7

Strength of soils

7.1 Introduction: Mohr–Coulomb failure

To most geotechnical engineers the phrase 'strength of soils' conjures up images of Mohr–Coulomb failure criteria. This chapter presents strength of soils in a rather more general way to show how classical notions of Mohr–Coulomb failure can be reconciled with the patterns of response that have been developed in earlier chapters. For convenience, discussion of the link between dilatancy and strength is deferred to Chapter 8; however, it is one of the messages of critical state soil mechanics that these two aspects of soil behaviour are inextricably entwined.

Mohr–Coulomb failure is concerned with stress conditions on potential rupture planes within the soil. The Mohr–Coulomb failure criterion says that failure of a soil mass will occur if the resolved shear stress $\tau$ on any plane in that soil mass reaches a critical value. It can be written as

$$\tau = \pm (c' + \sigma' \tan \phi')$$  \hspace{1cm} (7.1)

This defines a pair of straight lines in the $\sigma' : \tau$ stress plane (Fig. 7.1). If Mohr's circle of effective stress touches these lines, then failure of the soil will occur. It is assumed that, for sliding to occur on any plane, the shear stress has to overcome a frictional resistance $\sigma' \tan \phi'$, which is dependent on the effective normal stress $\sigma'$ acting on the plane and on a friction angle $\phi'$, together with a component $c'$, which is independent of the normal stress. This component $c'$ is often called cohesion but is more usefully regarded merely as an intercept on the shear stress axis which defines the position of the Mohr–Coulomb strength line. [Wroth and Houlsby (1985) prefer to use the symbol $s$ rather than $c$ in order to escape from the physical associations of the 'cohesion' intercept.]

Mohr–Coulomb failure can also be defined in terms of principal stresses.
From Fig. 7.1 the limiting relationship between the major and minor principal effective stresses $\sigma'_1$ and $\sigma''_3$ is

$$\frac{\sigma'_1 + c' \cot \phi'}{\sigma''_3 + c' \cot \phi'} = \frac{1 + \sin \phi'}{1 - \sin \phi'} \tag{7.2}$$

(Previously, principal stresses have been written with subscripts 1, 2, and 3 without any declaration about the relative magnitudes of the three principal stresses. Where it is important to specify the major, intermediate, or minor principal stress, then the subscripts I, II, and III, respectively, will be used.)

Mohr–Coulomb failure, because it deals with conditions on a plane, is opaque to the value of the intermediate principal effective stress $\sigma''_3$, which plays no part in (7.2). Thus, in Fig. 7.1, this intermediate principal stress can be equal to the minor principal stress $\sigma''_3 = \sigma''_1$ (Fig. 7.1a), equal to

Fig. 7.1 Mohr–Coulomb failure. Intermediate principal stress (a) equal to minor principal stress, (b) truly intermediate, and (c) equal to major principal stress.
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the major principal stress \( \sigma_{ii} = \sigma'_{i} \) (Fig. 7.1c), or take some truly intermediate value \( \sigma'_{i} > \sigma''_{ii} > \sigma''_{iii} \) (Fig. 7.1b) without affecting the position of the largest Mohr circle, containing the major and minor principal stresses, which controls the attainment of failure.

The stress conditions illustrated in Fig. 7.1a with \( \sigma''_{ii} = \sigma''_{iii} \) correspond to triaxial compression in which the cell pressure provides the minor (and equal intermediate) principal stress. Expression (7.2) can be rewritten in terms of triaxial stress variables \( \rho', q \), where

\[
\rho' = \frac{\sigma'_{i} + 2\sigma''_{iii}}{3}
\]

(7.3)

\[
q = \sigma'_{i} - \sigma''_{iii}
\]

(7.4)

and becomes (Fig. 7.2)

\[
\frac{q}{\rho' + c' \cot \phi'} = \frac{6 \sin \phi'}{3 - \sin \phi'}
\]

(7.5)

The stress conditions illustrated in Fig. 7.1c with \( \sigma''_{ii} = \sigma'_{i} \) correspond to triaxial extension in which the cell pressure provides the major (and equal intermediate) principal stress. The triaxial stress variable \( \rho' \) then becomes

\[
\rho' = \frac{2\sigma'_{i} + \sigma''_{iii}}{3}
\]

(7.6)

and (7.2) can be written as

\[
\frac{q}{\rho' + c' \cot \phi'} = \frac{-6 \sin \phi'}{3 + \sin \phi'}
\]

(7.7)

This is also plotted in Fig. 7.2, where, for convenience, a negative sign has been assigned to values of \( q \) in triaxial extension.

The critical states described in Chapter 6 and produced in triaxial

![Fig. 7.2 Mohr–Coulomb failure criterion.](image-url)
compression tests defined lines given for many soils by

$$\eta = \frac{q}{p'} = M$$

(7.8)

Comparison of (7.8) and (7.5) suggests that soils are failing in a purely frictional manner at the critical state, that is, with $c' = 0$. The deformations have been so large that the soil has been thoroughly churned up, and any bonding between particles which might have led to some cohesive strength has broken down. With $c' = 0$, comparison of (7.8) and (7.5) shows that for triaxial compression,

$$M = \frac{6 \sin \phi'}{3 - \sin \phi'}$$

(7.9)

or

$$\sin \phi' = \frac{3M}{6 + M}$$

(7.10)

Comparison of (7.7) and (7.5), however, shows that if the soil knows about a particular angle of friction $\phi'$, and if this soil reaches the critical states given by (7.8) in both triaxial compression and triaxial extension, then it is not possible for the value of $M$ to be the same in triaxial compression and extension. In triaxial extension,

$$M^* = \frac{6 \sin \phi'}{3 + \sin \phi'}$$

(7.11)

or

$$\sin \phi' = \frac{3M^*}{6 - M^*}$$

(7.12)

Conversely, if the soil knows about a particular value of $M$ at the critical state, by whatever mode of deformation that critical state may have been reached, then with $M = M^*$, the corresponding angles of friction must, from (7.10) and (7.12), be different. However, experimental evidence (e.g. from Gens, 1982) suggests that the critical state angle of friction is the same under conditions of triaxial compression and triaxial extension (and plane strain), and it would therefore be inappropriate to force $M = M^*$ in (7.10) and (7.12) to describe these ultimate failure conditions.

Expressions (7.9) and (7.11) have been plotted in Fig. 7.3. Although the concern here is with ultimate failure conditions, it will be useful subsequently to interpret them also as relationships between values of stress ratio $\eta = q/p'$ and mobilised angles of friction $\phi'$ at any stage of triaxial compression or extension. Expression (7.9) for triaxial compression...
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can be written approximately as

\[ M \approx \frac{\phi'}{25} \]  \hspace{1cm} (7.13)

and (7.11) can be written rather less accurately as

\[ M^* \approx \frac{\phi'}{35} \]  \hspace{1cm} (7.14)

where, in each expression, \( \phi' \) is measured in degrees.

7.2 Critical state line and undrained shear strength

So far, discussion of the Mohr–Coulomb failure criterion has been in terms of effective stresses. If a soil sample is not allowed to drain, then the Mohr circle of effective stress at failure (\( E \)) in Fig. 7.4) can be associated with an infinite number of possible total stress circles (\( T_1, T_2, \ldots \) in Fig. 7.4) displaced along the normal stress axis by an amount equal to

Fig. 7.3 Relationship between stress ratio \( \eta = q/p' \) and mobilised angle of shearing resistance \( \phi' \) in triaxial compression and extension.

Fig. 7.4 Mohr's circles of total stress and effective stress.
the pore pressure. The pore pressure does not affect the differences of stresses or shear stresses, so all these total stress circles must have the same size. Clay soils are often loaded sufficiently fast for drainage of shear-induced pore pressures to be impossible. It is then convenient to perform analysis of the stability of clay masses in terms of an undrained shear strength \( c_u \), which is the radius of all the Mohr circles in Fig. 7.4, whether effective stress or total stress circles. This undrained strength is the maximum shear stress that the soil can withstand, and the failure criterion for undrained conditions becomes [compare (7.1)]:

\[
\tau = \pm c_u \quad (7.15)
\]

A unique undrained shear strength for a given specific volume has here been presented as a consequence of the principle of effective stress. A critical state line provides a unique relation between specific volume (or void ratio or water content, for a saturated soil) and the ultimate value of deviator stress \( q_d \) for soil sheared in a particular mode, for example, triaxial compression. This ultimate value of deviator stress is the difference between the axial and radial total or effective stresses at the end of the test, which is the diameter of the Mohr circle of effective or total stress (Fig. 7.4) and is hence twice the undrained shear strength \( c_u \). The simple form of experimentally observed critical state lines implies a similarly simple mathematical description of the relationship between undrained

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Fig. 7.5 Conventional undrained triaxial compression test on overconsolidated soil: (a) \( p'q \) effective stress plane (ESP, effective stress path; TSP, total stress path), (b) \( w'p' \) compression plane.
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strength and specific volume and also permits the undrained strength of soil subjected to any specified consolidation history to be estimated.

A soil with a specific volume \( v \) (Fig. 7.5), no matter how that volume was arrived at, will end on the critical state line at a mean effective stress \( p'_c \) when tested in undrained triaxial compression. From (6.10),

\[
p'_c = \exp \left( \frac{\Gamma - v}{\lambda} \right)
\]

(7.16)

This implies an ultimate value of deviator stress

\[
q_t = M p'_c
\]

(7.17)

and hence an undrained shear strength

\[
c_u = \frac{M p'_c}{2} = \frac{M}{2} \exp \left( \frac{\Gamma - v}{\lambda} \right)
\]

(7.18)

To link undrained shear strength with consolidation history, some idealisations concerning compression and unloading of clays must be made. It was suggested in Chapter 6 that both the critical state line and normal compression line could be assumed, from experimental evidence, to be reasonably straight and parallel in a semi-logarithmic compression plane \( v \):\( \ln p' \) (Fig. 7.6) over a reasonable range of mean effective stress. A normal compression line, not necessarily for isotropic normal compression, can be written as

\[
v = v_1 - \lambda \ln p'
\]

(7.19)(4.2bis)

[For isotropic normal compression, \( v_1 \) would be equal to \( N \); compare (5.8).]

The critical state line can be written as

\[
v = \Gamma - \lambda \ln p'
\]

(7.20)(6.10bis)

It is convenient to assume that the unloading and reloading of soil are described by a similar expression,

\[
v = v_u - \kappa \ln p'
\]

(7.21)(4.3bis)

Fig. 7.6 Normal compression line (ncl), unloading-reloading line (url), and critical state line (cst).
giving straight unloading–reloading lines in the $v: \ln p'$ compression plane (Fig. 7.6).

