8

Stress–dilatancy

8.1 Introduction

A general framework on which to create elastic–plastic models of soil behaviour was set up in Chapter 4. The basic features of this framework included yield surfaces, which bound elastically attainable states of stress, and plastic potentials, which control the mode or mechanism of plastic deformation that occurs when the soil yields. Although examples of yield loci deduced from triaxial tests on soils were presented in Chapter 3, few data were produced to guide the choice of plastic potentials because, it was noted, it is often convenient to make plastic potentials and yield surfaces coincident. Indeed in the Cam clay model described in detail in Chapter 5, we assumed that the plastic potentials and yield surfaces were identical, so that plastic deformation of this model soil obeys an associated flow rule.

In this chapter, the forms of the plastic potentials which emerge from various models of material response are presented, and some of the factors controlling the modes of plastic deformation are discussed with reference to various sets of experimental data.

8.2 Plastic potentials, flow rules, and stress–dilatancy diagrams

Plastic potentials were introduced in Section 4.4.2 as curves in the $p':q$ effective stress plane to which, by definition, the vectors of plastic strain increment $\delta\varepsilon_p^p, \delta\varepsilon_q^p$ are orthogonal (Fig. 8.1). Many of the models of soil behaviour that have been proposed make the mechanism of plastic deformation, the value of the ratio $\delta\varepsilon_q^p/\delta\varepsilon_p^p$, dependent only on stress ratio $\eta = q/p'$ and not on the individual values of $q$ or $p'$. The ratio $\delta\varepsilon_q^p/\delta\varepsilon_p^p$ is defined in the direction of the normal to the plastic potential, and this is a function only of stress ratio $\eta$, all plastic potential curves for a soil, drawn in the $p':q$ effective stress plane, can be collapsed onto a
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A single curve. Any particular plastic potential curve can be obtained from any other one by radial scaling from the origin of the $p':q$ plane.

An alternative way of presenting information about plastic potentials is to plot stress ratio $\eta$ against the plastic strain increment ratio $\frac{\delta e_p}{\delta e_q}$ or $\frac{\delta e_q}{\delta e_p}$. However, use of the ratios $\frac{\delta e_p}{\delta e_q}$ or $\frac{\delta e_q}{\delta e_p}$ themselves is inconvenient if situations are to be included which involve yielding with no plastic shear strain ($\delta e_q = 0$) or with no plastic volumetric strain ($\delta e_p = 0$), respectively. A quantity which expresses the ratio between increments of plastic volumetric strain and shear strain but which always remains finite is the angle $\beta$ between the strain increment vector and the $p'$ axis (Fig. 8.1), where

$$\tan \beta = \frac{\delta e_q}{\delta e_p}$$

(8.1)

The relationship between plastic strain increment ratio and stress ratio is known as a flow rule governing the mode or mechanism of plastic deformation or flow of the soil. The ratio $\frac{\delta e_p}{\delta e_q}$ is the plastic dilatancy.

Fig. 8.1 Stress ratio $\eta = q/p'$ and dilatancy angle $\beta = \tan^{-1} \frac{\delta e_q}{\delta e_p}$.

- (a) Plastic strain increments for isotropic compression and critical states plotted in $p':q$ effective stress plane; (b) stress-dilatancy diagram $\eta;\beta$; isotropic compression ($I:\eta = 0$) and critical state ($C:\eta = M$).
of the soil, and the resulting plots in terms of $\beta$ and stress ratio $\eta$ (Fig. 8.2b) are called stress–dilatancy diagrams.

This method of presentation links strain increments with stresses, a familiar feature of plastic behaviour, but something which is alien to usual notions of elastic behaviour. It would not be appropriate to plot an elastic strain increment ratio on such a diagram because at any given stress ratio $\eta$, the ratio of elastic strain increments depends on the applied ratio of stress increments, which can take any value. It is not appropriate, either, to plot a ratio of total strain increments since this includes both plastic and elastic elements. Frequently, however, separation of elastic and plastic components of strain is not straightforward, and total strain increments

Fig. 8.3 (a) Plastic potential and (b) stress–dilatancy relationship for Cam clay model (drawn for $M = 1$).

(a)

(b)
are plotted in a stress–dilatancy diagram. For many situations the contribution of elastic strains to total strains may be negligible when yielding is occurring, and the difference between a plastic strain increment ratio and a total strain increment ratio may be small. However, though this may seem attractive, it is necessary to view total strain increment plots with caution: an undrained or constant volume test has $\delta e_p/\delta e_q = 0$ throughout; but if the soil is yielding, elastic and plastic volumetric strains are of opposite sign and, in general, are certainly not zero in magnitude, so that $\delta e_p^p/\delta e_q^p \neq 0$.

For soil which is being sheared without plastic volumetric strain, corresponding to the attainment of a critical state according to the Cam clay model, $\eta = M$ and $\delta e_p^p/\delta e_q^p = 0$. The plastic strain increment vectors are directed parallel to the $q$ axis (Fig. 8.2a) and, from (8.1), $\beta = \pi/2$ (point $C$ in Fig. 8.2b). A soil which has experienced only isotropic stresses in its past and is now being compressed isotropically ($q = 0$) changes in volume without changing in shape, $\delta e_q^p = 0$. The plastic strain increment vectors are directed parallel to the $p'$ axis (Fig. 8.2a), and since $\delta e_q^p = 0$, $\beta = 0$ (point $I$ in Fig. 8.2b). A plastic potential for isotropic soil can be converted into a flow rule which links points $C$ and $I$ in the $\beta: \eta$ diagram, but the precise form of the curve between $C$ and $I$ is a matter for assumption in any particular soil model.

The Cam clay model described in Chapter 5 is a model of particular relevance for isotropically compressed soil. The shape of the plastic potentials assumed in that model (identical to the elliptical yield loci) (Fig. 8.3a) leads to a flow rule:

$$\frac{\delta e_p^p}{\delta e_q^p} = \frac{M^2 - \eta^2}{2\eta} \quad (8.2)(5.7\text{bis})$$

or

$$\tan \beta = \frac{2\eta}{M^2 - \eta^2} \quad \quad (8.3)$$

This flow rule is plotted in Fig. 8.3b.

### 8.3 Stress--dilatancy in plane strain

The Cam clay model has been applied so far only to the description of the behaviour of soils in triaxial tests. Models which illustrate the interconnection between stress ratios and rates of dilation can be described more simply for conditions of plane strain. Parallels can then be drawn to devise related flow rules for triaxial conditions.

