Critical states

6.1 Introduction: critical state line

All of the tests in the previous chapter, for which the simple elastic-plastic model, Cam clay, was used to predict the response, eventually tended towards an ultimate condition in which plastic shearing could continue indefinitely without changes in volume or effective stresses. This condition of perfect plasticity has become known as a critical state, the attainment of which can be expressed by

\[ \frac{\partial \sigma'}{\partial \varepsilon_q} = \frac{\partial q}{\partial \varepsilon_q} = \frac{\partial v}{\partial \varepsilon_q} = 0 \]  \hspace{1cm} (6.1)

These critical states were reached with an effective stress ratio

\[ \frac{q_{cs}}{p'_{cs}} = \eta_{cs} = M \]  \hspace{1cm} (6.2)

In drained or undrained tests on normally compressed (or lightly overconsolidated) soil (AB and AC in Fig. 6.1; cf. Figs. 5.5 and 5.15), yielding first occurs with stress ratio \( \eta < M \). Continued loading, whether drained or undrained, is associated with plastic hardening, expansion of yield loci, and increase of stress ratio until ultimately the effective stress state is at the top of the current yield locus (y1 B or y1 C), the plastic strain increment vector is directed parallel to the \( q \) axis, \( \delta \varepsilon_p^p/\delta \varepsilon_q^p = 0 \), and a perfectly plastic critical state is reached with \( \eta = M \).

In drained and undrained tests on heavily overconsolidated soil (PQ and PR in Fig. 6.1; cf. Figs. 5.7 and 5.17), yielding first occurs with \( \eta > M \). Continued deformation is associated with plastic softening, contraction of yield loci, and decrease of stress ratio until ultimately the effective stress state is at the top of the current yield locus (y1 Q or y1 R), and a critical state is again reached with \( \eta = M \).
A moment's consideration shows that this stress ratio $\eta = M$ produces an ultimate limiting condition, a critical state, in all tests, provided plastic deformations are occurring, because with $\eta = M$ the effective stress state is at the top of the current yield locus, and so indefinite plastic shearing can occur without further expansion or contraction of the yield locus. The caveat “provided plastic deformations are occurring” is important because, with the shape of yield locus assumed in the Cam clay model, it is quite possible for stress ratios equal to $M$ to be reached elastically within the

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Fig. 6.1 Conventional drained and undrained triaxial compression tests on normally compressed and heavily overconsolidated samples.

(a)

(b)
current yield locus (e.g. at X in Fig. 6.2) and without the serious consequences of a critical state developing.*

The locus of critical states in the $p':q$ stress plane is the line joining the tops of yield loci (Fig. 6.3a), $\eta = M$ or

$$q_{cs} = M p'_{cs} \quad (6.3)$$

this line is called the critical state line (csl). The proportions of the Cam clay yield loci are independent of their size. The general equation for the

![Diagram of yield loci](image)

*Fig. 6.2 Points X and Y inside current yield locus and not at critical state.

*Although it is usually convenient to contemplate changes in effective stress in relation to yield loci, these conclusions about the existence of critical states for $\eta = M$ result because the plastic potentials in the Cam clay model have normals parallel to the $q$ axis, implying $\delta \varepsilon_q^p/\delta q^*_p = 0$, for $\eta = M$. It so happens that these plastic potentials have the same shape as the yield loci, so the critical state is reached with the effective stress state at the top of the current yield locus. For other elastic-plastic models of the more general form discussed in Chapter 4, the plastic potentials and yield loci can be distinct, and there is then no necessity for the condition $\delta \varepsilon_q^p/\delta q^*_p = 0$ to be associated with the top of the yield loci. Compare Exercise E4.4.
yield loci is

\[ \frac{p'}{p'_o} = \frac{M^2}{(M^2 + \eta^2)} \]  \hspace{1cm} (6.4) (5.3bis)

The size of a yield locus is controlled by \( p'_o \), and the top of the yield locus, where \( \eta = M \) (Fig. 6.3a), has

\[ p'_{cs} = \frac{p'_o}{2} \]  \hspace{1cm} (6.5)

Fig. 6.3 Critical state line (csl) and intersection of yield loci with line \( q/p' = \eta \).