The volume separation of the normal compression and critical state lines is $v_s - \Gamma$ (Fig. 7.6). It will be useful to describe the separation of these lines also in terms of the ratio $r$ of the pressures on any particular unloading–reloading line between the normal compression and the critical state lines. Referring to Fig. 7.6, one has

\[
    r = \frac{P_s}{P_c} \quad (7.22)
\]

or

\[
    r = \exp \left( \frac{v_s - \Gamma}{\lambda - \kappa} \right) \quad (7.23)
\]

This pressure ratio $r$ can be regarded as an extra soil parameter. In the description of elastic–plastic models for soil in Chapter 4, the value of $r$ is implicitly fixed by the relative geometry of yield loci and plastic potentials. In particular, for the Cam clay model described in Chapter 5, if isotropic normal compression is being considered, so that $v_s = N$, then $r$ is the ratio of tip pressure to critical state pressure for any yield locus and is fixed at $r = 2$.

Now consider an undrained test on a sample which has been normally compressed to $O$ (Fig. 7.5) with $p' = p'_o$, and then unloaded and allowed to swell to $I$ with $p' = p'_i$. Even if this loading and unloading history is not isotropic, an isotropic overconsolidation ratio $n_o$ can be defined as the ratio of these two mean effective stresses:

\[
    n_o = \frac{P'_o}{P'_i} \quad (7.24)
\]

The specific volume of the sample at $I$ is

\[
    v_i = v_s - \lambda \ln p'_o + \kappa \ln n_o \quad (7.25)
\]

Hence, from (7.18), the undrained strength reached at point $F$ on the critical state line (Fig. 7.5) is

\[
    c_u = \frac{M}{2} \exp \left[ \frac{(I - v_s)}{\lambda} + \ln p'_o - \left( \frac{\kappa}{\lambda} \right) \ln n_o \right] \quad (7.26)
\]

which, from (7.23) and (7.24), can be written as

\[
    c_u = \frac{P'_o}{P'_i} \left( \frac{n_o}{r} \right)^\Lambda \quad (7.27)
\]

where

\[
    \Lambda = \frac{\lambda - \kappa}{\lambda} \quad (7.28)(6.19\text{bis})
\]
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In (7.27), the undrained strength is made non-dimensional by dividing it by the mean effective stress which the soil experienced at the end of compression (or unloading) just before the undrained test was begun. This strength ratio is linked with the isotropic overconsolidation ratio \( n_p \), the other parameters in (7.27), \( M, \Lambda \), and \( r \), are all soil constants describing different aspects of the effective stress response of the soil. Expression (7.27) links a total stress characteristic of soils, the undrained shear strength, with effective stress parameters and the consolidation history of the soil.

The parameter \( \Lambda \) describes the relative slopes of the normal compression and unloading–reloading lines for the soil. For a typical clay with \( \Lambda = 0.85 \), \( r = 2 \), and \( M = 1 \) [implying \( \phi' \approx 25^\circ \) from (7.13)], the variation of \( c_u/p_i' \) with \( n_p \) is shown in Fig. 7.7. When both \( c_u/p_i' \) and \( n_p \) are plotted on logarithmic scales, a straight-line relationship emerges with slope \( \Lambda \).

For normally compressed samples, \( n_p = 1 \) and

\[
\frac{c_u}{p_i'} = \left( \frac{M}{2} \right)^{r-\Lambda} \text{ (7.29)}
\]

The ratio of the values of \( c_u/p_i' \) for overconsolidated and normally compressed samples is independent of \( M \) and \( r \):

\[
\frac{c_u/p_i'}{c_u/p_i'} = n_p^{\Lambda} \text{ (7.30)}
\]

This analysis has been presented only in terms of the isotropic overconsolidation ratio \( n_p = p_u'/p_i' \). A conventional overconsolidation ratio \( n = \sigma'_{UU}/\sigma''_{UU} \), defined in terms of the ratio of past and present vertical effective stresses, would be more familiar to practising engineers. The relationship between \( n_p \) and \( n \), of course, depends on the type of past loading. If the soil has known only isotropic stresses through laboratory testing, then \( n = n_p \). If the soil has a history of one-dimensional loading and unloading,

Fig. 7.7 Ratio of undrained strength to initial mean effective stress \( (c_u/p_i') \) varying with isotropic overconsolidation ratio.
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Fig. 7.8 Ratio of undrained strength to initial vertical effective stress \( (c_v/c_u) \) varying with overconsolidation ratio \( n \): (1) Drammen clay \( (I_p = 0.30) \) (after Andersen, Berre, Kleven, and Lunne, 1979); (2) Maine organic clay \( (I_p = 0.34) \); (3) Bangkok clay \( (I_p = 0.41) \); (4) Atchafalaya clay \( (I_p = 0.75) \); (5) AGS CH clay \( (I_p = 0.41) \); (6) Boston blue clay \( (I_p = 0.21) \); (7) Connecticut Valley varved clay \( (I_p = 0.39/0.12) \) (after Ladd, 1981).
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then the link between \( n \) and \( n_p \) is more complex (it is discussed in Section 10.3). Written in terms of vertical effective stresses instead of mean effective stresses, (7.30) becomes

\[
\frac{c_u/\sigma'_{vi}}{(c_u/\sigma'_{vi})_{ac}} = \left( \frac{n}{n_p} \right)^\lambda
\]  

(7.31)

Data of the strength ratio \( c_u/\sigma'_{vi} \) from undrained simple shear tests on six clays, reported by Ladd and Edgers (1972) and Ladd, Foot, Ishihara, Schlosser, and Poulos (1977), are shown in Fig. 7.8a. The samples in these tests were initially one-dimensionally compressed, and the strength ratio is plotted against the overconsolidation ratio \( n \). The same data are presented in Fig. 7.8b in the form of the ratio \((c_u/\sigma'_{vi})/(c_u/\sigma'_{vi})_{ac}\) plotted against \( n \). Ladd et al. remark that all the data then fit into a narrow band which is reasonably well defined by the expression

\[
\frac{c_u/\sigma'_{vi}}{(c_u/\sigma'_{vi})_{ac}} = n^\mu
\]  

(7.32)

where \( \mu \approx 0.8 \), though a better fit is obtained if \( \mu \) reduces from 0.85 to 0.75 with increasing overconsolidation.

Although the expressions (7.30) and (7.31) were deduced with reference to undrained strengths in triaxial compression tests, the ratio of strengths of overconsolidated and normally compressed soils contained in this expression should not depend on the type of test used to measure the strengths (the same test being used for all strengths). It will emerge in Section 10.3.2 that the values of \( n^\mu \) and \((n/n_p)n^\lambda_p\) are typically fairly close for values of \( n_p \) less than 16. That the empirical expression (7.32) has the same form as (7.30) provides support for the analysis presented in this section. The value of the exponent \( \Lambda \) depends on the soil being investigated; it may be coincidence that the seven clays studied by Ladd et al. provide such a narrow band of data in Fig. 7.8b.

It will be seen in Section 9.4.5 that there is a problem in determining reliable values of \( \Lambda \), the slope in the \( \nu: \ln p \) compression plane of unloading–reloading lines. Consequently, Mayne and Swanson (1981) have made use of (7.30) [or, strictly, expression (7.32) with \( \mu \) set equal to \( \Lambda \), since they have analysed data from 95 clays subjected to a variety of tests on isotropically and one-dimensionally consolidated samples] to deduce the average value of \( \Lambda \) of which each soil is aware. They report a large range of values of \( \Lambda \) (see Section 9.4.5). The value of \( \Lambda \) thus deduced can in principle be used in other analyses using models based within the framework of critical state soil mechanics.
7.3 Critical state line and pore pressures at failure

In Section 1.6, a pore pressure parameter \( a \) was introduced to link pore pressure changes with changes in applied total stresses:

\[
\delta u = \delta p + a \delta q \tag{7.33}(1.65\text{bis})
\]

It was shown in Sections 1.6 and 5.4 that \( a \) is an indication of the current slope of the undrained effective stress path. Since undrained effective stress paths will not usually be straight, the value of \( a \) will certainly not be a soil constant but will depend on the current effective stress state in the soil, as well as on the history of consolidation of the soil. The link between effective stress and total stress descriptions of soil strength requires some knowledge of the pore pressure at failure. Even though the pore pressure parameter \( a \) may not be a constant, an average value \( a_t \) can be defined as the average slope of the effective stress path during the undrained loading:

\[
a_t = -\frac{\Delta p'}{\Delta q} \tag{7.34}
\]

It has already been shown (Section 7.2) that the mere proposal of the existence of a critical state line permits an estimate to be made of the undrained triaxial compression strength of overconsolidated samples. The pore pressure that would be observed when a critical state was reached in a conventional undrained test, and the corresponding value of \( a_t \), can be calculated by an extension of the same analysis.

Consider a sample with initial isotropic overconsolidation ratio \( n_p \) at \( I \) (Fig. 7.5):

\[
n_p = \frac{p'_o}{p_i} \tag{7.24\text{bis}}
\]

When subjected to an undrained compression test, the path in the compression plane moves from \( I \) to \( F \), on the critical state line (Fig. 7.5b). The end point of the test is thus fixed in the stress plane 100 (Fig. 7.5a), though the shape of the effective stress path between \( I \) and \( F \) is not known. The value of deviator stress \( q_t \) at \( F \) is given from (7.27) as

\[
q_t = M p \left( \frac{n_p}{r} \right)^{\frac{1}{r}} \tag{7.35}
\]

For a conventional triaxial compression test, the total stress path (TSP in Fig. 7.5a) has a gradient \( \Delta q/\Delta p = 3 \). Thus, when the undrained test reaches failure, the total stress changes will be:

\[
\Delta q = q_t \tag{7.36a}
\]

\[
\Delta p = \frac{q_t}{3} \tag{7.36b}
\]
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The effective mean stress at the end of the test, \( \sigma^' \) is

\[
\sigma^' = \frac{q}{M} = \sigma_0 \left( \frac{n_p}{r} \right)^\Lambda
\]

(7.37)

and the pore pressure change during the test will be (Fig. 7.5a)

\[
\Delta u = \sigma^' + \Delta P - \sigma^'
\]

\[
= \sigma^' + q \left( \frac{1}{3} - \frac{1}{M} \right)
\]

\[
\Delta u = \sigma_0 \left[ 1 + M \left( \frac{n_p}{r} \right)^\Lambda \left( \frac{1}{3} - \frac{1}{M} \right) \right]
\]

(7.38)

Then, from (7.34),

\[
\frac{a_t}{q_u} = -\frac{\left( \sigma^' - \sigma_0 \right)}{q_u}
\]

\[
a_t = \frac{\left( n_p/r \right)^\Lambda - 1}{M}
\]

(7.39)

Data of pore pressures at failure in triaxial compression tests on samples of Weald clay tested at different isotropic overconsolidation ratios are given by Bishop and Henkel (1957) and shown in Fig. 7.9 in terms of the pore pressure parameter \( a_t \). Values of \( a_t \) have been calculated and plotted for comparison, using (7.39), with the soil parameters \( \lambda = 0.091 \) and

Fig. 7.9 Dependence of pore pressure parameter \( a_t \) at failure on overconsolidation ratio for isotropically overconsolidated Weald clay (after Bishop and Henkel, 1957).
$M = 0.95$ (as found in Section 6.3), $\kappa = 0.034$ (from an average slope of the unloading lines in Fig. 6.8), and $r = 2.0$ (according to the Cam clay model). Then, from (7.28), $\Lambda = 0.63$. Agreement is good, particularly at lower overconsolidation ratios.