A simple analogy of dilating soil is provided by the interlocking saw blades shown in Fig. 8.4. Because the soil is expanding as it is sheared, it
is supposed that sliding within the soil takes place, not on horizontal planes, but on planes inclined at an angle of dilation $\psi$ to the horizontal (Fig. 8.4a). Sliding between adjacent soil particles occurs on these planes. Looking at the forces involved in this sliding process (Fig. 8.4b), if the angle of friction resisting sliding on the inclined planes is $\phi'_{es}$, then the apparent externally mobilised angle of friction on horizontal planes, $\phi'_m$, is

$$\phi'_m = \phi'_{es} + \psi$$

(8.4)

If it is supposed that the angle of friction resisting motion between layers of soil particles is always $\phi'_{es}$, then $\phi'_{es}$ can be seen as a soil constant, and (8.4) becomes a stress–dilatancy relation linking the mobilised friction $\phi'_m$ with an angle of dilation $\psi$. [Equation (8.4) is not a rigorous statement about the equilibrium of the saw tooth shown in Fig. 8.4 because it results from an argument based on forces when an argument based on stresses would be more appropriate. However, it does immediately suggest that some relationship between mobilised friction, or stress ratio, and dilatancy is to be expected.]

The saw-tooth failure or sliding surface of Fig. 8.4 could be imagined forming across a direct shear box test on a soil sample (Fig. 8.5). The quantities measured in such a test are the normal load $P$, the shear load $Q$, and the corresponding displacements $y$ and $x$ of the boundaries of the shear box. The work done by the applied loads $P$ and $Q$ on the soil sample during incremental displacements $\delta y$ and $\delta x$ is

$$\delta W_T = P \delta y + Q \delta x$$

(8.5)

The term $Q \delta x$ represents the work done in shearing the sample. The term $P \delta y$ represents the work that is done because the soil sample is changing in volume as it is sheared. For a soil such as dense sand that is dilating as it is sheared, $\delta y < 0$ and the normal load $P$ is being lifted up as the

Fig. 8.4  (a) Sliding of interlocking saw blades on inclined rough surfaces; (b) resultant force on inclined surface.
soil is sheared. The soil expands because of the interlocking of the particles: deformation can proceed only if some particles are able to ride up over other particles. Because of this interlocking, not all of the shearing work $Q \delta x$ is absorbed by the soil; some of it is required to lift the normal load and overcome the interlocking of the soil particles. The remainder, the nett work input $\delta W_T$ goes into the sample; some may be stored in elastic deformations of the soil particles, but most is dissipated in frictional resistance between the grains as they roll and slide on each other.

Following Taylor (1948), a simple stress–dilatancy equation can be obtained if it is assumed that all of this nett work input is dissipated in friction (no energy is stored in elastic deformations) and if it is also assumed that this frictional dissipation is controlled by the normal load $P$ and a frictional constant $\mu$, so that

$$\delta W_T = \mu P \delta x$$  \hspace{1cm} (8.6)

Putting (8.5) and (8.6) together and rearranging gives

$$\frac{Q}{P} + \frac{\delta y}{\delta x} = \mu$$  \hspace{1cm} (8.7)

The first term is the externally mobilised friction on horizontal planes

$$\frac{Q}{P} = \tan \phi_m'$$  \hspace{1cm} (8.8)

The second term describes the dilatancy of the sample; by comparison with Fig. 8.4:

$$\frac{\delta y}{\delta x} = -\tan \psi$$  \hspace{1cm} (8.9)

When the soil particles are sliding and rotating in such a way that the volume of the soil remains constant, so that the soil has reached a critical state, then $\delta y/\delta x = 0$ and $Q/P = \mu$. Expression (8.7) can be written as

$$\tan \phi_m' = \mu + \tan \psi$$  \hspace{1cm} (8.10)

or, strength equals friction plus dilatancy. (The relationship between $\mu$

Fig. 8.5 Inclined rough surfaces in shear box.
and a critical state angle of friction $\phi'_s$ has been left deliberately vague: the details of the link between the two require a number of assumptions to be made; see, for example, Airey, Budhu, and Wood, 1985.)

Relationship (8.10) can be tested on data from direct shear box tests on dense and loose sands reported by Taylor (1948) (Fig. 8.6). The slope of the graph of displacement data $x:y$ (Fig. 8.6b) is used to generate values of $\delta y/\delta x$, which are plotted with the ratio of shear load to normal load $Q/P$ in Fig. 8.7a. The sum $Q/P + \delta y/\delta x$ is plotted against horizontal displacement $x$ in Fig. 8.7b. It is evident that the initial points in Figs. 8.7a, b do not match the simple expression (8.7) or (8.10). But a flow rule should

Fig. 8.6 Direct shear tests on Ottawa sand with normal stress 287 kPa (×, dense, $\rho_s = 1.562$; •, loose, $\rho_s = 1.652$): (a) mobilised friction $Q/P$ on horizontal plane and shear displacement $x$; (b) vertical displacement $y$ and shear displacement $\chi$ (after Taylor, 1948).
8.3 Stress–dilatancy in plane strain

contain information about plastic deformations rather than total deformations; and in the initial stages of a shear test, some work is probably being done by the shear load in causing elastic deformation of the soil particles, so some deviation from (8.7) or (8.10) is to be expected. Beyond the points of peak load ratio (or peak mobilised friction) the data are

Fig. 8.7 Stress ratio and dilatancy in direct shear tests on Ottawa sand with expression (8.7) superimposed (x, dense; *, loose): (a) friction $Q/P$ and dilatancy $\frac{\delta y}{\delta x}$; (b) $Q/P + \delta y/\delta x$ and shear displacement $x$ (data from Taylor, 1948).
more consistent, and a value of $\mu \approx 0.49$ might be proposed. Expression (8.7), with $\mu = 0.49$, is plotted in Fig. 8.7a and describes fairly well the softening of the sand as the stress ratio drops from its peak value.