(a)

(b)

(c)
Each yield locus (yl) is associated with an unloading–reloading line (url) in the compression plane $p' : v$ (Fig. 6.3b) which has its tip at $p' = p'_o$ on the isotropic normal compression line (iso-ncl). This normal compression line has the equation

$$v = N - \lambda \ln p'$$  \hspace{1cm} (6.6) (5.8bis)

and is straight in the semi-logarithmic compression plane $v : \ln p'$ (Fig. 6.3c). The unloading–reloading lines are also straight in this form of the compression plane, with general equation

$$v = v_\kappa - \kappa \ln p'$$  \hspace{1cm} (6.7) (4.3bis)

Thus, the particular unloading–reloading line which corresponds to the yield locus with size $p'_o$ is

$$v = N - \lambda \ln p'_o + \kappa \ln \frac{p'_o}{p}$$  \hspace{1cm} (6.8)

At a mean stress $p' = p'_{cs} = p'_o/2$, the specific volume is then

$$v_{cs} = N - \lambda \ln 2p'_{cs} + \kappa \ln 2$$

or

$$v_{cs} = N - (\lambda - \kappa) \ln 2 - \lambda \ln p'_{cs}$$  \hspace{1cm} (6.9)

Each critical state combination of $p'_{cs}$ and $q_{cs}$ in the effective stress plane is associated with a critical state combination of $p'_{cs}$ and $v_{cs}$ in the

Fig. 6.4  Three-dimensional view of normal compression line (ncl), critical state line (csl), and series of Cam clay yield loci.
6 Critical states

The compression plane (Figs. 6.3a, b). The line joining these critical states has the expression (6.9), which can be rewritten as

\[ v_{es} = \Gamma - \lambda \ln p'_{es} \]  \hspace{1cm} (6.10)

where

\[ \Gamma = N - (\lambda - \kappa) \ln 2 \]  \hspace{1cm} (6.11)

This is a line in the compression plane at constant vertical separation, \((\lambda - \kappa) \ln 2\), from the normal compression line (parallel to the normal compression line in the \(v: \ln p'\) plane of Fig. 6.3c) which links the combinations of \(p'_{es}\) and \(v_{es}\) corresponding to the effective stress states in which plastic deformations are occurring with \(\eta = M\). The constant \(\Gamma\) is the specific volume intercept at \(p'_{es} = 1\). Like \(N\), its value, unfortunately, depends on the units of stress. Throughout this book the unit of stress is the kilopascal (kPa), so \(v_{es} = \Gamma\) for \(p'_{es} = 1\) kPa. The two expressions (6.3) and (6.10) provide a complete specification of the critical state line.

Combinations of \(p'_{es}, q_{es},\) and \(v_{es}\) which simultaneously satisfy (6.3) and (6.10) are critical states. Again, it is possible to discover combinations of \(p', q,\) and \(v\) which satisfy one but not both (6.3) and (6.10). Point \(X\) in Fig. 6.2a has \(q = M p'\); it appears to be on the critical state line in the stress plane but lies inside the current yield locus, so its position on the current unloading–reloading line in the compression plane (Fig. 6.2b) is not on the critical state line. Point \(Y\) in Fig. 6.2b, reached perhaps by isotropic compression and unloading, appears to lie on the critical state line in the compression plane but also lies inside the current yield locus, so its position in the stress plane (Fig. 6.2a) is not on the critical state line.

If \(p', q,\) and \(v\) are thought of as three orthogonal axes, the combinations of stresses and volume can be plotted in the three-dimensional space defined by these axes. The critical state line becomes a single curved line (Fig. 6.4) of which (6.3) and (6.10) are projections onto the \(p': q\) plane \((v = 0)\) and \(p': v\) plane \((q = 0)\), respectively. Only combinations of \(p', q,\) and \(v\) lying on this curved three-dimensional line are critical states.