For isotropically normally consolidated Weald clay ($n_r = 1$), $a_r \approx \frac{3}{4}$. This value is typical of many clays and indicates, from comparison of (7.34) and (7.36), that for the conventional triaxial compression test in which the deviator stress is increased at constant cell pressure, the pore pressure at failure is approximately equal to the deviator stress at failure (Fig. 7.10). It follows that the average slope of the effective stress path from $I$ to $F$ is about $-\frac{1}{3}$. For other total stress paths, the pore pressure at failure would be quite different even if the values of deviator stress and pore pressure parameter $a_r$ were the same, because the contribution to the pore pressure of the change of total mean stress $\Delta p$ would be quite different.

For $n_r = 2$, the value of $a_r$ in Fig. 7.9 is calculated and observed to be zero. For this overconsolidation ratio, the Cam clay model predicts a purely elastic response all the way from the initial state to failure at a critical state, and there is no tendency for dilatancy of the soil to contribute to the observed pore pressure.

### 7.4 Peak strengths

In the previous two sections, the strength of soil has been associated with conditions at the critical state. At a critical state, shear deformations can continue at constant effective stresses and constant volume: the soil is being continuously churned up or remoulded. This is in principle an ultimate state, and a critical state strength should be an ultimate strength. (Practical exceptions will be discussed in Section 7.7.)

According to the Cam clay model described in Chapter 5, all tests
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eventually reach the critical state line. However, it was seen in Section 5.3 and Fig. 5.7 that drained tests on heavily overconsolidated samples pass through a peak value of deviator stress followed by a subsequent drop in deviator stress to a critical state. This occurs because the yield locus in the $p':q$ stress plane lies above the critical state line at high values of overconsolidation ratio. If conventional drained triaxial compression tests were performed on a large number of samples, each subjected to the same maximum past isotropic consolidation pressure but unloaded to different overconsolidation ratios, then the locus of the maximum values of deviator stress reached in these tests would be as shown in Fig. 7.11a. This locus of peak points is made up of two sections:

1. Normally compressed and lightly overconsolidated samples (up to an overconsolidation ratio of about 2 according to the Cam

Fig. 7.11 Points of peak deviator stress $q$ in conventional drained triaxial compression tests on isotropically overconsolidated samples having same past maximum preconsolidation pressure: (a) $p':q$ effective stress plane; (b) $v:p'$ compression plane.
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clay model) end up on the critical state line between C and A, and the critical state condition represents the maximum value of deviator stress that they have experienced.

2. For more heavily overconsolidated samples, the peak deviator stress comes at the intersection of the applied stress path ($\delta q/\delta p = 3$ for conventional drained compression) with the initial yield locus between C and B. (The point B would be reached for a sample tested at zero cell pressure.)

Clearly, it would be inappropriate to try to fit a single Mohr–Coulomb strength equation such as (7.5) to this set of data.

These data represent soil samples with a single past maximum consolidation pressure. Similar sets of tests on soil samples with different past maximum consolidation pressures would produce similar loci of peak deviator stresses, with normally compressed and lightly overconsolidated samples still ending on the critical state line, but with more heavily overconsolidated samples reaching a peak on distinct segments of different critical yield loci (Fig. 7.12). Conventional drained tests on samples with all possible histories would produce peak points lying in the fan-shaped region TOC (Fig. 7.12), and it would not be rational to try to pick a single Mohr–Coulomb strength line to represent an average of all these data.

One way in which the peak strength locus BCA in Fig. 7.11 could be made to apply to all soils would be to make the axes non-dimensional by dividing by the maximum past consolidation pressure $p'_c$. All the results of Fig. 7.12 would then become a single locus BCA in the $p'/p'_c:q/p'_c$ plane (Fig. 7.13). Given a particular sample of soil, it is not easy to deduce what the past maximum consolidation pressure might have been. However, it is feasible to measure the water content (for saturated soil) and thus, with

Fig. 7.12 Loci of points of peak deviator stress $q$ in conventional drained triaxial compression tests on isotropically overconsolidated samples having different past maximum preconsolidation pressures.
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knowledge of the normal compression characteristics of the soil, to deduce the equivalent consolidation pressure $p'_c$ (Fig. 6.6 and Section 6.2).

The data of peak deviator stress were discussed in Fig. 7.11a without reference to paths in the compression plane. The tests that reach their peak on the critical state line between $C$ and $A$ in the stress plane will, at that peak deviator stress, be on the critical state line between $C$ and $A$ in the compression plane (Fig. 7.11a, b). The tests that reach their peak on the initial yield locus between $B$ and $C$ in the stress plane will, at that peak, be on the initial unloading–reloading line between $B$ and $C$ in the compression plane (Fig. 7.11a, b). The two-part nature of this locus of peak points is again clear. The specific volume (or water content) of each sample changes during these drained tests. When the stresses $p'_c:q$ at the point of peak deviator stress are divided by the value of the equivalent consolidation pressure $p'_c$ appropriate to the volume of the sample at that point, the data of $p'/p'_c; q/p'_c$ again lie on a single, curved segment ($CB$ in Fig. 7.14). As described in Section 6.2, the critical state line becomes a single point $C$ in this plot: All the tests that reach their peak on the critical state line, on $OC$ in Fig. 7.12, plot at point $C$ in Fig. 7.14. All the tests that reach their peak within the fan $TOC$ in Fig. 7.12, plot on the curve $CB$ in Fig. 7.14. The use of the non-dimensional stress quantities $p'/p'_c; q/p'_c$ takes account of differences in water content at failure in different samples and makes apparently incompatible information compatible.

In the predictions of the response of heavily overconsolidated soil in drained triaxial compression made using the Cam clay model, the peak deviator stress was followed by a drop of deviator stress (towards a critical state) as deformation was continued (see Figs. 5.7 and 7.15). Such a falling stress: strain curve can only be followed to the critical state ($FC$) in a test.

Fig. 7.13 Data of peak deviator stress normalised with respect to past maximum preconsolidation pressure $p'_c$.
in which the axial compression $\varepsilon_\text{a}$ of the sample is steadily increased, a strain-controlled test. In a test in which the deviator stress $q$ is steadily increased (a stress-controlled test), catastrophic failure occurs as soon as the peak deviator stress is reached ($FG$ in Fig. 7.15). Even in a strain-controlled test, uniform deformation of the sample is unlikely beyond the peak. Theoretical treatment of the development of non-uniform deformations in test specimens is complex (e.g. see Vardoulakis, 1978; Vermeer, 1982), but a qualitative discussion is possible with reference to conditions on a potentially critical plane within the specimen.

Just as the critical state line in the $p':q$ plane can be equated with a purely frictional strength criterion $OC$ in the normal stress:shear stress $\sigma':\tau$ plane (Fig. 7.16a), so the peak strength envelope for heavily overconsolidated samples, $BC$ in Fig. 7.11, can be converted into a strength criterion $BC$ in the $\sigma':\tau$ plane (Fig. 7.16a). If behaviour in terms of $\sigma'$ and $\tau$ is thought of as being broadly equivalent to behaviour in terms of $p'$ and $q$, then the response expected on this potentially critical plane is as follows. The initial stresses are isotropic, so the initial shear stress at $P$ is zero. As the drained

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**Fig. 7.14** Data of peak deviator stress normalised with respect to equivalent consolidation pressure $p_e$.

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**Fig. 7.15** Strain softening response after peak.
7.4 Peak strengths

Fig. 7.16 Bifurcation of response as deformation becomes concentrated in thin region: (a) $r':\sigma'$ effective stress plane; (b) $\sigma':\sigma'$ compression plane; (c) shear stress $\tau$ and specific volume $\psi$; (d) shear stress $\tau$ and shear displacement $x$; (e) distribution of specific volume before formation of failure plane; (f) distribution of specific volume after formation of failure plane.
compression test proceeds, the shear stress and the normal stress on this plane increase, \( PQ \) (Fig. 7.16a). This is associated with essentially elastic changes in specific volume (Fig. 7.16b, c) and elastic deformations between points on opposite sides of the plane (Fig. 7.16d). After the peak strength envelope \( BC \) has been reached at \( Q \), then softening to the critical state at \( R \) requires increase in specific volume or water content of the soil. As the water content or specific volume of the soil increases, the corresponding strength curve changes (compare Fig. 7.12), and the soil becomes weaker. This will be an unstable process because it is extremely unlikely that the soil sample will be absolutely homogeneous; it only requires that some parts of the sample should start this post-peak softening fractionally ahead of other parts for further deformation to become concentrated in the weaker parts of the sample. Shearing then occurs almost solely on critical planes in the soil. In Fig. 7.16, the soil in the critical plane (which is likely to be a thin zone of failure) may continue to deform towards the critical state \( QR \). If the soil is saturated, then the expansion in volume in this failure region may occur at the expense of a decrease in volume of the adjacent soil, which will unload stiffly, \( QS \). There is a bifurcation of the response similar to that which occurs when a strut buckles and similar to the uncertainty of response illustrated in Fig. 5.8. The expected spatial distribution of water content in a saturated soil before and after this bifurcation of response and localisation of deformation is shown in Figs. 7.16e, f. A field observation of just such a distribution of water content will be discussed in Section 7.7.