Although external measurements of forces $P$ and $Q$ and displacements $x$ and $y$ can be made in tests on soils in a direct shear box (Fig. 8.5), interpretation of these quantities in terms of stresses and strains is not feasible because the soil in the shear box is clearly not deforming homogeneously. The simple shear apparatus is another plane strain apparatus which attempts to impose a more uniform deformation on soil samples. This mode of deformation is shown in Fig. 8.8a: the length of the sample remains constant, but its height may change as the sample is sheared. Stress conditions in the simple shear apparatus are not particularly uniform (see Section 1.4.2), but with suitable instrumentation, measurements of the normal and shear stresses $\sigma_{yy}$ and $\tau_{yx}$ acting on the soil can be made. Measurements of horizontal and vertical displacements $x$ and $y$ can be converted to strains:

$$\delta \varepsilon_{yy} = \frac{\delta y}{h} \quad (8.11)$$

and

$$\delta \gamma_{yx} = \frac{\delta x}{h} \quad (8.12)$$

Fig. 8.8 (a) Simple shear deformation and (b) definition of angle of dilation $\psi$; (c) mobilised angle of friction $\phi^\prime_m$. 

\[ (a) \]

\[ (b) \]

\[ (c) \]
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where \( h \) is the height of the sample. The constant length of the sample fixes \( \delta e_{xx} = 0 \). The angle \( \psi \), which for the shear box was the direction of movement of the top platen of the shear box relative to the bottom platen, now has a particular geometrical significance in the Mohr circle of strain increment (Fig. 8.8b),

\[
\tan \psi = -\frac{\delta e_{zz}}{\delta y_{yx}}
\]  \hspace{1cm} (8.13)

or

\[
\sin \psi = -\frac{\delta e_{z}}{\delta e_{y}}
\]  \hspace{1cm} (8.14)

where \( \delta e_{z} \) is the increment of volumetric strain and \( \delta e_{y} \) the increment of shear strain (that is, the diameter of the Mohr circle in Fig. 8.8b; these strain variables for plane strain situations were introduced in Section 1.5). The angle \( \psi \) is called the angle of dilation; it is also, as seen in Fig. 8.8b, the slope of the tangent to this Mohr circle at the points where the circle cuts the line of zero direct strain, \( \delta e = 0 \). It is often helpful to think of \( \psi \) as a strain increment equivalent of angle of friction \( \phi' \) (Fig. 8.8c). It is, however, only useful under conditions of plane strain; and although, following (8.14), an angle \( \theta \) might be defined for triaxial conditions as

\[
\sin \theta = -\frac{\delta e_{zz}}{\delta e_{y}}
\]  \hspace{1cm} (8.15)

Fig. 8.9 Sum of stress ratio and dilatancy in simple shear tests on Leighton Buzzard sand (\( \bullet \), dense, \( \phi_{c} = 45.5^\circ \); \( \times \), loose, \( \phi_{c} = 1.78^\circ \) (after Stroud, 1971).
neither this nor the angle $\beta$ that was used in Section 8.2 has any geometrical interpretation in Mohr circles of strain increment for conditions of axial symmetry.

In terms of the stress and strain quantities that can be determined in the simple shear apparatus, (8.7) becomes

$$\frac{\tau_{yx}}{\sigma_{yy}} + \frac{\delta e_{yy}}{\delta \gamma_{yx}} = \mu$$  \hspace{1cm} (8.16)

Data from simple shear tests on dense and loose Leighton Buzzard sand performed by Stroud (1971) have been plotted in Fig. 8.9 as values of the sum $\tau_{yx}/\sigma_{yy} + \delta e_{yy}/\delta \gamma_{yx}$ against measured values of shear strain $\gamma_{yx}$. This plot is equivalent to Fig. 8.7b for the direct shear box data. The simple shear data seem to support more closely a constant value for $\mu \approx 0.575$, though there is a slight tendency for the sum to increase with increasing shear strain. The quantity $\delta e_{yy}/\delta \gamma_{yx}$ has been calculated in terms of total strain increments with no allowance for elastic recoverable deformations. Nevertheless, both these examples show that when volume change and stress:strain data are brought together, the response of initially dense and initially loose samples of sand again falls into a single clear picture.

8.4 Work equations: 'original' Cam clay

Equations (8.5) and (8.6) describe the way in which the work done by the forces acting on the boundaries of the shear box is dissipated as friction in the soil. The resulting stress–dilatancy equation (8.7) is specific to the shear box, but it can be used to suggest how a stress–dilatancy equation might be generated which is appropriate for the axisymmetric conditions of the triaxial test.

The obvious parallels to be drawn are between

- normal load $P$ and mean effective stress $p'$,
- shear load $Q$ and deviator stress $q$,
- shear deformation $x$ and triaxial shear strain $\varepsilon$, and
- volumetric deformation $y$ and volumetric strain $\varepsilon_v$.

The total work input per unit volume to a triaxial sample supporting stresses $p', q$ as it undergoes strains $\delta \varepsilon_p, \delta \varepsilon_q$ is

$$\delta W = p' \delta \varepsilon_p + q \delta \varepsilon_q$$  \hspace{1cm} (8.17)(1.32bis)

but of this work input, part is stored in elastic deformations of the soil. The energy available for dissipation is

$$\delta E = p' \delta \varepsilon_p^e + q \delta \varepsilon_v^e$$  \hspace{1cm} (8.18)

Following Taylor's (1948) analysis of the shear box, we might assume that this energy is dissipated entirely in friction according to a simple expression

$$\delta E = M p' \delta \varepsilon_v^e$$  \hspace{1cm} (8.19)
8.4 Work equations: 'original' Cam clay

Then the combination of (8.18) and (8.19) can be rearranged in a form equivalent to (8.7) or (8.16):

\[ p' \delta \varepsilon_p + q \delta \varepsilon_q = M p' \delta \varepsilon_q \]  \hspace{1cm} (8.20)

or

\[ \frac{q}{p'} + \frac{\delta \varepsilon_p}{\delta \varepsilon_q} = M \]  \hspace{1cm} (8.21)

The critical state parameter \( M \) is appropriate because at the critical state \( \delta \varepsilon_p / \delta \varepsilon_q = 0 \) and \( q/p' = \eta = M \). Expression (8.21) can again be broadly interpreted as proposing that friction plus dilatancy equals a constant.

Expression (8.21) can be plotted as a stress–dilatancy relationship in a \( \eta : \beta(= \tan^{-1} \delta \varepsilon_q / \delta \varepsilon_p) \) diagram: curve 2 in Fig. 8.10. The corresponding shape of plastic potentials in the \( p' : q \) plane can be obtained by integration since the direction of the strain increment vector, controlled by the ratio \( \delta \varepsilon_q / \delta \varepsilon_p \), is by definition the same as the direction of the normal to the plastic potential. The equation to be integrated then becomes

\[ \frac{q}{p'} \frac{dq}{dp'} = M \]  \hspace{1cm} (8.22)

and the equation of the plastic potential is then

\[ \eta = \ln \frac{p'_e}{p'} \] \hspace{1cm} (8.23)

where \( p'_e \) merely indicates the size of a particular plastic potential curve and is the value of \( p' \) for \( \eta = 0 \). This curve is plotted in Fig. 8.11.