6.2 Two-dimensional representations of \(p': q : v\) information

Display of information about (triaxial) states of stress \(p': q\) and about specific volumes \(v\) requires two two-dimensional plots – the stress plane and the compression plane – or one three-dimensional plot such as that shown in Fig. 6.4. The Cam clay model can point the way to two-dimensional devices for displaying this information.

The equation of the Cam clay yield locus is

\[ \frac{p'}{p'_c} = \frac{M^2}{M^2 + \eta^2} \]  \hspace{1cm} (6.4bis)
and involves only ratios of stresses. For a given ratio of deviator stress to mean stress \(q/p' = \eta\), there is a certain ratio of mean stress to tip stress \((p'/p)\). Constant ratios become constant separations in logarithmic plots. Therefore, just as the stress ratio \(\eta = M\) which defines the tops of the yield loci produces the critical state line parallel to the normal compression line in the compression plane \(v:ln p'\) (Fig. 6.3c), so any other stress ratio which defines a series of geometrically similar points on a series of yield loci produces a line which is particular to that stress ratio and also parallel to the normal compression line (Fig. 6.3c).

Each of these lines has an equation of the form
\[
v = v_1 - \lambda \ln p'
\]
(6.12)

A way of converting specific volume and mean effective stress information to a single variable is to use
\[
v_1 = v + \lambda \ln p'
\]
(6.13)
or, in other words, to project the information of \(v\) and \(\ln p'\) parallel to the normal compression line up to the line \(p' = 1\). The value of \(v_1\) for soil which is deforming plastically (yielding) depends only on the stress ratio \(\eta = q/p'\); that is, the pair of variables \(\eta; v_1\) can be used to display information about the three quantities \(p':q:v\) defining the state of the soil. Evidently, by comparison with (6.6), for isotropic normal compression,
\[
\eta = 0, \quad v_1 = N
\]
(N in Fig. 6.5); and by comparison with (6.10), at the critical state,
\[
\eta = M, \quad v_1 = \Gamma
\]
(C in Fig. 6.5). The values of \(v_1\) corresponding to other values of \(\eta\) can be found by combining the expression for the specific volume at a point on an unloading–reloading line,
\[
v = N - \lambda \ln p' + \kappa \ln \frac{p^2}{p'}
\]
(6.8bis)
with the definition of \(v_1\) (6.13) to give
\[
v_1 = N - (\lambda - \kappa) \ln \frac{p^2}{p'}
\]
(6.14)
This, with the equation of the yield locus (6.4), gives
\[
v_1 = N - (\lambda - \kappa) \ln \frac{M^2 + \eta^2}{M^2}
\]
(6.15)
or
\[
\frac{\eta^2}{M^2} = \exp \left( \frac{N - v_1}{\lambda - \kappa} \right) - 1
\]
(6.16)
This relationship is plotted in Fig. 6.5 as curve NCX. Any stress state which is on a current yield locus lies on this curve.

The path of any conventional drained or undrained compression test on isotropically normally compressed soil starts at N and moves along curve NC to end at a critical state at C. The path of an undrained test on an isotropically overconsolidated soil starts at a point such as P (lightly overconsolidated) or R (heavily overconsolidated) in Fig. 6.5 with \( \eta = 0 \). Loading within the yield locus occurs at constant mean stress \( p' \). Since \( v \) is also constant in an undrained test, from (6.13) \( v_\lambda \) remains constant until the yield locus is reached (PQ or RS in Fig. 6.5). The path then moves along the yield locus until a critical state is reached (QC or SC in Fig. 6.5).

The path of any conventional drained test on an isotropically overconsolidated soil also starts at a point such as P or R. Initial loading within the yield locus is associated with increase in \( p' \) and decrease in \( v \) down an unloading–reloading line. The slope \( \kappa \) of the unloading–reloading line is lower than the slope \( \lambda \) of the normal compression line, and, hence, this initial loading produces an increase in \( v_\lambda \) (PF and RG in Fig. 6.5) until the yield locus is reached. Then the yield locus is followed up or down to a critical state (FC and GC in Fig. 6.5).