A nice example of the contrast between soil which deforms steadily to a critical state and soil which develops failure planes after a peak strength is shown in Fig. 7.17, taken from Taylor (1948). The loose sample of sand

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Fig. 7.17 Failed samples of (a) dense and (b) loose Fort Peck sand tested in conventional drained triaxial compression (from Taylor, 1948).
7.4 Peak strengths

(Fig. 7.17b) hardens as it is sheared and bulges symmetrically at large deformations. No critical failure planes are being mobilised in this sample; and although the restraint provided by the end platens is producing some axial non-uniformities, variables based on principal stresses and strains such as $p':q$ and $\varepsilon_p':\varepsilon_q$ will be appropriate for describing the behaviour to large deformations. The dense sample of the same sand tested under similar conditions (Fig. 7.17a) develops a failure plane. Although variables such as $p':q$ and $\varepsilon_p':\varepsilon_q$ may be appropriate for describing the behaviour before the failure plane develops, once it has developed, it is only logical to describe subsequent response in terms of the normal and shear stresses acting on the plane and the components of strain within the thin failure zone, or the deformations across the failure plane.

Such a study of stress and deformation conditions obtaining in a failure zone formed in a dense sand sample has been carried out by Vardoulakis (1978); some typical results are shown in Fig. 7.18. In Fig. 7.18a, the friction mobilised on the failure plane is plotted against the relative sliding

![Diagram](attachment:image.png)

Fig. 7.18  (a) Mobilised friction and (b) volume change of thin failure zone shearing to critical state (medium grained Karlsruhe sand, $v_s = 1.57$) (after Vardoulakis, 1978).
movement \( x \) that occurs across the plane. The friction is initially high but drops to a steady value as deformation continues. In Fig. 7.18b, \( \delta y/\delta x \), the rate of change of separation \( y \) across the failure plane with sliding movement \( x \), is plotted against \( x \). This ratio is a measure of the volume change occurring in the failure zone and is high as sliding begins but falls to zero as deformation continues. At larger deformations, the dense sand has actually reached a critical state at which it deforms at constant volume and constant mobilised friction.

7.4.1 Peak strengths for clay

Once localisation of deformation occurs in a test sample (Fig. 7.16), then statements about deformations and specific volumes made on the basis of boundary measurements will no longer be reliable. Hvorslev (1937) studied the strength of clays using a direct shear box. He was aware of the localisation of deformation that occurred and took care to dismantle his apparatus rapidly after each test so that samples of soil could be obtained from the region of failure, for water-content determination. He was then able to plot his data of failure in both a stress plane and a compression plane.

In the shear box (Fig. 1.24), the stress components that are measured are the average normal effective stress, \( \sigma'_n = P/A \), and the average shear stress, \( \tau_h = Q/A \), acting on the horizontal plane of failure. Typically, shear box tests are performed with the normal stress \( \sigma'_n \) held constant. The stress plane in which data from shear box tests can be plotted has axes of normal stress \( \sigma'_n \) and shear stress \( \tau_h \), and the compression plane also has abscissa \( \sigma'_n \). One set of results obtained by Hvorslev (1937), for Vienna clay, is shown in Fig. 7.19. The peak shear stresses that were observed are shown in the \( \sigma'_n : \tau_h \) stress plane in Fig. 7.19a. The data were obtained from specimens sheared from various one-dimensionally normally compressed states or at various overconsolidation ratios on a single unloading-reloading loop. The pattern found is broadly similar to that shown for the Cam clay model in Fig. 7.11a.

The water contents measured in the thin failure zone of the shear box are plotted in the \( \sigma'_n : w \) compression plane in Fig. 7.19b. Again, the pattern can be related to that shown for the Cam clay model. Normally compressed samples, 1 and 2, show equal drops in water content from the one-dimensional normal compression line to the critical state line; such equal drops are implied by the parallelism of these lines assumed in the Cam clay model. Overconsolidated samples reach failure at points in the compression plane below the critical state line; compare Figs. 7.19b and 7.11b.
Fig. 7.19 Shear box tests on normally compressed and overconsolidated Vienna clay: (a) $\tau_n$, $\sigma'_v$, effective stress plane; (b) $w$; $\sigma'_c$, compression plane; (c) equivalent consolidation pressure $\sigma''_c$ and vertical effective stress $\sigma'_v$ (after Hvorslev, 1937).
Since complete information about stress states is not available from this apparatus, it is not possible to calculate values of mean effective stress at any stage of a test, and hence it is not possible to calculate values of the equivalent consolidation pressure \( p'_e \). However, an equivalent one-dimensional consolidation pressure \( \sigma'_{ve} \) can be defined as shown in Fig. 7.20 (compare Fig. 6.6). Values of the equivalent consolidation pressure \( \sigma'_{ve} \) relevant at failure in these shear box tests are shown in Fig. 7.19c. It is immediately apparent that the pattern of variation of \( \sigma'_{ve} \) at failure is essentially the same as the pattern of variation of \( \tau_b \) at failure.

By plotting the ratios \( \sigma'_{ve}/\sigma'_{ve} \) and \( \tau_b/\sigma'_{ve} \) at failure, the data are brought together into a single picture which is equivalent to that shown in Fig. 7.14 for Cam clay. The data for Vienna clay are shown in Fig. 7.21. The data lie on a straight line of the form

\[
\frac{\tau_b}{\sigma'_{ve}} = c'_{ve} + \frac{\sigma'_{ve}}{\sigma'_{ve}} \tan \phi'_{e}
\]  

(7.40)

where \( c'_{ve} \) and \( \phi'_{e} \) are soil parameters. Comparing this with the conventional

**Fig. 7.20** Equivalent consolidation pressure \( \sigma'_{ve} \) on one-dimensional normal compression line (1-D ncl).

![Diagram](image)

**Fig. 7.21** Failure data from shear box tests on Vienna clay (after Hvorslev, 1937).

![Diagram](image)
7.4 Peak strengths

Mohr–Coulomb failure criterion,

\[ \tau = c' + \sigma' \tan \phi' \]  \hspace{1cm} (7.1bis)

we see that

\[ c' = c_{ve}' \sigma_{ve}' \]  \hspace{1cm} (7.41)

and

\[ \phi' = \phi'_e \]  \hspace{1cm} (7.42)

In other words, the apparent cohesion to be used in the Mohr–Coulomb strength equation depends linearly on the equivalent consolidation pressure; hence, if a linear relation is assumed between water content and logarithm of pressure during normal compression, the cohesion increases exponentially as the water content decreases.

This provides a good illustration of the need to consider a volumetric quantity as well as effective stress variables when trying to assemble strength data. If strength data follow the relationship found by Hvorslev (7.40), then it is only samples which fail at the same water content or specific volume – and hence at the same equivalent consolidation pressure – that will lie on a single, simple Mohr–Coulomb line (7.1). A series of samples which have been unloaded from a single maximum past pressure, or a series of samples taken from a single profile in the ground, can have quite different water contents and hence can produce failure points which lie each on a quite different strength line.

The normally compressed and lightly overconsolidated samples tested by Hvorslev did actually fail and reach their greatest strength on what, in the light of discussions in Chapter 6, can be called a critical state line. At the critical state, the strength seen in the shear box should be purely frictional and of the form

\[ \tau_c = \sigma' \tan \phi'_c \]  \hspace{1cm} (7.43)

This critical state condition imposes a limit on the range of validity of (7.40), and Fig. 7.21 shows that there is a cluster of points at the right-hand end of the line. The values of the strength quantities \( c_{ve}', \phi'_e, \) and \( \phi'_c, \) in (7.40) and (7.43) for Vienna clay are given in Table 7.1, together with values for the Little Belt clay, also tested by Hvorslev.

The significance of the limited extent of the Hvorslev failure line (7.40) may become clearer when peak strength points from triaxial tests are considered and a diagram which is directly comparable with Fig. 7.14 can be obtained. For triaxial tests on isotropically normally compressed and overconsolidated samples, the equivalent consolidation pressure to be used to normalise the stress results is \( p'_e, \) as defined in Fig. 6.6 and Section 6.2. The peak strength data obtained by Parry (1956) in a large number of
Table 7.1. Hvorslev strength parameters

<table>
<thead>
<tr>
<th>Soil</th>
<th>(c'_{ve})</th>
<th>(c'_{pe})</th>
<th>(\phi'_v) (deg)</th>
<th>(\phi'_{ve}) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vienna clay</td>
<td>0.100</td>
<td>0.141*</td>
<td>17.5</td>
<td>26.0</td>
</tr>
<tr>
<td>Little Belt clay</td>
<td>0.145</td>
<td>0.187*</td>
<td>9.9</td>
<td>19.6</td>
</tr>
<tr>
<td>Weald clay</td>
<td>0.034*</td>
<td>0.046</td>
<td>18.8</td>
<td>22.4</td>
</tr>
</tbody>
</table>

*These values have been calculated assuming that \(p'_c/\sigma'_{ve} = (1 + 2K_{one})/3 = (3 - 2\sin\phi'_v)/3\), where \(K_{one}\) is the ratio of horizontal to vertical effective stress during one-dimensional normal compression.

Triaxial tests on Weald clay are shown in Fig. 7.22 in the dimensionless stress plane \(p'/p'_c - q/p'_c\). These tests include drained and undrained, compression and extension tests with various total stress paths, but the failure points lie closely around two straight lines: one for compression tests and one for extension tests. (Extension tests, having axial stress less than radial stress, are plotted with negative values of \(q = \sigma_x - \sigma_r\).) Although these lines have slopes of different magnitude in the \(p'/p'_c - q/p'_c\) plane, they are in fact matched by a single set of Hvorslev strength parameters [refer to Section 7.1 and expressions (7.5) and (7.7)], as shown by Schofield and Wroth (1968). Expression (7.40) is now slightly modified to

\[
\frac{\tau}{p'_c} = c'_{pe} + \frac{\sigma'}{p'_c} \tan \phi'_v \tag{7.44}
\]

Fig. 7.22 Failure data from triaxial compression and extension tests on Weald clay (data from Parry, 1956).
7.4 Peak strengths

so that (7.41) becomes
\[ c' = c'_{pe}p'_e \]  
(7.45)

In triaxial compression, the failure data lie on the line
\[ \frac{q}{p'_e} = h_e \left( g + \frac{p'}{p'_e} \right) \]  
(7.46)

and in triaxial extension the failure data lie on the line
\[ \frac{q}{p'_e} = -h_e \left( g + \frac{p'}{p'_e} \right) \]  
(7.47)

with
\[ g = c'_{pe} \cot \phi'_e \]  
(7.48)

and, by comparison with (7.5) and (7.7),
\[ h_e = \frac{6 \sin \phi'_e}{3 - \sin \phi'_e} \]  
(7.49)

and
\[ h_e = \frac{6 \sin \phi'_e}{3 + \sin \phi'_e} \]  
(7.50)

The values of the strength parameters \( c'_{pe}, \phi'_e, \) and \( \phi'_{os} \) for Weald clay are also given in Table 7.1. The final figure, for \( \phi'_{os} \), comes from (6.25) via (7.9). The critical state condition again provides a limit to the line, just as the critical state point \( C \) provided a limit to the peak strength line predicted by the Cam clay model in Fig. 7.14. Normally compressed and lightly overconsolidated samples are expected to proceed stably to a critical state without a premature peak.