Fig. 8.10 Stress–dilatancy relationships: (1) Cam clay, (2) original Cam clay, and (3) Rowe’s stress–dilatancy (drawn for \( M = 1 \)).
If it is assumed that a soil whose plastic flow is governed by (8.21) also obeys the principle of normality, then yield loci for the soil also have the shape given by (8.23). If these yield loci and identical plastic potentials are then placed in the framework of volumetric hardening elastic–plastic models of soil behaviour described in Chapter 4, then the 'original' Cam clay model, which was described by Roscoe and Schofield (1963) and mentioned in passing in Section 5.1, is recreated.

The flow rule of the Cam clay model described in Chapter 5,
\[
\frac{\delta \varepsilon_p^p}{\delta \varepsilon_s^p} = \frac{M^2 - \eta^2}{2\eta}
\]
(8.2bis)
can also be recast as a plastic dissipation equation,
\[
p' \delta \varepsilon_p^p + q \delta \varepsilon_s^p = p' (\delta \varepsilon_p^p)^2 + (M \delta \varepsilon_s^p)^2
\]
(8.24)
though this lacks the simplicity of (8.20). The flow rule and elliptical plastic potential for the Cam clay model are shown in Fig. 8.10 (curve 1) and Fig. 8.11 for comparison with original Cam clay. The most obvious difference appears at low values of stress ratio \(\eta\): the original Cam clay flow rule does not pass through the isotropic point (\(\eta = 0, \beta = 0\)) in Fig. 8.10. This implies that significant plastic shear strains will develop even for isotropic compression at zero stress ratio.

Schofield and Wroth (1968) cover this problem by suggesting that the plastic potentials in triaxial extension are mirror images of the curves in triaxial compression (Fig. 8.11) so that the flow rule has a discontinuity at \(\eta = 0\) (Fig. 8.10). The soil then makes the theoretically acceptable decision to adopt the mid-value at the discontinuity, i.e. \(\beta = 0\). Roscoe,

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Fig. 8.11 Plastic potentials: (1) Cam clay, (2) original Cam clay, and (3) Rowe's stress–dilatancy (drawn for \(M = 1\)).
8.5 Rowe's stress–dilatancy relation

Schofield, and Thurairajah (1963) propose heterogeneous soil deformations and suggest that 'A sample in this state may contain local regions in which plastic distortion occurs in different directions in such a manner that the whole sample does not exhibit any overall distortion'. Neither of these proposals is particularly satisfactory from a homogeneous computational point of view because the model still implies that for very small but non-zero stress ratios, significant plastic shear strains will develop. The discontinuous plastic potential function can also cause computational difficulties in numerical analyses; careful study of the relevant subroutines in the finite element program CRISP described by Britto and Gunn (1987) shows that, for practical use in numerical calculations, the point of the original Cam clay plastic potential has to be rounded off and the discontinuity eliminated.

8.5 Rowe’s stress–dilatancy relation

An alternative justification of a flow rule is provided by the stress–dilatancy relation proposed by Rowe (1962). Rowe produces an expression which states that, for a soil sample that is being sheared, the ratio of the work done by the driving stress to the work done by the driven stress in any strain increment should be a constant. This constant $K$ is supposed to be the same for triaxial and plane strain conditions:

$$\frac{\text{work put in by driving stress}}{\text{work taken out by driven stress}} = -K$$

(8.25)

The constant $K$ is related to an angle of soil friction $\phi_r^\prime$ by the expression

$$K = \tan^2 \left( \frac{\pi}{4} + \frac{\phi_r^\prime}{2} \right) = \frac{1 + \sin \phi_r^\prime}{1 - \sin \phi_r^\prime}$$

(8.26)

Rowe suggests that the angle $\phi_r^\prime$ lies in the range

$$\phi_\mu \leq \phi_r^\prime \leq \phi_{cs}^\prime$$

(8.27)

where $\phi_{cs}^\prime$ is the critical state angle of friction for constant volume shearing, and $\phi_\mu$ is the angle of interparticle sliding friction.

In triaxial compression, the axial stress $\sigma_s^\prime$ is the driving stress (with an associated compressive strain increment $\delta e_s$), and the radial stress $\sigma_r^\prime$ is the driven stress (with an associated tensile strain increment $-\delta e_r$). Rowe’s stress–dilatancy relation then states that for triaxial compression,

$$\frac{\sigma_s^\prime}{-2\sigma_r^\prime} \frac{\delta e_s}{\delta e_r} = K$$

(8.28)

Equation (8.28) can be rewritten in terms of the stress variables $p'$ and $q$ (which appear only as stress ratio $\eta = q/p'$) and the strain increment
variables $\delta e_p$ and $\delta e_q$ which are preferred in this book:

$$
\frac{\delta e_p}{\delta e_q} = \frac{3\eta(2 + K) - 9(K - 1)}{2\eta(K - 1) - 3(2K + 1)}
$$

(8.29)

Neglecting elastic strains, we can integrate (8.29) to give a plastic potential

$$
\frac{p'}{p'_o} = 3 \left[ \frac{3 - \eta}{(2\eta + 3)^K} \right]^{1/(K - 1)}
$$

(8.30)

It is convenient to fix the value of $\phi'_c$ at the ultimate critical state value $\phi'_{cs}$. Then, since $\phi'_{cs}$ and $M$ are related for triaxial compression by

$$
\sin \phi'_{cs} = \frac{3M}{6 + M}
$$

(8.31)(cf. 7.10)

expression (8.26) becomes

$$
K = \frac{3 + 2M}{3 - M}
$$

(8.32)

and expression (8.29) can be written as

$$
\frac{\delta e_p}{\delta e_q} = \frac{9(M - \eta)}{9 + 3M - 2M\eta}
$$

(8.33)

which has similarities to the original Cam clay flow rule obtained by rearranging (8.21):

$$
\frac{\delta e_p}{\delta e_q} = M - \eta
$$

(8.34)

The flow rule (8.33) is plotted in terms of $\eta$ and $\beta$ (ignoring the difference between total and plastic strains) in Fig. 8.10 (curve 3). The plastic potential curve (8.30) is plotted in the $p':q$ effective stress plane in Fig. 8.11.