The quantity \( v_\lambda \) converts data of \( p' \) and \( v \) to a constant mean stress section. An alternative which is equivalent to converting these data to a
constant volume section is to normalise the effective stresses $p':q$ with respect to the so-called equivalent consolidation pressure $p'_e$. This equivalent consolidation pressure is the pressure which, in isotropic normal compression, would give the soil its current specific volume (Fig. 6.6). The isotropic normal compression line is

$$v = N - \lambda \ln p'$$  \hspace{1cm} (6.6bis)

and hence $p'_e$ is

$$p'_e = \exp \frac{N - v}{\lambda}$$  \hspace{1cm} (6.17)

With a mean stress $p'$ inside a current yield locus of size $p'_o$,

$$v = N - \lambda \ln p'_o + \kappa \ln \frac{p'_o}{p'}$$  \hspace{1cm} (6.8bis)

and hence, after some manipulation,

$$\frac{p'}{p'_e} = \left( \frac{p'}{p'_o} \right)^{\Lambda}$$  \hspace{1cm} (6.18)

where

$$\Lambda = \frac{\lambda - \kappa}{\lambda}$$  \hspace{1cm} (6.19)(5.20bis)

---

**Fig. 6.6** Equivalent consolidation pressure $p'_e$. 

\[ \text{Diagram showing the relationship between } p', \text{ and } p'_e. \]
For stress states on the current yield locus,
\[ \frac{p'}{p_e'} = \frac{M^2}{M^2 + \eta^2} \]  \hspace{1cm} (6.4bis)
and
\[ \frac{p'}{p_e'} = \left( \frac{M^2}{M^2 + \eta^2} \right)^{\frac{1}{\nu}} \]  \hspace{1cm} (6.20)
which is essentially the same as (5.19), the equation of the undrained effective stress path. Evidently,
\[ \frac{q}{p_e'} = \frac{\eta p'}{p_e'} \]  \hspace{1cm} (6.21)
and the pair of expressions (6.20) and (6.21) can be used to generate the curve in the \( p'/p_e'; q/p_e' \) plane corresponding to the Cam clay yield locus (Fig. 6.7). There is a point \( N(p'/p_e' = 1; q/p_e' = 0) \), corresponding to isotropic normal compression, and a point \( C(p'/p_e' = 2^{-\frac{1}{2}}; q/p_e' = M2^{-\frac{1}{2}}) \), corresponding to the critical state.

Because \( p_e' \) remains unchanged during a constant volume undrained test, undrained effective stress paths scale directly into the \( p'/p_e'; q/p_e' \) diagram. So an undrained test on an overconsolidated sample which shows no change of mean effective stress \( p' \) until the current yield locus is reached rises at constant \( p'/p_e' \) until the yield locus is reached (\( PQ \) and \( RS \) in Fig. 6.7) and then moves round the yield curve in the \( p'/p_e'; q/p_e' \) plane to a critical state (\( QC \) and \( SC \) in Fig. 6.7).

A conventional drained test on a normally compressed sample follows

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**Fig. 6.7** Paths of conventional drained and undrained triaxial compression tests plotted in non-dimensional effective stress space \( p'/p_e'; q/p_e' \).
6.3 Critical states for clays

the path NC in the compression plane shown in Fig. 5.5b. The mean stress \( p' \) increases because of the imposed stress path, but the equivalent consolidation pressure increases more rapidly as the soil compresses and the ratio \( p'/p'_e \) falls. In fact, the test follows round the curve NC in the \( p'/p'_e : q/p'_e \) plane (Fig. 6.7). (The point C is not at the top of the curve NCX for precisely the same reason that the effective stress path followed in an undrained test does not have zero slope, \( \delta q/\delta p' \neq 0 \), at the critical state, which is a consequence of the non-negligible elastic volumetric strains that are assumed to occur in the model. If there are no elastic volumetric strains, \( \kappa = 0 \) and \( \Lambda = 1 \), and the point C is then at the summit of the curve in Fig. 6.7.)