There will be a limit to the extent of the Hvorslev failure line also at low values of \( p'/p'_e \). If, for example, it is supposed that the soil can withstand no tensile effective stresses, then the condition of zero effective radial stress defines a limiting line in triaxial compression \( OA \) in Fig. 7.22,
\[ q = 3p' \]  
(7.51)

and the condition of zero effective axial stress defines a limiting line in triaxial extension \( OB \) in Fig. 7.22,
\[ q = -\frac{3p'}{2} \]  
(7.52)

The Hvorslev lines then span between the critical state points and the no-tension lines.

Comparison of Fig. 7.14 with Fig. 7.22 (and indirectly with Fig. 7.21) suggests that although the Cam clay model as formulated in Chapter 5
is correctly indicating that peak strengths are to be expected for heavily overconsolidated samples, the peak strengths that are being predicted by the model look too high.

A complete picture of the triaxial compression response of the Weald clay can be produced in the dimensionless $p'/p'_e:q/p'_e$ plane, as shown in Fig. 7.23a. The constant volume locus of the Cam clay model (Fig. 6.7) has been drawn between the isotropic normal compression point $N$ and the critical state point $C$, using the same values of soil parameters as in Section 7.3 ($M = 0.872$, $\lambda = 0.091$, and $\kappa = 0.034$). The Hvorslev line from Fig. 7.22 has been drawn through $C$ to meet the no-tension line $OT$ at $T$.

![Triaxial test paths and Hvorslev failure for Weald clay (+, drained tests; ×, undrained tests): (a) $p'/p'_e:q/p'_e$; (b) $\eta:v_\lambda$ (data from Bishop and Henkel, 1957).](image)
7.4 Peak strengths

The data of the two drained and two undrained tests on Weald clay that were replotted in the stress plane and compression plane in Fig. 6.9 have also been included in Fig. 7.23a.

It is clear that in this two-dimensional plot the data of drained and undrained tests are brought together. The two tests on the normally compressed samples lie close to the Cam clay curve NC. The two tests on the heavily overconsolidated samples rise towards the Hvorslev line TC and curl over towards the critical state point, which they fail to reach before the tests are terminated.

This two-dimensional diagram can be used, following Schofield (1980), to illustrate the regions of different expected characters of response. At high values of \( p'/p'_e \), the soil yields stably according to an elastic-plastic soil model such as Cam clay and eventually reaches a critical state \( C \). At values of \( p'/p'_e \) less than about \( \frac{2}{3} \) for Weald clay (the values for other clays depend on the relevant values of \( M, \lambda, \) and \( k \)), rupture of the soil can occur on critical planes according to a Hvorslev strength criterion (7.44), \( CT \) in Fig. 7.23 - or Mohr-Coulomb strength criterion (7.1) with strength parameters given by (7.45) and (7.42). At very low values of \( p'/p'_e \) (perhaps less than about \( \frac{1}{2} \)), fracture of the soil may occur as the soil fails in tension, \( TO \) in Fig. 7.23.

Writing the Hvorslev expression for the peak strength using the form (7.46) or (7.44) suggests an apparent cohesion dependent on current water content, through an equivalent consolidation pressure \( p'_e \). Equation (7.46) can be rearranged as

\[
\eta = \frac{q}{p'} = h_e \left( 1 + g \frac{p'_e}{p'} \right) \tag{7.53}
\]

which suggests a peak stress ratio, or peak mobilised angle of friction, which is dependent on mean effective stress. The alternative two-dimensional plot of Section 6.2 uses stress ratio \( \eta \) and the volumetric variable \( v_\lambda \):

\[
v_\lambda = v + \lambda \ln p' \tag{6.13bis}
\]

If the equivalent consolidation pressure \( p'_e \) is determined on the isotropic normal compression line, then from (6.13) and (6.17),

\[
v_\lambda = N + \lambda \ln \frac{p'}{p'_e} \tag{7.54}
\]

and (7.53) becomes

\[
\eta = h_e \left( 1 + g \exp \frac{N - v_\lambda}{\lambda} \right) \tag{7.55}
\]

Then, since at the critical state \( v_\lambda = \Gamma \) and \( \eta = M \), the four soil parameters
$M, \Gamma, g, \text{ and } h_e$ are related through

$$M = h_e \left( 1 + g \exp \frac{N - \Gamma}{\lambda} \right) \quad (7.56)$$

and only three of the four parameters are in fact independent.

The Weald clay data of Fig. 7.23a have been replotted in terms of $\eta$ and $v_\lambda$ in Fig. 7.23b, and the coalescence of data of drained and undrained tests is again apparent. (The value of $\Gamma$ implied in Fig. 7.23b is different from that deduced in Section 6.3. The separation $N - \Gamma$ of the critical state line and isotropic normal compression line is found experimentally to be greater than that deduced from Cam clay with $\lambda = 0.091$ and $\kappa = 0.034$. This value of $\kappa$ was deduced from the average slope of the unloading curves in Fig. 6.8. The observed value of $N - \Gamma$ is consistent with a much lower value of $\kappa$, perhaps corresponding to the initial slope of the unloading curves where they leave the normal compression line.)

The two-dimensional plots $p'/p'_e:q/p'_e$ or $\eta:v_\lambda$ were presented in Section 6.2 as convenient ways of displaying information about three variables: $p', q$, and specific volume $v$. The two-dimensional diagrams of Fig. 7.23 can be transformed back into a three-dimensional surface in $p':q:v$ space (Fig. 7.24). The segments $NC$, $CT$, and $TO$ now become surfaces corresponding to Cam clay yielding, Hvorslev rupture, and tensile fracture; these essentially limit the combinations of stress and specific volume that can be reached in triaxial compression tests. This composite surface can be called a state boundary surface.
7.4 Peak strengths

7.4.2 Interpretation of peak strength data

This section discusses some strength data from triaxial tests on 42 samples (3 samples from each of 14 locations and depths) of London clay to illustrate how the Hvorslev strength equation (7.44) can help to focus attention on the factors controlling the strength of the soil. These data have been discussed also by Wroth and Houlsby (1985).

All the tests were consolidated drained triaxial compression tests. A standard procedure was adopted of taking sets of three samples from a single depth and testing them with three cell pressures. A traditional way of interpreting the failure data is to draw a straight line touching as nearly as possible the three Mohr circles of effective stress at failure (Fig. 7.25a). This line is then treated as a Mohr–Coulomb strength criterion of the form (7.1), giving values of apparent cohesion $c'$ and friction angle $\phi'$ for

![Diagram showing selection of strength parameters from drained triaxial tests on London clay: (a) Mohr-Coulomb failure line fitted to Mohr's circles of effective stress; (b) range of values of cohesion $c'$ and angles of friction $\phi'$; (c) failure points in $\sigma'$-$q$ effective stress plane.](image-url)
this trio of samples. The range of Mohr–Coulomb strength values deduced from the 14 sets of samples is shown in Fig. 7.25b; the range of apparent cohesions is from 6.9 to 56.2 kPa with an average of 30.8 kPa, and the range of angles of friction is from 16.2° to 25.0° with an average of 21.2°.

An alternative, fairly traditional way of interpreting the failure data is to plot the failure values of mean effective stress \( p' \) and deviator stress \( q \) for all 42 samples (Fig. 7.25c) and then to fit the best straight line through all these points. The slope and intercept of this line can be converted, from (7.5), to values of apparent cohesion \( c' = 27.4 \) kPa and angle of friction \( \phi' = 21.3^\circ \).

Neither of these methods of interpretation makes allowance for the fact that the various sets of samples may be different even though they are samples from one geological stratum and one locality, so that the different strengths seen in Fig. 7.25 may be related to differences in the past histories of the soil. One manifestation of differences in the past history of the soil is the different values of water content of the various samples. If information about the normal compression of the clay is available, then water content differences can be accommodated through the equivalent consolidation pressure \( p'_e \). The 42 failure points are plotted in Fig. 7.26a as values of \( p'/p'_e \cdot q/p'_e \), and again the best line can be fitted through these data. Using (7.48) and (7.49) produces \( c'_{pe} = 0.031 \) and \( \phi'_e = 19.7^\circ \).

The angle of friction has dropped a little from the previous averages, but the value of \( c'_{pe} \) can be converted to a relationship between apparent

Fig. 7.26  Failure data for London clay (a) plotted in normalised effective stress plane \( p'/p'_e ; q/p'_e \), (b) implied dependence of cohesion on water content.

![Diagram](image)
7.4 Peak strengths

cohesion $c'$ in (7.1) and water content (Fig. 7.26b). On the one hand, this
suggests that a little of the scatter of the strength data can be ascribed to
variations in water content, though the variations of water content between
the various samples were not great, and the amount of scatter is perhaps
not significant; a case could probably be made for supporting either
interpretation. On the other hand, whether or not this final interpretation
is the preferred interpretation, the discussion of Section 7.4.1 should serve
to emphasise the importance of controlling water content if peak cohesive
strengths are to be replied upon.

7.4.3 Peak strengths for sand

The message that has emerged from Section 7.4.1 is that peak
strengths of clay can be properly understood only if account is taken of
both the effective mean stress $p'_t$ and the specific volume $v_t$ at failure, since
both of these will influence the failure value of deviator stress $q_t$. The
interdependence of these three quantities was explored in two ways. The
first exploration was in a plot of $p'/p'_t:q/p'_t$, using the Hvorslev analysis
and interpreting the data as an indication that clays have an apparent
cohesion that is dependent on water content. The second exploration was
in a plot of $\eta:q$. In this plot, since $\eta = q/p'$ is equivalent to mobilised
friction, the data can be interpreted to indicate an angle of friction
dependent on the failure state, as summarised by the variable
$v_{st} = v_t + \lambda \ln p'_t$, which combines the information of mean effective stress
and of specific volume. The two interpretations are entirely equivalent,
and it may ultimately be personal preference which will guide the choice
of variable apparent cohesion or of variable friction as a means of
describing the failure conditions. Sands are usually thought of as frictional
materials, and it is an adaptation of this second approach that appears
to lead to the most convenient description of the strength of sands. The
picture will be built up in stages.