For conditions of triaxial extension, the radial stress is now the driving stress and the axial stress is the driven stress, and (8.25) becomes

$$
\frac{2\sigma'_r \delta e_r}{-\sigma'_s \delta e_s} = K
$$

(8.35)

The equivalent of (8.29) for triaxial extension is then

$$
\frac{\delta e_p}{\delta e_q} = \frac{3\eta(2K + 1) + 9(K - 1)}{2\eta(1 - K) - 3(K + 2)}
$$

(8.36)

and the plastic potential, equivalent of (8.30), is

$$
\frac{p'}{p'_o} = 3 \left[ \frac{3 + 2\eta}{(3 - \eta)^K} \right]^{1/(K - 1)}
$$

(8.37)

Evidently, the plastic potential and flow rule do not have the same shapes in triaxial extension and triaxial compression. The corresponding curves,
8.5 Rowe's stress-dilatancy relation

with $\eta < 0$ and $q < 0$, have been plotted in Figs. 8.11 and 8.10 (curve 3 in each figure). Like original Cam clay, the plastic potential has a vertex and the flow rule has a discontinuity for $\eta = 0$.

For conditions of plane strain, provided principal axes of strain increment and of stress are coaxial, (8.25) becomes

$$\frac{\sigma'_1 \delta \varepsilon_1}{-\sigma'_3 \delta \varepsilon_3} = K$$  \hspace{1cm} (8.38)

since the intermediate principal stress does no work. Now, from the Mohr

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**Fig. 8.12** Conventional drained triaxial compression test on dense Fontainebleau sand ($c_u = 1.61$, $\sigma = 100$ kPa): (a) stress ratio $\eta$ and triaxial shear strain $\varepsilon_\sigma$; (b) volumetric strain $\varepsilon_v$ and triaxial shear strain $\varepsilon_\sigma$ (data from Luong, 1979).
circle of effective stresses (Fig. 8.8c),
\[
\frac{\sigma_1'}{\sigma_3'} = \frac{1 + \sin \phi_m'}{1 - \sin \phi_m'}
\] (8.39)

where $\phi_m'$ is the mobilised friction in the soil. From the Mohr circle of strain increments (Fig. 8.8b),
\[
-\delta e_1 = \frac{1 - \sin \psi}{1 + \sin \psi} \delta e_3
\] (8.40)

where $\psi$ is the angle of dilation for the soil. From (8.26) with $\phi_c' = \phi_c'$, the critical state angle of friction, we obtain
\[
K = \frac{1 + \sin \phi_c'}{1 - \sin \phi_c'}
\] (8.41)

Equation (8.38) can then be rearranged to give
\[
\sin \phi_m' = \frac{\sin \phi_c' + \sin \psi}{1 + \sin \phi_c' \sin \psi}
\] (8.42)

Considerations of dilatancy, volume change on shearing, through the variables $\beta$ or $\psi$, rather than ratios of principal strain increments such as $\delta e_a / \delta e$, in a triaxial test, give a better feel for the way in which the soil is responding to a particular state of stress: that is, whether it is choosing to expand or contract as it is sheared. The stress–dilatancy relation in the form (8.33) makes it clear that the volumetric strain increments are zero when the stress ratio $\eta = M$. Dense sands usually show a peak strength $\eta > M$ before deforming to an ultimate critical state with $\eta = M$ (Fig. 8.12a and see Sections 6.5 and 7.4.3). The stress ratio has passed through the value $\eta = M$ at an early stage of the test before the peak is reached. If the soil follows the stress–dilatancy relation (8.33) throughout the test (and the assumption of a certain form of flow rule or plastic potential places no restrictions on the form of the yield locus), then at this stage too the sand is deforming instantaneously at constant volume, $\delta e_p / \delta e_q = 0$ (Fig. 8.12b).

Studies of the behaviour of sand under cyclic loading such as those of Luong (1979) and Tatsuoka and Ishihara (1974b) show that this stress ratio, at which the sand deforms instantaneously at constant volume, plays an important role in governing the behaviour of the sand, controlling whether cyclic loading tends to stabilise the soil or leads to catastrophe. Tatsuoka and Ishihara call this stress ratio the ‘phase transformation stress ratio’, and Luong calls it the ‘characteristic stress ratio’; but according to stress–dilatancy ideas, it should be the same as the critical state stress ratio. (Luong notes that his characteristic stress ratio is easier
8.5 Rowe's stress–dilatancy relation

to determine because it is reached at an early stage of tests, whereas the critical state is usually reached only by extrapolation at a late stage.

The role played by this stress ratio in controlling response of sand to small cycles of loading is illustrated in Fig. 8.13. In drained cyclic loading (Fig. 8.13a), cycling below the critical stress ratio tends to produce cumulative volumetric compression and densification of the sand; cycling above the critical stress ratio tends to produce cumulative volumetric expansion. In undrained cyclic loading (Fig. 8.13b) cycling below the critical stress ratio tends to produce positive pore pressures, because volumetric compression is prevented. These positive pore pressures reduce the effective mean stress and lead to liquefaction or related phenomena of large deformations. Cycling above the critical stress ratio tends to

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Fig. 8.13 Phenomena observed in (a) drained and (b) undrained cyclic loading of sand (S, shakedown; I, incremental collapse; C, cyclic mobility; L, liquefaction of loose sand) (after Luong, 1979).

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Fig. 8.14 Inclined toothed plane of separation in triaxial sample (after de Josselin de Jong, 1976).
produce negative pore pressures because volumetric expansion is prevented. These negative pore pressures increase the effective mean stress and strengthen the soil.

Rowe originally deduced his stress-dilatancy relation from minimum energy considerations of particle sliding. However, de Josselin de Jong (1976) shows that the relation emerges also from combining considerations of equilibrium and kinematics of particles sliding on inclined saw-tooth surfaces (Fig. 8.14). The important feature is that the relation arises from analysis of sliding deformations in granular materials and hence that it is likely to be of most relevance in describing deformations of soils in which sliding between particles provides the principal contribution to the deformation. It is thus not surprising that this flow rule gives a paradoxical prediction of significant shear strains for purely isotropic stresses ($\eta = 0$), where particle sliding does not dominate; the flow rule is being stretched beyond its region of legitimate application. Given the similarity between the flow rules of Rowe's stress-dilatancy and original Cam clay, which is evident in Fig. 8.10, one might suggest that original Cam clay is also out of its region of legitimate application at low stress ratios, $\eta \approx 0$. The difference in concept between the two flow rules is that original Cam clay is based on continuum considerations, whereas Rowe's stress-dilatancy is based on considerations of equilibrium and kinematics of particular planes of sliding.