A conventional drained test on an overconsolidated sample begins with a small volumetric compression down the current unloading–reloading line as the stress path rises to the current yield locus (Figs. 5.6b, 5.7b). The equivalent consolidation pressure \( p'_e \) increases slightly with this volumetric compression, but not as rapidly as the mean stress; \( p'/p'_e \) increases, until the yield locus is reached (PF and RG in Fig. 6.7). Yielding of the lightly overconsolidated sample is associated with plastic volumetric compression to a critical state, \( p'/p'_e \) decreases (FC in Fig. 6.7; compare Fig. 5.6b). Yielding of the heavily overconsolidated sample is associated with plastic volumetric expansion to a critical state, \( p'/p'_e \) increases (GC in Fig. 6.7; compare Fig. 5.7b).

These two-dimensional representations bring together drained and undrained response into a single diagram and help to emphasise the fact that, considered in terms of effective stresses, different modes of testing are merely probing different parts of a single unified picture of soil behaviour.

6.3 Critical states for clays

Data from four conventional triaxial compression tests on samples of reconstituted Weald clay (taken from Bishop and Henkel, 1957) were used in Sections 5.3 and 5.4 to demonstrate that the simple Cam clay model was reproducing commonly observed features of soil response. These data were presented in the form of the standard plots that are used to display triaxial test results: plots of deviator stress \( q \) and volumetric strain \( \varepsilon_v \) or pore pressure change \( \Delta u \) against axial strain \( \varepsilon_a \) (Figs. 5.9, 5.10, 5.18, and 5.19). Axial strain is a quantity which can increase without limit, subject only to restrictions imposed by the apparatus. In the search for experimental evidence of critical states, it is more helpful to look at data in terms of quantities which are limited in their variation and to plot the paths of tests in the effective stress plane \( p' : q \) and the compression plane
$p' : v$, or in the combined two-dimensional representations of these two planes that have been discussed in the previous section.

Enough information is usually available to achieve the conversion. The deviator stress $q$ is given directly. The constant cell pressure at which a test is conducted is the initial mean effective stress $p'_i$ (assuming zero pore pressure and zero deviator stress at the start of the test). At subsequent stages of the test,

$$p' = p'_i + \frac{q}{3} - \Delta u$$  \hspace{1cm} (6.22)

where $\Delta u$ is the measured change in pore pressure, which is zero for drained tests. Conversion of volumetric strains to specific volumes requires a value of a volumetric variable at some stage of the test, usually the initial water content $w_i$ for saturated clays. The initial specific volume is then

$$v_i = 1 + G_s w_i$$  \hspace{1cm} (6.23)

where $G_s$ is the specific gravity of the soil particles; at subsequent stages of the test,

$$v = v_i (1 - \epsilon_p)$$  \hspace{1cm} (6.24)

For the reconstituted Weald clay, $G_s = 2.75$, and data of isotropic compression and unloading are reported by Henkel (1959) (Fig. 6.8). The two normally compressed samples (1 and 2) were compressed isotropically to a mean effective stress $p' = 207\text{kPa}$ (30 pounds force per square inch, lbf/in$^2$) before being sheared. The two overconsolidated samples (3 and

Fig. 6.8 Isotropic compression and unloading of Weald clay (after Henkel, 1959).
Table 6.1

<table>
<thead>
<tr>
<th>Test</th>
<th>Point</th>
<th>$e_p$ (%)</th>
<th>$v$</th>
<th>$q$ (kPa)</th>
<th>$\Delta u$ (kPa)</th>
<th>$p'$ (kPa)</th>
<th>Remarks</th>
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<td>207</td>
<td></td>
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<td>253</td>
<td></td>
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<td>290</td>
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<td>1.640</td>
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<td>0</td>
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</tr>
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<tr>
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</table>

4) were compressed isotropically to a mean effective stress $p' = 827$ kPa (120 lbf/in$^2$) and then allowed to swell back isotropically to a mean effective stress $p' = 34$ kPa (5 lbf/in$^2$), resulting in an overconsolidation ratio of 24.