‘Loose sands and gravels are known to have less resistance to shear
than the same soils in a dense state’ (Winterkorn and Fang, 1975). A first
estimate of the peak angle of friction of a sand might be obtained from
charts such as those produced by Winterkorn and Fang (1975) or the U.S.
Department of the Navy (1971), which require knowledge only of the
packing of the sand (and some basic information about particle shape
and size). The packing of the sand is indicated by its relative
density $I_D$:

$$I_D = \frac{v_{\text{max}} - v}{v_{\text{max}} - v_{\text{min}}} \quad (7.57)$$
where \( v_{\text{max}} \) and \( v_{\text{min}} \) are so-called maximum and minimum values of specific volume, determined by standard procedures (e.g. see Kolbuszewski, 1948). This dependence on relative density is illustrated for Chattahoochee River sand in Fig. 6.20, where the dense samples were prepared with \( I_D = 0.84 \) and the loose sample with \( I_D = 0.14 \).

It is evident from Fig. 6.20, however, that relative density on its own is not sufficient since the dense sample tested at a high stress level shows a much lower strength, close to that of the loose sample. The data shown in Fig. 6.20 were obtained from tests in which the mean stress was kept constant. Most test data for soils have come from conventional triaxial compression tests in which the cell pressure is held constant, with the consequence that in a drained test the mean stress level increases from the start of the test until failure occurs. If only the strength of sands in conventional triaxial compression tests is of concern, then it may be acceptable to seek correlation of strength with initial densities and confining stresses.

The steady decrease in the peak angle of friction of dense Chattahoochee River sand as the mean effective stress at failure increases is shown in Fig. 7.27 (from Vesić and Clough, 1968). At high stresses, the dense sand shows no peak strength and proceeds to a critical state angle of friction \( \phi'_{\text{cs}} \), as the loose sand does at all stress levels. The character of this response has been examined for many sands by Bolton (1986), and he has produced the expression

\[
\phi' - \phi'_{\text{cs}} = 3I_D(10 - \ln p') - 3
\]

(7.58)

Fig. 7.27 Variation of peak angles of shearing resistance \( \phi' \) with mean effective stress \( p' \) for initially dense (•) and initially loose (○) samples of Chattahoochee River sand (after Vesić and Clough, 1968) with expression (7.58) superimposed.
as a best fit to a wide range of data. In this expression, mean stress has
to be measured in kilopascals and \( \phi' \) in degrees. Bolton suggests that it
should be used only where it leads to values of \( \phi' \) in the range
\[ 12^\circ > (\phi' - \phi'_{\text{cs}}) > 0. \] The resulting chart is superimposed on the data in
Fig. 7.27.

The problem with the use of relative density as an index of sand
behaviour is that it is conventionally computed using the specific volume
of the sample as it has been prepared, with no confining pressure.
Consequently, it does not reflect the changes in volume that may occur
either as an initial stress state is applied or as the sand is sheared. For
clays, the composite volumetric variable \( v_\lambda \) was used to reflect the current
volumetric and stress state. The data in Fig. 7.23b can be described by
the expression
\[ \phi' - \phi'_{\text{cs}} = f(v_\lambda - \Gamma) \quad (7.59) \]
If a critical state line for a sand of the same form
\[ v = \Gamma - \lambda \ln p' \quad (7.20\text{bis}) \]
can be located in the \( p' : v \) compression plane, then the difference
\[ v_\lambda - \Gamma = v + \lambda \ln p' - \Gamma \quad (7.60) \]
can be calculated at any stage of a test.

An extensive study of the use of the quantity \( (v_\lambda - \Gamma) \) to characterise
the strength (and dilatancy) of sands has been made by Been and Jefferies
(1985, 1986). They have managed to locate straight critical state lines in
the \( v : \ln p' \) compression plane for many sands and sandy silts and have
calculated values of \( (v_\lambda - \Gamma) \), which they call the 'state parameter', from
the volume and mean stress obtaining when the sand is about to be sheared
in a triaxial test. Thus, they include the effect of the volume change that
has occurred as the sample is compressed but not the effect of dilatancy
during shear. The strength data for sand that they have accumulated
reveal a fairly narrow spread (Fig. 7.28). As a result, if the initial value of
the state parameter \( (v_\lambda - \Gamma) \) is known, then the peak strength to be
expected in triaxial compression tests can be estimated to an accuracy of
\[ \pm 2.5^\circ. \] Subsequent work (Been, Crooks, Becker, and Jefferies, 1986; Been,
Jefefries, Crooks, and Rothenburg, 1987) has indicated that this state
parameter \( (v_\lambda - \Gamma) \) is useful also in understanding results of cone
penetration tests in sands and sandy silts.

However, the use of initial values of \( (v_\lambda - \Gamma) \) is not satisfactory if a
rational picture of sand response is to be built up because, in general,
volume changes and stress changes on relevant field stress paths (which
may bear little resemblance to triaxial compression stress paths) will lead
to continuous and major variations in \( v_\lambda \).
Consideration of the data for the Chattahoochee River sand in Fig. 6.21a shows that the critical state line that has been estimated for this sand can be considered only locally straight in the \( v : \ln p' \) compression plane. If the specific volume \( v_{cs} \) on the critical state line, of whatever actual shape, at the current mean effective stress can be determined, then the quantity \( (v - v_{cs}) \) becomes a more general state variable which has wider application than \( (v_{cs} - \Gamma) \). If the critical state line is straight in the \( v : \ln p' \) compression plane, then

\[
v_{cs} = \Gamma - \lambda \ln p'
\]  

(7.61)

Fig. 7.28  Variation of peak angles of shearing resistance of sands with state parameter (after Been and Jefferies, 1986).

Fig. 7.29  Isotropic compression and conventional triaxial compression tests (●●) on Sacramento River sand (after Lee and Seed, 1967).
and, from (7.60),

\[ v - v_{\text{es}} = v_1 - \Gamma \]  \hspace{1cm} (7.62)

Examples of the use of \( v_1 \) to show the progress of triaxial and simple shear tests on sand have been given by Stroud (1971) and Atkinson and Bransby (1978). The use of the state variable \((v - v_{\text{es}})\) to produce a diagram for a sand similar to Fig. 7.23b for Weald clay will now be described, using data for Sacramento River sand reported by Lee and Seed (1967).

Lee and Seed report results of conventional triaxial compression tests performed at constant cell pressures between 98 kPa and 12 MPa. They report triaxial tests on samples prepared at two initial densities and isotropic compression tests on samples prepared at four initial densities. All the available compression plane information is shown in Fig. 7.29, where the arrows indicate the progress from initial state to failure of the triaxial tests. The steady reduction in dilatancy of the dense samples is apparent. The loose samples also show some dilatancy when tested at the lowest stress level.

An approximate location for a curved critical state line in the compression plane is suggested in Fig. 7.29; a straight line would not fit well with these data. The isotropic compression tests have a smaller slope than the critical state line at the same stress level. Major volume changes only occur in isotropic compression at very high mean stress levels, when particle crushing becomes a major feature of the response (compare Fig. 6.22). At very high stress levels, greater than about 10 MPa for this Sacramento River sand, the isotropic compression and critical state lines may be becoming approximately parallel, and the behaviour starts to resemble the compression behaviour of clays. At lower stress levels, the changing slope of the critical state line is significant, and in this detail the behaviours of sand and clay diverge, though the general pattern is broadly similar.

The curved critical state line in Fig. 7.29 has been used to calculate values of \( v_{\text{es}} \) for particular values of mean effective stress. The failure data reported by Lee and Seed are presented in Fig. 7.30 in terms of peak angle of friction as a function of \( v_r - v_{\text{es}} \), calculated from the failure values of specific volume. An approximate description of these failure data is

\[ \phi' - \phi'_{\text{es}} \approx -55(v_r - v_{\text{es}}) \quad \text{for} \; v_{\text{es}} > v_r \]  \hspace{1cm} (7.63a)

\[ \phi' - \phi'_{\text{es}} = 0 \quad \text{for} \; v_{\text{es}} < v_r \]  \hspace{1cm} (7.63b)

where \( \phi' \) is measured in degrees and \( \phi'_{\text{es}} \) is the critical state value. The expectation from the work of Been and Jefferies (1985, 1986) is that these relationships should hold for samples of this sand prepared at any initial density.
Complete test paths for two tests on initially dense and two tests on initially loose samples are shown in a \((v - v_e) \cdot \eta\) plot in Fig. 7.31, which is equivalent to Fig. 7.23b for Weald clay. The loose sample tested at low pressure and the dense sample tested at a moderately high pressure show essentially no change in state variable \((v - v_e)\) as they are sheared; they start and remain very close to the critical state line in the \(v: \ln p'\) plane. The dense sample tested at low pressure rises to a peak and then heads down towards the critical state. The loose sample tested at high pressure

Fig. 7.30  Dependence of peak angles of shearing resistance on state variable at failure for Sacramento River sand: •, dense, \(v_e = 1.61\); ○, loose, \(v_e = 1.87\) (data from Lee and Seed, 1967).

Fig. 7.31  Triaxial test paths for Sacramento River sand: (1) \(v_e = 1.609\), \(\sigma'_f = 98.1\, \text{kPa}\); (2) \(v_e = 1.576\), \(\sigma'_f = 293.3\, \text{kPa}\); (3) \(v_e = 1.870\), \(\sigma'_f = 98.1\, \text{kPa}\); (4) \(v_e = 1.769\), \(\sigma'_f = 393.4\, \text{kPa}\) (data from Lee and Seed, 1967).
7.5 Stability and collapse calculations

rises steadily towards the critical state. In effect, (7.63b) is redundant because samples with $v > v_{cr}$ are not expected to show a peak before the critical state is reached.

Plotting information in terms of $\eta$ and $(v - v_{cr})$, where $v_{cr}$ is deduced from a curved critical state line, amounts to a geometric distortion of the $\eta: v_t$ plot used as a two-dimensional representation of effective stress and specific volume information for clays but serves the same purpose in bringing together data from samples with widely differing densities (corresponding to samples of clay with widely differing histories of overconsolidation). The state variable $(v - v_{cr})$ introduces mean effective stress in a rational way, and a limiting relationship such as (7.63a) (shown as curve A in Fig. 7.31) could be converted back into a limiting surface in $p':q:u$ space for the sand, very similar to Fig. 7.24 for a clay.

Relative density is not a sufficient quantity for characterising sand behaviour. It might be suggested that $(v - v_{cr})$ should be normalised by dividing it by $(v_{max} - v_{min})$ to produce a composite state variable which can bring together data for sands of differing mineralogy, angularity, and particle size (compare Hird and Hassona, 1986; Been and Jefferies, 1986). The specific volume range $(v_{max} - v_{min})$ gives an indication of the range of packings available at low stress levels but does not appear to relate directly either to the slope of the critical state line at low stress levels or to the slope of isotropic compression curves at higher stress levels, where particle crushing starts to become important; these are both factors that, through the state variable $(v - v_{cr})$, appear to have a controlling influence on sand behaviour in general and on the strength of sands in particular.