8.6 Experimental findings

So far, the presentation of flow rules has proceeded almost entirely along theoretical lines, with a brief interlude for the re-analysis of some direct shear and simple shear data following Taylor's (1948) simple flow rule. In this section, several sets of published data are produced for comparison with these theoretical relationships. It must be emphasised again that in presentation of data for exploration of flow rules, most authors either explicitly or implicitly ignore the presence of elastic deformations and base calculations of dilatancies on total strain increments. This approach tends to be most in error when the effective stress path followed in a test is moving most nearly tangentially to the current yield surface.

Rowe's stress-dilatancy relation has met with greatest success in describing the deformation of sands and other granular media. Data from simple shear tests on Leighton Buzzard sand performed by Stroud (1971) have already been introduced. The apparatus used by Stroud was heavily instrumented so that he was able to compute values of principal stresses in his tests. For these tests, an interpretation in terms of $\sin \phi'_m$ and
8.6 Experimental findings

\( \sin \psi \) is possible, where \( \phi'_m \) is the current angle of friction mobilised in the soil, which is not usually the same as the angle of friction mobilised on horizontal planes parallel to the top platen. The narrow spread of all his data on initially loose and dense samples is shown in Fig. 8.15. It is apparent that the agreement with the stress–dilatancy relation (8.42) is

**Fig. 8.15** Range of data from simple shear tests on Leighton Buzzard sand compared with Rowe’s stress–dilatancy relationship (R) (after Stroud, 1971).

**Fig. 8.16** Data from drained plane strain compression tests on Mersey River quartz sand (\( \bullet \), loose, \( \nu_s = 1.66 \); \( + \), dense, \( \nu_s = 1.54 \)) and feldspar (\( \circ \), loose, \( \nu_s = 1.79 \); \( \times \), dense, \( \nu_s = 1.64 \)) (data from Rowe, 1971).
good, even though the principal axes of strain increment and of stress are rotating and not coincident during the tests.

Data from plane strain tests in which no rotations of principal axes occur are reported by Rowe (1971). These have been replotted in terms of $\sin \phi'_m$ and $\sin \psi$ in Fig. 8.16. Equation (8.42) has been plotted in this figure with two values of $\sin \phi'_m$ (0.5 and 0.7) to show the shape of Rowe's stress-dilatancy relationship. The data refer to two sands, each sheared from two initial densities. The data for each sand lie within a narrow band.

Data from triaxial tests on sands are also reported by Rowe (1971). These have been replotted in terms of $\eta$ and $\beta$ in Fig. 8.17. For comparison, the Cam clay flow rule (8.3) and Rowe's stress-dilatancy relationship, deduced from (8.33), have been plotted for $M = 1.0$ and 1.5; the data seem to be following a trend which is closer to that deduced from (8.33). Although these values of $\beta$ have been calculated from total strain increments, it is widely accepted that Rowe's stress-dilatancy relationship, suitably interpreted, provides a reasonable description of the plastic flow of sands, particularly when particle sliding is the dominant mechanism of irrecoverable deformation.

The Cam clay flow rule has not been included in Fig. 8.16. Rowe's stress-dilatancy relationship specifically ignores all elastic deformations. However, to apply the Cam clay flow rule to plane strain situations, an assumption about the elastic behaviour of the soil is required. Although the total strain increments $\delta e_2$ are zero, these are made up of elastic and
8.6 Experimental findings

plastic components $\delta e_2^i$ and $\delta e_2^p$, and there is no reason why these individual components should also be zero. Looking at the relative shapes of the Cam clay and Rowe flow rules for triaxial compression in Fig. 8.17, we cannot expect the plane strain version of Cam clay (even if elastic strains are ignored) to be more successful than Rowe's stress–dilatancy relationship in matching the plane strain data in Fig. 8.16.

For clays, the picture is less clear. Roscoe, Schofield, and Thurairajah (1963) have presented data from triaxial tests on isotropically normally compressed kaolin ($w_L = 0.7, w_p = 0.4$) in support of the original Cam clay flow rule (8.34). Their data have been replotted in terms of $\eta$ and $\beta$ in Fig. 8.18. These data come from conventional drained and conventional undrained compression tests; elastic volumetric strains but not elastic shear strains have been taken into account. The elastic shear strains are likely to be most significant at low stress ratios, where the effective stress path progresses most nearly tangentially to the initial yield locus, and can be expected to be more significant in the undrained test than the drained test. Overestimation of $\delta e_2^p$ leads to overestimation of $\beta$ (8.1).

A more direct study of flow rules for clay is reported by Lewin (1973). Lewin prepared samples of silty clay ($w_L = 0.31, w_p = 0.18$) in the laboratory by consolidation from a slurry of slate dust and water. One group of samples was compressed isotropically ($OA$ in Fig. 8.19a), another group was compressed one-dimensionally ($\delta e_t = 0$) ($OP$ in Fig. 8.19b). For

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**Fig. 8.18** Data from conventional drained (○) and undrained (●) triaxial compression tests on isotropically normally compressed spestone kaolin [stress–dilatancy relationships: (1) Cam clay, (2) original Cam clay] (data from Roscoe, Schofield, and Thurairajah, 1963).
each sample the stress ratio $\eta = q/p'$ was then changed at constant mean effective stress $p'$ ($AB, AC$, etc., in Fig. 8.19a; $PQ, PR$, etc., in Fig. 8.19b), and the strains were measured on a subsequent stress probe at constant stress ratio ($BB', CC'$, etc., in Fig. 8.19a; $QQ', RR'$, etc., in Fig. 8.19b). Lewin states that elastic strains were small so that total strain increments could be assumed to be essentially equal to plastic strain increments. Since the stress increments with which he was probing the yield surfaces in Fig. 8.19 are not far from being orthogonal to probable yield loci, plastic effects are likely to dominate.

Fig. 8.19 (a), (b) Stress paths used to investigate plastic potential and (c) resulting stress–dilatancy relationships for Llyn Brianne slate dust; (a) and (c, o) isotropically compressed; (b) and (c, •) one-dimensionally compressed (data from Lewin, 1973).
The results are shown in terms of $\eta$ and $\beta$ for the two sets of samples in Fig. 8.19c. For this clay there is a large difference between the results for the isotropically compressed and the one-dimensionally compressed clays, and the degree of anisotropy that the one-dimensional history produces in the flow rule is very significant. For example, from Fig. 8.19c, non-distortional compression ($\delta e_q = 0, \beta = 0$) corresponds to a stress ratio $\eta \approx 0.4$, and deformation under isotropic stresses ($q = 0, \eta = 0$) corresponds to a value of $\beta \approx -20^\circ$ and a strain increment ratio $\delta e_q/\delta e_p \approx -0.36$.

