bmatrix}
\]

III.2.3. Stress-displacement Matrix

The element stress and load displacement are related as:
\[
\{\sigma\} = [C][B]\{d\} = [C.B]\{d\}
\]

In the case of isotropic materials, the constitutive relation is given by [1,22]. Imposing the condition that \(\sigma_{zz} = 0\), the following relation is obtained for the stress-strain relation in \(x', y', z'\) coordinates
\[
\begin{bmatrix}
\sigma_x' \\
\sigma_y' \\
\sigma_z' \\
\tau_{xx'} \\
\tau_{yy'} \\
\tau_{zz'} \\
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1-\nu}{2} & 0 & 0 & 0 \\
0 & 0 & \alpha(1-\nu) & 2 & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha(1-\nu)}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha(1-\nu)}{2} & 0 \\
\end{bmatrix} \begin{bmatrix}
\varepsilon_x' \\
\varepsilon_y' \\
\varepsilon_z' \\
\gamma_{xx'} \\
\gamma_{yy'} \\
\gamma_{zz'} \\
\end{bmatrix}
\]

(3.43)

where: \(\alpha\) - shear factor, \(\alpha = 5/6\).

To facilitate adoption of different numerical integration schemes for bending and shear contributions to the stiffness matrix, the constitutive matrix is split into [Cm] and [C1] as:
II.2.4. Element stiffness matrix

It is convenient to split the stiffness matrix into two parts: the bending and membrane effects and transverse shear effects. This will allow the use of appropriate order of number integration of each part. Thus,

\[
[K] = [K]_m + [K]_s
\]

i.e.

\[
[K] = \sum_{i=1}^{4} \sum_{j=1}^{4} \left[ [K_{ij}]_m + [K_{ij}]_s \right]
\]

\[3.44a\]

\[3.44b\]

Substitution from (3.42) for bending and membrane contribution into (3.45a)

\[
[K_{ij}]_m = \int \left[ [B_{1mi}]^T [C_m] [B_{1mj}] \right] \left[ B_{1mi} \right]^T dV
\]

\[3.45a\]

\[
[K_{ij}]_s = \int \left[ [B_{si}]^T [C_s] [B_{sj}] \right] dV
\]

\[3.45b\]

Symplifying the above equation and expressing it in natural coordinate

\[
[K_{ij}] = \int \int \int \int \left[ [B_{1mi}]^T [C_m] [B_{1mj}] \right] \left[ B_{1mi} \right]^T dV
\]

\[3.47\]

where:

\[ J \] - det. of the Jacobean matrix, to be consistent with shell assumption

\[ J_{(r,s,t)} \] can be approximated by \[ J_{(r,s,\theta)} \]

Since \([B_{1mi}]\) and \([B_{3mi}]\) are functions of \(r\) and \(s\) only, the integral of 3.7 can be integrate with respect to \(r\) and \(s\) thus,

\[
[K_{ij}]_m = \int \int \left[ 2[B_{1mi}]^T [C_m] [B_{1mj}] \right] \left[ B_{1mi} \right]^T dV
\]

\[3.48\]

\[
[K_{ij}] = \int \left[ [B_{2si}]^T [C_s] [B_{2sj}] + [B_{3si}]^T [C_s] [B_{3sj}] \right] dV
\]

\[3.49\]
Integrating across the thickness as above

\[
[K_{ij}]_s = \int \int \left[ 2[B_{1a}]^T[C_s][B_{1j}] \right] \left[ 2[B_{1a}]^T[C_s][B_{2j}] \right] + \frac{2}{3}[B_{3a}]^T[C_s][B_{3j}] \right] J_{(r,s,\theta)} \|drds
\]  

(3.50)

The size of each submatrix in (3.48) and (3.49) is 6x6. Thus the bending and membrane and shear stiffness contribution to the element stiffness matrix can be computed as :

\[
[K]_m \text{ or } [K]_s = \begin{bmatrix}
[K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] \\
[K_{21}] & [K_{22}] & [K_{23}] & [K_{24}] \\
[K_{31}] & [K_{32}] & [K_{33}] & [K_{34}] \\
[K_{41}] & [K_{42}] & [K_{43}] & [K_{44}] 
\end{bmatrix}
\]  

(3.51)

A 2x2 Gauss quadrature is used to evaluate the integral (3.48), bending and membrane contribution, one point quadrature is used to evaluate the integral (3.50), shear contribution to the stiffness matrix.

\textit{II.2.5. Torsional stiffness}

For the four nodes shell element, the rotation of the normal and midsurface field are independent. The rotation of the midsurface is 

\[
\frac{1}{2} \left( \frac{\partial \nu'}{\partial x'} - \frac{\partial u'}{\partial y'} \right)
\]  

show as fig. 3.3c. The derivation of the torsional rotation of the normal from that of the midsurface is assumed to have the governing train energy,

\[
U_t = \alpha_t \int \int_{A} \left[ \alpha_3' - \frac{1}{2} \left( \frac{\partial \nu'}{\partial x'} - \frac{\partial u'}{\partial y'} \right) \right]^2 \left[ \frac{\partial \nu'}{\partial x'} - \frac{\partial u'}{\partial y'} \right] J_{(r,s,\theta)} \|drds
\]  

(3.52)

where : \( \alpha_t \) - torsional coefficient

If \( \alpha_tGh \) is chosen to be large relative to the factor \( Eh^3 \) used in banding energy calculation, (3.52) will play the role of penalty function and results in the desired constrain at Gauss points as :

\[
\alpha_3' \approx \frac{1}{2} \left( \frac{\partial \nu'}{\partial x'} - \frac{\partial u'}{\partial y'} \right)
\]  