7.5 Status of stability and collapse calculations

Values of strength parameters for soils are required for analysis of the stability and collapse of geotechnical structures. There are essentially three principal approaches to the estimation of the loads that cause collapse of geotechnical structures; they have been well described and applied by Atkinson (1981) and are only briefly summarised here. The first two, stress fields and collapse mechanisms, are well founded in the theory of plasticity; the third, limit equilibrium, has no such theoretical basis but has been found to provide plausible results in many applications and is widely used.

1. Stress fields. If a distribution of stress within the soil can be found which is in equilibrium with the applied loads and which does not violate the failure criterion for the soil, then the applied loads will not cause collapse. It may be supposed that the soil will always be cleverer in distributing its stresses and hence that a
human estimate of safe applied loads will always be a lower bound to the actual collapse load of the soil mass.

2. **Collapse mechanisms.** If a mechanism of collapse for the soil can be postulated, then the corresponding applied loads can be calculated as the loads necessary to drive the collapse mechanism, that is, to provide a work input which just balances the work absorbed by the collapse mechanism in the soil. However, the soil will always be cleverer in finding other, more efficient modes of collapse than those that we postulate, and hence a human estimate of collapse loads will always be an upper bound to the actual collapse load of the soil mass.

3. **Limit equilibrium.** In the limit-equilibrium method, the soil is typically divided into a number of blocks separated by failure planes. It is assumed that the stresses on these failure planes cannot violate the failure criterion for the soil. Calculations of equilibrium of the blocks are linked together to estimate, from equilibrium considerations alone, the collapse loads. The mechanism of collapse – the arrangement of the failure planes – can be varied and optimised to obtain the most pessimistic estimate of the collapse load. There is no attempt to generate distributions of stress within the sliding blocks and no attempt to satisfy any kinematic constraints in choosing the mechanisms.

That the first two methods do indeed provide lower and upper bounds to collapse loads can be readily proved (e.g. see Calladine, 1985; Davis, 1968), but the proof is subject to the condition that the plastic deformations of the soil that occur at collapse should be associated with the failure criterion. The concept of normality or associated flow was discussed in Section 4.4.3. It was noted that to postulate associated flow for soils is often convenient but is not always justifiable; an example of a sliding frictional block which did not follow a rule of associated flow was introduced in Section 4.4.1. Normality was incorporated into the elastic–plastic models for soil in Chapter 4 in terms of normality of plastic strain increment vectors to yield loci.

The critical state line provides a locus of failure points, states of stress at which indefinite shearing can occur, but the plastic deformations that occur at failure are associated not with this critical state line, which provides the failure criterion, but with the yield loci. Normality to the critical state line would imply very high rates of dilation, volumetric expansion, at failure (Fig. 7.32a). (The rates of dilation that might be associated with peak strengths rather than ultimate critical state strengths
will be discussed in Chapter 8.) In general, therefore, there is a difficulty in applying the theorems of plastic collapse to justify the designation of estimates of collapse loads for soil structures as upper or lower bounds. One exception exists, however, in the analysis of the undrained collapse of clay masses. Analyses of undrained failure are usually conducted in terms of total stresses. It was shown in Section 7.2 that a consequence of the principle of effective stress is that the undrained strength of soil is independent of the applied total stresses. In the \( p:q \) total stress plane, the undrained failure criterion becomes a straight line parallel to the \( p \) axis (Fig. 7.32b). When the critical state is reached in undrained shearing, plastic shear deformation continues at constant effective stresses and constant volume. Thus vectors of plastic strain increment plot normal to the failure criterion \( (\delta e_p^e = 0, \text{Fig. 7.32b}) \), and the condition of associated flow is satisfied. For this case only, techniques of collapse analysis based on study of stress fields and collapse mechanisms do indeed lead to demonstrable lower or upper bounds to the collapse loads.

### 7.6 Total and effective stress analyses

Although it has been emphasised throughout previous chapters that the response of soils should be studied in terms of effective stresses, it is often easier to analyse equilibrium of geotechnical structures in terms of total stresses. For example, at depth \( z \) in a soil deposit (Fig. 7.33), the total vertical stress will be

\[
\sigma_v = \gamma z
\]  

(7.64)

where \( \gamma \) is the unit weight of the overlying soil. However, the effective vertical stress cannot be known without some information about the pore

---

**Fig. 7.32** (a) Normality of plastic strain increment vectors to critical state line (csl) implies continuing large volumetric expansion at failure; (b) total stress strength criterion for undrained failure and corresponding plastic strain increment vectors.
pressure, and the pore pressure may consist of separable parts due to the presence of a water table or a steady seepage flow regime, and due to the shearing of the soil.

Stability analyses can be carried out in terms of either total stresses or effective stresses. Clearly, a geotechnical structure that is on the point of collapse should appear critical whether the stability analysis is carried out in terms of total stresses or effective stresses. For the effective stress analysis to be correct, it is necessary that the distribution of pore pressures should be correctly known; for this reason, the idea of pore pressure parameters linking changes in pore pressure with changes in applied total stresses was introduced in Section 1.6. For the total stress analysis to be correct, it is necessary that the variation of soil strength should be correctly known.

Total stress analyses are usually performed for situations in which undrained loading or response is of interest and a single undrained shear strength can be assigned to the soil. Then, at different points within the soil mass, different total stress Mohr circles may be operating at failure, but they will all have the same radius and will be displaced by appropriate pore pressures from a single effective stress Mohr circle, as shown in Fig. 7.4. Once some drainage has been permitted, then changes of water content or specific volume will have occurred, and it may no longer be reasonable to argue that the total stress Mohr circles should all have the same size: where the water content has fallen, the strength will have increased and vice versa.

An example of parallel total and effective stress stability calculations is provided by back-analysis of a slope failure which occurred at Jackfield, Shropshire after a period of heavy rain, during the winter 1952–3 (Henkel and Skempton, 1955). A plan and section of the landslide are shown in Fig. 7.34. About 300,000 tons of overconsolidated clay soil slid between 10 and 20 m down a plane slope inclined at 10.5° to the horizontal. Investigation revealed the existence of a thin but extensive plane of failure parallel to the plane of the slope and at a uniform depth of about 5 m.

An element of clay of depth $z$ from the surface of an infinite slope of angle $\beta$, width $b$, unit thickness, and weight $W = \gamma bz$, where $\gamma$ is the total unit weight of the clay, is shown in Fig. 7.35a. Consideration of
equilibrium of the forces acting on the element shows that the total normal stress $\sigma$ and shear stress $\tau$ acting on a plane at depth $z$ parallel to the slope (Fig. 7.35a) are

$$\sigma = \gamma z \cos^2 \beta$$  \hspace{1cm} (7.65)

$$\tau = \gamma z \cos \beta \sin \beta$$  \hspace{1cm} (7.66)

A total stress approach to the analysis of the stability of the slope would say that failure would be expected if the shear stress (7.66) reached the undrained strength of the clay.

The clay at Jackfield had liquid limit $w_L = 0.45$, plastic limit $w_p = 0.2$, and natural water content, away from the failure zone, $w = 0.2$. The undrained strength of this clay was 76.6 kPa. The saturated unit weight of the clay was 20.4 kN/m$^3$. At a depth of 5 m the shear stress would have been $\tau = 18.3$ kPa, giving an apparent factor of safety of 4.2 by comparison with the strength of the bulk of the clay.

Fig. 7.34  Landslide at Jackfield, Shropshire (after Henkel and Skempton, 1955).
Tests on specimens of clay from the failure zone showed that the soil there had a water content \( w = 0.30 \) and an undrained strength 21.5 kPa. Using this strength, we obtain an apparent factor of safety of 1.17, which is sufficiently close to unity to confirm concern for the stability of the slope.

Drained triaxial tests gave effective stress data from which Henkel and Skempton deduced that the clay failed according to a Mohr–Coulomb failure criterion

\[
\tau = c' + \sigma' \tan \phi' \quad (7.1\text{bis})
\]

with \( c' = 7.2 \) kPa and \( \phi' = 21^\circ \). The shear strength on any plane depends on the effective normal stress \( \sigma' \), where

\[
\sigma' = \sigma - u \quad (7.67)
\]

and \( u \) is the pore pressure. The total stress on the failure plane can be calculated from (7.65) as \( \sigma = 98.6 \) kPa, and the necessary pore pressure to precipitate failure according to the failure criterion with these strength parameters is then \( u = 69.7 \) kPa, corresponding to a head of 7.1 m of water on the failure surface.

High positive pore pressures are unlikely to result from the shearing of an overconsolidated clay (see Section 7.3). The maximum steady in situ pore pressures occur when there is a state of steady seepage parallel to the slope, down the hillside (Fig. 7.35b). The flowlines for this seepage

Fig. 7.35  (a) Element of soil in infinite slope; (b) seepage parallel to slope.
flow are then parallel to the slope, the equipotentials orthogonal to the slope, and the total pore pressure at depth z is (from Fig. 7.35b)

\[ u = \gamma_w z \cos^2 \beta \]  
(7.68)

At a depth of 5 m, then, a maximum pore pressure of \( u = 47.4 \text{ kPa} \) could be anticipated.

The values of the strength parameters \( c' \) and \( \phi' \) quoted above relate to peak strengths seen in drained tests. It seems from the water content evidence that in the failure zone the clay had an opportunity to soften towards an ultimate critical state condition, for which it would be expected that a failure criterion with \( c' = 0 \) would be more appropriate. Accepting the angle of friction \( \phi' = 21^\circ \) as a critical state angle of friction and assuming that a pore pressure \( u = 47.4 \text{ kPa} \) was present on the failure surface, we can calculate a shear strength of 19.6 kPa, giving a factor of safety of 1.07 by comparison with the shear stress \( \tau = 18.3 \text{ kPa} \) and also confirming an expectation of instability. (The critical state angles of friction \( \phi'_{cs} \) quoted in Section 7.4.1 are in general somewhat higher than Hvorslev angles of friction \( \phi' \) used to describe peak strength conditions. However, the strength parameters quoted by Henkel and Skempton have not been deduced from a plot in which water content differences have been taken into account, and the angle of friction quoted is probably intermediate between \( \phi' \) and \( \phi'_{cs} \).)

Thus total and effective stress analyses of this slope failure both give sensible results, provided the appropriate undrained strength is used in the former and the appropriate strength parameters and appropriate pore pressures are used in the latter. A moral from the effective stress analysis is that peak strengths may not be reliable for design purposes: their existence relies on softening of the soil not occurring or being prevented.