To illustrate the conversion of the experimental data to the stress plane and the compression plane from the conventional plots of Figs. 5.9, 5.10, 5.18, and 5.19, four points have been marked for each test, and values of $p':q:v$ have been computed for each point. These values are tabulated in Table 6.1.

The paths of these tests are plotted in the $p':q$ effective stress plane and the $p':v$ compression plane in Fig. 6.9. The effective stress paths of the drained tests (1 and 3) both have to rise at gradient $\delta q/\delta p' = 3$ (Fig. 6.9a). However, whereas test 1 rises steadily, $ABCD$, test 3 rises to a peak $KLM$ and then falls, $MN$. Test 1 shows a steady fall in specific volume (Fig. 6.9b) whereas test 3 shows an initial drop in volume $KL$, followed by expansion $LMN$.

The effective stress paths for drained tests 1 and 3 form the total stress paths for the undrained tests 2 and 4: the pore pressure at each stage is the horizontal ($p'$) separation in the stress plane between the two corresponding paths. The effective stress path for test 2 lies to the left of its corresponding total stress path (test 1): positive pore pressures build up, and the mean effective stress falls. The effective stress path for test 4 lies initially to the left and then to the right of its corresponding total stress.
path (test 3): the pore pressure is initially positive but then becomes negative. The initial pore pressure variation in this test is slightly deceptive. Looking at the tabulated values for points $P$ and $Q$, we can see that the mean effective stress $p'$ is initially almost constant (as expected for undrained constant volume shearing of isotropic elastic soil, and Cam clay would predict that soil with this history of overconsolidation is initially elastic) in spite of the initial positive pore pressures. The pore

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**Fig. 6.9** Conventional drained and undrained triaxial compression tests on Weald clay: (a) $q:p'$ effective stress plane; (b) $v:p'$ compression plane (data from Bishop and Henkel, 1957).
6.3 Critical states for clays

pressure more or less balances the increase in total mean stress imposed in this constant cell pressure test: \( \delta u \approx \delta p = \delta q/3 \), as shown in Fig. 5.19b.

It was noted in Chapter 5 that there is a parallelism between volume changes in drained tests and pore pressure changes in undrained tests on samples with the same consolidation history. Pore pressures develop in undrained tests only because the soil wants to change in volume as it is sheared but is prevented from doing so by the test configuration. The normally consolidated soil wants to collapse as it is sheared (test 1, Figs. 5.9 and 6.9b): positive pore pressures build up in the undrained test to reduce the mean effective stress carried by the soil particles. The resulting expansion just balances the collapse that could not occur (test 2, Figs. 5.18 and 6.9a). The heavily overconsolidated soil ultimately shows negative pore pressures in the undrained test (test 4, Figs. 5.19 and 6.9a): the soil skeleton is having to be held back by the pore pressure to prevent it from exploding (compare the drained test 3, Figs. 5.10 and 6.9b).

Do these test data support the existence of critical states for Weald clay? In the effective stress plane (Fig. 6.9a), a straight line can be drawn through the origin and through, or close to, the end points \( D, H, N, \) and \( S \) of the four tests. In the compression plane (Fig. 6.9b), these four tests do not define a line or curve of end points particularly closely. However, a curved zone can be indicated towards which the tests appear to be heading: test 1 by decrease in volume, test 2 by decrease in mean effective stress, test 3 by increase in volume, and test 4 by increase in mean effective stress.