Natural soils have certainly experienced non-isotropic stresses in their past, and although they may show a stress ratio, $\eta = M$, at which yielding occurs without plastic volumetric strain, there is no necessity for increases of isotropic stress in the absence of deviator stress to cause only compression without distortion.* Flow rules for natural soils can thus be expected to pass through point C ($\eta = M, \beta = \pi/2$) in Fig. 8.2b but not necessarily to pass through the origin I ($\eta = 0, \beta = 0$).

Strain increment vectors were shown in Fig. 4.14a for natural clay ($w_L \approx 0.77, w_p \approx 0.26$) from Winnipeg, Canada. For this clay Graham, Noonan, and Lew (1983) separated plastic and elastic components of strain increment and plotted plastic strain increment vectors. These data have been transferred to a $\eta;\beta$ stress–dilatancy diagram in Fig. 8.20.

The flow rule for this Winnipeg clay is well defined from a large number

![Fig. 8.20 Stress–dilatancy relationship $\eta;\beta$ observed for undisturbed Winnipeg clay (data from Graham, Noonan, and Lew, 1983).](image)

*The stress ratio at which yielding occurs without plastic volumetric strain can be loosely termed a critical state in the context of the present models: it is a feature of the initial yielding of the undisturbed soil. If major changes in the structure of the soil occur as the soil yields, which is the case for sensitive natural soils, then the basic assumption that yielding occurs without change of shape of the yield loci or plastic potentials may no longer be reasonable; and an ultimate critical state is a feature of a subsequent plastic potential rather than the initial plastic potential.
of triaxial compression tests. The data extend well beyond the point of zero plastic volumetric strain increment $\eta = M, \beta = \pi/2$, and $\delta e_p^v/\delta e_p^v = \infty$. At higher stress ratios, values of $\beta$ greater than $\pi/2$ indicate that shearing is accompanied by plastic volumetric expansion. Although the data show a certain amount of scatter, they do suggest that the flow rule does not pass through the origin. Purely compressive, non-distortional plastic deformation $\beta = 0$ occurs for a small positive stress ratio, and yielding under isotropic stresses $\eta = 0$ corresponds to a small negative value of $\beta$, indicating plastic volumetric compression with negative plastic shear strain ($\delta e_p^v > \delta e_p^v$). On the other hand, the Cam clay flow rule (8.3) with $M = 0.88$ is also plotted in Fig. 8.20 and provides a moderately good fit to these data; in fact, the margin by which the flow rule defined by the experimental data misses the origin $I (\eta = 0, \beta = 0)$ in Fig. 8.20 is really very small. The error in assuming that plastic flow could be described by the Cam clay flow rule would be small. The anisotropy that this implies in the plastic potentials is smaller than that evident in the yield loci (Fig. 3.22).

These data were used in Section 4.4.3 in support of a framework for elastic–plastic soil models based on the hypothesis of associated flow, identity of yield loci, and plastic potentials. Perhaps for clays too an improved description would be obtained by allowing non-associated flow – but the departure from normality does not seem to be as great as for sands.

No particular conclusion can be drawn about the appropriateness of any one theoretical flow rule for matching experimental data for clays. Choice of flow rule must be guided by some experimental observation. It is usually important to make some allowance for the anisotropy that has been introduced during the depositional compression of the clay, though the extent of this anisotropy may be related to the plasticity of the clay.

8.7 Strength and dilatancy

It was shown in Section 7.4 that heavily overconsolidated clays and dense sands can show peak strengths before ultimate critical states are reached. At the critical state, by hypothesis, shearing continues at constant volume. The flow rules or stress–dilatancy relations that govern the deformation behaviour of soils indicate that shearing at a stress ratio not equal to the critical state stress ratio ($\eta \neq M$) is accompanied by volume change. All the flow rules that have been presented here are in broad agreement that the amount of that volume change depends on how far the stress ratio is from the critical state value. For soils showing a peak strength, deforming with $\eta > M$, the shearing is accompanied by volumetric expansion.
8.8 Conclusion

Data of peak strength of sands were presented in Section 7.4.3. Rowe's stress–dilatancy relation (Section 8.5) was found to describe the dilatancy of many granular materials quite satisfactorily, and, in particular, it can be used to estimate the volumetric expansion expected for any observed peak stress ratio. The stress–dilatancy behaviour of sands is not much influenced by stress level, but, as discussed in Section 7.4.3, stress level has a strong influence on the peak strength of sands. The peak dilatancy \( \frac{\delta e_p}{\delta e_q} \) is dependent on stress level only because the peak strength is dependent on stress level.

More comment can now be given about the desirability of relying on peak strengths in calculating the stability of geotechnical structures. Peak strengths can exist as a soil is deformed only if volumetric expansion is prevented. The examples of back analysis of slope and retaining-wall failures in Sections 7.6 and 7.7 showed that if it is possible for expansion (and consequent softening) to occur in a thin failure region, then a critical state strength rather than a peak strength is the only strength that can be relied upon in the long term.

Another comment can also be made about the status of analyses of plastic collapse of drained soils showing frictional strength characteristics. It was noted in Section 7.5 that the validity of plasticity analyses, which could potentially provide upper or lower bounds to actual collapse loads of geotechnical structures, depends on the deformations occurring during plastic failure being linked with the stress states under which plastic failure can occur. Most plasticity analyses are performed for situations of plane strain. For plane strain this requirement for associated flow implies that the angle of dilation \( \psi \) (Fig. 8.8b) and angle of friction \( \phi' \) (Fig. 8.8c) should be equal. Clearly, when the soil is deforming at a critical state \( \phi' = \phi'_{\text{cs}} \) and no volume change is occurring so that \( \psi = 0 \), and the condition of associated flow is not satisfied.

When the soil is deforming at mobilised angles of friction greater than the critical state angle \( \phi'_m > \phi'_{\text{cs}} \), a stress–dilatancy relation such as (8.42) could be used to estimate expected angles of dilation. A typical example of this stress–dilatancy relation for \( \phi'_{\text{cs}} = 35^\circ \) was shown in Fig. 8.16. A peak angle of dilation of, say, \( 20^\circ \) might be measured; the corresponding mobilised angle of friction would be, from (8.42), \( \phi'_m = 49.9^\circ \). Angles of dilation are always considerably lower than angles of friction.

8.8 Conclusion

This chapter has been concerned with stress–dilatancy relationships or flow rules. The primary intention has not been to champion any particular flow rule or stress–dilatancy relation but to illustrate the
principle, which is generally accepted, that the volume-change characteristics and the stress-strain characteristics of soils are linked. Whether this link is thought of as a description of the way in which energy is dissipated as a soil is sheared, or as a shape of a curve (plastic potential) in an effective stress plane which indicates the ratio of components of (plastic) deformation, is secondary.