(3.53)

The torsional stiffness coefficient is derived from (3.52)
\[
\left(\frac{\partial \alpha'}{\partial x} - \frac{\partial \alpha'}{\partial y}\right) = \sum_{i=1}^{4} \left[ l_i \left( \frac{\partial N_i}{\partial x} l_2 u_i + \frac{\partial N_i}{\partial x} m_2 v_i + \frac{\partial N_i}{\partial x} n_2 w_i \right) + m_i \left( \frac{\partial N_i}{\partial y} l_2 u_i + \frac{\partial N_i}{\partial y} m_2 v_i + \frac{\partial N_i}{\partial y} n_2 w_i \right) \right]
\]

\[
= \sum_{i=1}^{4} \left[ l_i \left( \frac{\partial N_i}{\partial z} l_2 u_i + \frac{\partial N_i}{\partial z} m_2 v_i + \frac{\partial N_i}{\partial z} n_2 w_i \right) - l_2 \left( \frac{\partial N_i}{\partial x} l_1 u_i + \frac{\partial N_i}{\partial x} m_1 v_i + \frac{\partial N_i}{\partial x} n_1 w_i \right) \right]
\]

\[
+ m_i \left( \frac{\partial N_i}{\partial y} l_1 u_i + \frac{\partial N_i}{\partial y} m_1 v_i + \frac{\partial N_i}{\partial y} n_1 w_i \right) \]

\[
+ n_i \left( \frac{\partial N_i}{\partial x} l_1 u_i + \frac{\partial N_i}{\partial x} m_1 v_i + \frac{\partial N_i}{\partial x} n_1 w_i \right) \]

\[
- l_2 \left( \frac{\partial N_i}{\partial x} l_1 u_i + \frac{\partial N_i}{\partial x} m_1 v_i + \frac{\partial N_i}{\partial x} n_1 w_i \right) \]

\[
+ m_i \left( \frac{\partial N_i}{\partial y} l_1 u_i + \frac{\partial N_i}{\partial y} m_1 v_i + \frac{\partial N_i}{\partial y} n_1 w_i \right) \]

\[
+ n_i \left( \frac{\partial N_i}{\partial x} l_1 u_i + \frac{\partial N_i}{\partial x} m_1 v_i + \frac{\partial N_i}{\partial x} n_1 w_i \right) \]

\[
(3.54)
\]

If \( \alpha_{3i} \) is the rotation about the local \( z' \) axis at node I, then it can be expressed in terms of global rotation \( \theta_{xi}, \theta_{yi}, \theta_{zi} \) as:

\[
\alpha_{3i}' = l_3 \theta_{xi} + m_3 \theta_{yi} + n_3 \theta_{zi}
\]

\[
(3.55)
\]

If \( \alpha_3 \) is the rotation at any point \((r,s)\) on the midsurface, then variation can be expressed through the same function \( N_i \), as

\[
\alpha_3' = N_i(l_3 \theta_{xi} + m_3 \theta_{yi} + n_3 \theta_{zi})
\]

\[
(3.56)
\]

Arranging (3.14) and (3.15) and substituting in (3.12), we can express \( U_t \) in terms of torisional stiffness matrix as,

\[
U_t = \{d\}^T [K]_{t} \{d\}
\]

\[
(3.57)
\]

\[
[K]_{ij} = \alpha_t G h \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} [R_{mi}]^T [R_{mj}] & [R_{mi}]^T [R_{nj}] \end{bmatrix} [R_{ni}]^T [R_{nj}] |d| d\alpha d\beta
\]

\[
(3.58)
\]

where

- \( \alpha_t \) - torisional coefficient
- \( G \) - shear modulus,
- \( h \) - thickness

\[
[R_{mi}] = \frac{1}{2} [B'(2, i) - B'(1, i) 0] \begin{bmatrix} D \end{bmatrix}^T ; [R_{ni}] = N_i \begin{bmatrix} l_3 & m_3 & n_3 \end{bmatrix}
\]

\[
dA = |e_1 \times e_2| = |a| d\alpha d\beta
\]

\[
|a| = \sqrt{ \left( \frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \alpha} \frac{\partial y}{\partial \beta} \right)^2 + \left( \frac{\partial x}{\partial \alpha} \frac{\partial z}{\partial \beta} - \frac{\partial x}{\partial \alpha} \frac{\partial z}{\partial \beta} \right)^2 + \left( \frac{\partial y}{\partial \alpha} \frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \alpha} \frac{\partial z}{\partial \beta} \right)^2}
\]

\[
|a| = J \sqrt{(J_{13})^2 + (J_{23})^2 + (J_{33})^2}
\]

\[
(3.60)
\]

A 1x1 Gauss point quadrature is used in evaluating \([K]_t\) at center of the element.

\[\text{II.3. Computer program for finite beam and shell elements.}\]
Base on the source codes of the BK-XD [38], subroutine for 3D frame element is re-developed and used in this study. In the program to analysis 3DFrame and BDS shell element are Objects. Details and descriptions can be found in the reference No[42].

For the BDS shell element, Its source code is first developed on 1978 by W.N. Nukulchai [26] in the XFEAP software. The first revision on November 1981 for the ETA project by P. Tantiprabha, in these times, the element lumped mass matrix is modified to analysis eigenvalue problem. The second revision on (2) JAN.1990 by L. Wann-Her, the geometry stiffness matrix is added to analysis to solve buckling problems.

In this thesis inherits source codes of BSD shell element and re-develop into Shell Object by using OOP technique in C++. If everyone needs it, can be found in private web site of the author.