7.7 Critical state strength and residual strength

When a cutting is excavated in a stiff, heavily overconsolidated clay and a retaining wall is constructed to support the remaining soil, negative pore pressures are left in the clay behind the wall as a result of the reduction of the lateral stress and the associated shearing (and repressed dilatation, assuming the construction process to be rapid) of the soil. A negative pore pressure makes effective stresses higher than total stresses and contributes beneficially to the strength of the clay. However, with time, these negative pore pressures tend to cause water to be drawn in from the nearby soil at a rate dependent on the swelling and permeability characteristics of the clay and on the structure of the soil. The water content or specific volume of the clay increases and its strength decreases.
If the design of the wall has not taken this potential softening into account, then failure is likely to occur some time after the construction of the wall.

An example is provided by the failure in 1954 of a retaining wall built in 1912 on a London underground railway line, described by Watson (1956) and quoted by Schofield and Wroth (1968). A section through the wall is shown in Fig. 7.36a. Investigations indicated the presence of discontinuous slip planes behind the wall, and water content studies reported by Henkel (1956) showed significant increases of water content

Fig. 7.36  Failure of retaining wall near Uxbridge: (a) section (after Watson, 1956); (b) water content variations around slip zone (after Henkel, 1956).
in the slip zone (Fig. 7.36b), of a form similar to those proposed on theoretical grounds in Fig. 7.16 and Section 7.4. Just as for the slope at Jackfield, described in the previous section, calculations of the stability of the wall that do not allow for the softening of the soil to a critical state in the failure region do not lead to sensible conclusions.

Data gathered by Skempton (1970b) for a number of slope failures in London clay in general confirm this picture (Fig. 7.37): failures are consistent with a critical state angle of friction \( \phi'_{cs} = 20^\circ \) and zero cohesion

![Fig. 7.37 Peak strength envelope, critical state line, and residual strength envelope for London clay (adapted from Skempton, 1970b).](image)

**Fig. 7.37** Peak strength envelope, critical state line, and residual strength envelope for London clay (adapted from Skempton, 1970b).

![Fig. 7.38 (a) Diagram of ring shear test; (b) drop of shearing resistance to residual value in ring shear test on Kalabagh clay \( (I_p = 0.36) \) (after Skempton, 1985); (c) orientation of clay particles on residual sliding surface.](image)

**Fig. 7.38** (a) Diagram of ring shear test; (b) drop of shearing resistance to residual value in ring shear test on Kalabagh clay \( (I_p = 0.36) \) (after Skempton, 1985); (c) orientation of clay particles on residual sliding surface.
$c' = 0$. Exceptions are provided, however, by slope failures which turn out to be reactivations of previous slope instabilities within historic or geologic past. For these, the angle of friction mobilised is considerably lower than the critical state value.

Such low angles of friction can be reproduced in the laboratory if ring shear tests are carried out in which very large displacements between two parts of a clay sample are applied. The ring shear apparatus is rather like a long shear box in which the two ends of the box have been bent round and joined together (Fig. 7.38a) so that, like the shear box, failure is forced to occur in a thin central region but, unlike the shear box, the failure surface through the soil specimen has no ends to it. Typical results of a ring shear test on Kalabagh clay taken to large displacement are shown in Fig. 7.38b from Skempton (1985): after a displacement of 100–200 mm, the angle of friction has dropped from a peak value of about 22° to a residual value $\phi'_r \sim 9^\circ$. Skempton notes that water content changes in the shear zone seem generally to have ceased after displacements of 5–10 mm. Skempton (1970b) notes that although sections of failure planes were found at the retaining wall failure in Fig. 7.36, there was no continuous failure surface; movements prior to failure had been small and insufficient to produce a deterioration of friction to the residual value.

When the mechanisms of residual shearing and shearing to a critical state are compared, it becomes clear that they are very different. When soil is at a critical state, it is being continuously remoulded and churned up, and its structure remains random. When a clay is sheared to a residual state on a failure surface, the deformations have been so large that the clay particles on both sides of the failure surface have become oriented parallel to the failure (Fig. 7.38c), so it is not surprising that the friction generated on such polished, slickensided surfaces is much lower than the friction mobilised when soil particles are still being stirred up.

The critical state strength is thus not always a lower bound to the strength of the soil; but whether residual strength is actually important depends on whether it is possible for a smooth, continuous failure surface to form through the soil. This possibility depends on the composition of the soil. Most soils are made up a range of particle sizes and shapes. The formation of a smooth failure surface requires the presence of an adequate proportion of clay particles; platelike particles which are capable of realignment. A soil with only a small proportion of clay particles is not able to form a continuous failure surface because the dominant rotund particles get in the way.

Lupini, Skinner, and Vaughan (1981) suggest that the most satisfactory correlation is with a volumetric variable, granular specific volume $v'_s$,
7.7 Critical state and residual strength

where, for saturated soil [compare (1.6)],

\[ v^* = 1 + \frac{\text{volume of water} + \text{volume of platelike particles}}{\text{volume of rotund particles}} \] (7.69)

This variable is broadly equivalent to plasticity or clay content; but, unlike these quantities, granular specific volume can make some allowance for the change in volume occupied by clay particles that may occur when the stress level is changed. A plot of residual friction against granular specific volume for a number of soils – some natural soils and some artificial sand and clay or glacial till and clay mixtures – is shown in Fig. 7.39a. Low residual angles of friction are expected for granular specific volumes in excess of about 3. To confirm this observation, Lupini et al. performed microscopic examinations of thin sections of the soil samples which had been sheared to large displacements. They were able to distinguish the different modes of failure that are illustrated in Figs. 7.39b, c. With a low clay content and low granular specific volume, no preferred orientation of clay particles developed, and the structure showed evidence of 'turbulent' flow to a critical state (Fig. 7.39b). With a high clay content and high granular specific volume, slickensided failure surfaces were found with strong orientation of clay particles (Fig. 7.39c). The observations of turbulent, transitional, or sliding behaviour are noted on Fig. 7.39a.

Residual strength is of particular importance where a new geotechnical structure is likely to load soil which has been previously sheared to very

Fig. 7.39 (a) Angles of shearing resistance and modes of failure related to granular specific volume \( v^*_g \) (○, turbulent; ×, transitional; ●, sliding); (b) and (c) mixtures of London clay and Happisburgh till; (b) clay fraction 0.2; (c) clay fraction 0.4 (after Lupini, Skinner, and Vaughan, 1981).
large deformations; frequently this will be the result of landslides which have occurred in the geological history of a deposit, but it may also result from solifluction, erosion, or other more recent processes. An engineer will need to be careful to look out for possible existing shear surfaces during the site investigation and need to study the geological record in detail.

Residual strength may not control the occurrence of first-time failures in previously intact clay, but the movements that occur when a slip takes place may be all the larger because the strength that can be mobilised on a failure plane drops dramatically as deformations develop.

7.8 Conclusion

Various aspects of the strength of soils have been discussed. The existence of a critical state line and its relationship to a normal compression line allow statements to be made about the variation of undrained strength with overconsolidation ratio. It has been shown that the form of the elastic–plastic models of soil behaviour developed in Chapters 4 and 5 leads to the existence of peak strengths. The need for strain softening to occur as the soil weakens from a peak strength to a critical state makes it very likely that localisation of deformation in thin rupture zones will occur, and this will tend to obscure overall observation of attainment of critical states. The critical state represents in many ways a lower bound to the strength of soils, but a lower, residual strength may be seen if it is possible for orientation of particles parallel to a failure plane to develop. It is, however, very important to distinguish between the drop of strength which arises because of particle reorientation and the drop of strength which occurs in overconsolidated soils as they suck in water and soften on shearing. That softening emerges naturally in an elastic–plastic model of soil behaviour such as Cam clay. Once a localised plane or thin zone of failure has developed, then such continuum models of soil behaviour lose their attraction; subsequent response must be described in terms of stresses on and displacements across that thin zone. All of this discussion indicates the importance of a volumetric variable as well as stresses: it makes little sense to obtain a single set of strength parameters from test data which are not comparable because the specific volumes or water contents of various samples are widely different. Apparent scatter of strength data may be understood when variations of volume are recognised.

Exercises

E7.1. A set of samples of Weald clay, for which the compression parameter \( \lambda = 0.091 \) and the specific gravity \( G_s = 2.75 \), have been
isotropically normally compressed under a pressure \( p' = 827 \text{ kPa} \) (1201bf/in\(^2\)), and the samples have been allowed to swell to a number of pressures along the lowest swelling line in Fig. 6.8. The samples have been subjected to conventional drained or conventional undrained triaxial compression tests, and their conditions at failure have been examined. Failure has been defined as the condition when the deviator stress reaches its maximum value \( q_t \); this is found experimentally for Weald clay to be related to the mean effective stress at failure \( p'_e \) by the expression

\[
q_t = 0.72(p'_e + 0.13p'_c)
\]

where \( p'_c \) is the equivalent pressure on the normal compression line corresponding to the water content of the sample at failure.

Plot a diagram showing the variation, with overconsolidation ratio, of the ratio of strengths in conventional drained and undrained triaxial compression tests. What implications does this diagram have for the design engineer? Does the point at which the ratio of strengths is equal to 1 have any special significance?

E7.2. A soil fails according to a Hvorslev surface described by parameters \( c'_{pe} \) and \( \phi'_e \). For soil which has been isotropically normally compressed to a maximum pressure \( p'_e \) and then unloaded isotropically to a given overconsolidation ratio \( n \), derive an expression for the deviator stress \( q_t \) at failure in a conventional undrained triaxial compression test, in terms of the current mean effective stress \( p'_e \) (assumed constant during the undrained test), \( n \), \( \Lambda = 1 - \kappa/\lambda \), and the Hvorslev strength parameters.

For \( c'_{pe} = 0.046 \), \( \phi'_e = 18.8^\circ \), and \( \Lambda = 0.63 \), calculate values of \( q_t \) for \( p'_e = 400 \text{ kPa} \) and values of \( n = 2, 4, 8, 16, \) and 32. Compare these values with peak values obtained using the Cam clay model, taking \( M = 0.87 \). Plot the two strength envelopes in the shear stress:effective normal stress plane \((\tau:\sigma')\).

E7.3. A sample of saturated clay is normally compressed in a triaxial cell by increasing the cell pressure and holding the axial length constant. The axial effective stress is \( K_1 \) times the radial effective stress, and the specific volume \( v \) is given by

\[
v = v_a - \lambda \ln p'
\]

where \( v_a \) and \( \lambda \) are soil constants, and \( p' \) is the mean normal stress. The samples are then subjected to undrained compression tests with the cell pressure held constant. Derive an expression for the normalised pore pressure at failure \((u_f/q_t)\), in terms of \( K_1, v_2, \lambda, \) and the soil constants \( M \) and \( \Gamma \) which define the positions of the critical state line for the clay.