Experimental data have been presented in support of some of the theoretical flow rules or stress-dilatancy relationships, but Fig. 8.10 shows that since a critical state point is common to Cam clay, original Cam clay, and Rowe's stress-dilatancy, the likelihood of significant differences between these theories that can be detected experimentally is small for stress ratios in the range, say, of $0.6M < \eta < 1.8M$.

The importance of past history on the nature of plastic flow has been noted, and for clay a contrast has been drawn between soil which has a history of isotropic compression and soil which has a history of one-dimensional compression. The structure of a sand sample is determined by the sample preparation procedure, which in nature means the manner of deposition of the sand. Sample preparation usually takes place under a gravitational stress field which is directional and not isotropic. Deformations of sands are very much governed by the ways in which their structure can change, hence the success of flow rules such as Rowe's stress-dilatancy relation. It is not easy to eradicate evidence of the initial structure until the sand has been made to flow at a critical state. It may not be feasible to prepare truly isotropic samples of sands (that is, samples which have an unbroken history of isotropic compression) except in an orbiting space laboratory.

Finally, it is important to remember that a flow rule only describes the mode of plastic deformation that occurs when yielding takes place at any particular state of stress; it does not say anything about whether yielding will actually occur. This is the distinction between yield loci and plastic potentials that was drawn in Chapter 4. Consequently, experimental data in support of or defining any particular flow rule cannot provide evidence either for or against the possibility that the soil may obey the postulate of normality: the identity of yield loci and plastic potentials.

**Exercises**

E8.1. Separate versions of Rowe's stress-dilatancy relationship were produced for triaxial compression and for plane strain conditions in Section 8.5. For conditions of plane strain, an expression relating mobilised angles of dilation and friction has been quoted (8.42). Show that, for this expression to be valid also for conditions of
triaxial compression, the angle $\psi$ would have to be defined as 
(Tatsuoka, 1987)

$$\sin \psi = \frac{-3 \delta e_p}{6 \delta e_q - \delta e_p}.$$ 

E8.2. Translation between plane strain and triaxial compression conditions requires some assumed model of soil response. An alternative hypothesis to Rowe's stress–dilatancy, used in exercise E8.1, would be that, at any particular mobilised angle of friction, the Cam clay flow rule (8.2) was appropriate, irrespective of the test conditions, with generalised strain increments $\delta e_q$ and $\delta e_p$ being defined as in Section 1.4.1.

Neglecting elastic strains, deduce the plane strain form of the Cam clay flow rule and compare this with Rowe's stress–dilatancy relationship.

(Note that if we assume dependence of dilatancy on angle of friction, the value of the intermediate principal stress under plane-strain conditions is not of concern.)

E8.3. Shibata (1963) presents experimental data for a normally compressed Japanese clay tested in drained triaxial compression with constant mean effective stress $p'$. He found that the volume decrease was proportional to the effective stress ratio

$$-\frac{\delta v}{v_i} = k \delta \eta$$

where $v_i$ is the initial specific volume, and $k$ is observed to be a constant. Show that this finding implies that this clay has yield loci of the form given by the original Cam clay model (8.23), provided that

$$k = \frac{\lambda - \kappa}{Mv_i}$$

Assume that the clay is to be modelled within the elastic–plastic framework of Chapter 4, with yield loci of size $p'_o$ which varies only with irrecoverable volumetric strain:

$$\delta p'_o = \frac{vp'_o}{\lambda - \kappa} \delta e_p$$

Assume also that the clay follows the principle of normality or associated flow.

E8.4. In discussion of the existence of critical states for clays in Section 6.3, some data obtained by Parry (1958) were presented (Fig. 6.13) which indicated that soil samples – which in drained
trial compression tests failed at values of mean effective stress $p'_e$ different from the critical state value $p'_{es}$ appropriate to the failure value of specific volume of the sample – were still changing in volume and still heading towards the critical state at failure, according to an expression

$$\frac{\delta\epsilon_p}{\delta\epsilon_q} = k \ln \frac{p'_e}{p'_{es}}$$

Ignoring elastic deformations, deduce the implied flow rule or stress–dilatancy relation

$$\frac{\delta\epsilon_p}{\delta\epsilon_q} = k \ln \frac{M - h_c}{\eta - h_c} \quad \text{for} \quad \eta > M$$

It will be necessary to refer to the failure data for Weald clay which were presented in Section 7.4.1. These data, in terms of effective stresses at failure, supported an expression

$$\frac{q}{p'_e} = h_c \left( g + \frac{p'}{p'_{es}} \right)$$

Use the requirement that the critical state point $(\eta = M, p' = p'_{es})$ must lie on this failure line to obtain an expression for $p'_{es}/p'_e$.

Taking values of $k, M$, and $h_c$ from the data presented in Sections 6.3 and 7.4.1, sketch this flow rule in a $\eta: \beta$ diagram and compare it with the Cam clay flow rule.

E8.5. Investigate the following flow rule for describing the plastic behaviour of clay:

$$\frac{\delta\epsilon_p}{\delta\epsilon_p} = \frac{\eta}{M^2 - \eta^2}$$

Plot this as a stress–dilatancy relationship and compare it with other flow rules discussed in this chapter. Derive an expression for the corresponding plastic potential, sketch the curve, and compare it with that for Cam clay.

For one-dimensional normal compression, derive an expression for the constant stress ratio $\eta_K$, assuming (i) that the elastic shear strains are negligible and (ii) that the changes in volume are given by the usual linear relationships between specific volume and the logarithm of pressure. Adopting values of $\lambda/\kappa = 5$ and $M = 1$, calculate the relevant value of $K_{onc} = \sigma'_{on} / \sigma'_e$ during one-dimensional normal compression and comment on whether this is a reasonable value for a clay with these soil parameters.

E8.6. The results of constant stress ratio tests on Newfield clay performed
Exercises

by Namy (1970) fit the following flow rule:
\[
\frac{\delta e^p}{\delta e^p} = A \eta
\]
where \( A \) is a constant and \( \delta e^p, \delta e^p \), and \( \eta \) have their usual meanings. Plot this flow rule in a \( \eta: \beta \) stress–dilatancy diagram and compare it with the other flow rules introduced in this chapter.

Comment on the compatibility of this flow rule with the possible existence of critical states for this clay. Discuss the errors in the observations of strain increments that could have resulted from the fact that Namy actually performed his constant stress ratio tests by applying alternate large increments of cell pressure and axial stress.