To draw conclusions about general patterns of behaviour from the results of just four tests would be risky. Roscoe, Schofield, and Wroth (1958), however, gathered data of end points of a large number of drained and undrained triaxial compression tests on reconstituted Weald clay. Data from undrained tests are shown in the stress plane and the compression plane in Fig. 6.10, and data from drained tests in Fig. 6.11. In both cases, data from tests with overconsolidation ratios greater than 8 at the start of shearing have been excluded because there is a tendency for non-uniformities to develop in these tests, as is explained in Section 7.4. (Thus, the end points of tests 3 and 4 from Fig. 6.9 do not appear in Figs. 6.10 and 6.11.)

The end points from undrained tests on normally compressed samples have been used to define a locus of end points in both the stress plane (Fig. 6.10a), in which this locus is a straight line through the origin, and in the compression plane (Fig. 6.10b), in which the points lie on a smooth curve. It is clear, however, that the end points from all the undrained tests lie on or close to these lines.
These same lines have been drawn again in Figs. 6.11 a, b for comparison with the data of end points from drained tests on Weald clay. It is clear that the same relationships match both sets of data. This does not, perhaps, now seem a surprising result. The response in drained and undrained tests can be predicted with a single model of soil behaviour based on effective stresses, such as Cam clay, and these tests are expected to fit into a single

Fig. 6.10  End points of conventional undrained triaxial compression tests on Weald clay (o normally compressed samples; ● overconsolidated samples):
(a) $p':q$ effective stress plane; (b) $v:p'$ compression plane; (c) $v:p'$ compression plane ($p'$ plotted on logarithmic scale) (after Roscoe, Schofield, and Wroth, 1958).
pattern of effective stress: strain response. Other drained tests could be conducted with other effective stress paths, so there is nothing particularly significant about the division between drained and undrained response. This finding should, however, be seen in the historical context of the state of soil mechanics thought in the 1950s when there was a tendency to regard the results of drained and undrained tests as quite unrelated and incompatible.

The collection of data in Figs. 6.10 and 6.11 presents more convincing evidence for the existence of a line of values of $p', q$, and $v$ towards which all the tests have headed, irrespective of the type of test (which controls

Fig. 6.11 End points of conventional drained triaxial compression tests on Weald clay (○ normally compressed samples; ● overconsolidated samples): (a) $p':q$ effective stress plane; (b) $v$; $p'$ compression plane (after Roscoe, Schofield, and Wroth, 1958).
to some extent the path of the test) and irrespective of the consolidation history (which controls the position of the starting points of the tests relative to the line of end points). This line is described in the $p' : q$ effective stress plane (Figs. 6.10a and 6.11a) by an equation,

$$q = 0.872p'$$  \hspace{1cm} (6.25)

Drawing the line in a semi-logarithmic compression plane $\ln p'$ together with the isotropic normal compression line (Fig. 6.10b), we can see that the two lines are parallel and essentially straight and that the line of ultimate states can be described by an equation,

$$v = 2.072 - 0.091 \ln p'$$  \hspace{1cm} (6.26)

The form of this line thus matches the form of the critical state line that emerged from the Cam clay model, (6.3) and (6.10); it is convenient to refer to it as the critical state line for Weald clay.

The critical state line operates as a limit on the changes of $p', q$, and $v$ that occur in a test. The formal definition of a critical state requires that

$$\frac{\partial p'}{\partial \varepsilon_q} = \frac{\partial q}{\partial \varepsilon_q} = \frac{\partial v}{\partial \varepsilon_q} = 0$$  \hspace{1cm} (6.1bis)

Arrival at the condition described by (6.1) may require very large strains since, for states close to true critical states, the derivatives $\partial p'/\partial \varepsilon_q, \partial q/\partial \varepsilon_q$, and $\partial v/\partial \varepsilon_q$ (which include the tangent shear stiffness $\partial q/\partial \varepsilon_q$) are close to zero. Test conditions may not permit sample uniformity to be retained to

Fig. 6.12   (a) Conventional drained and (b) conventional undrained triaxial compression tests ending off the critical state line but still trying to head towards it.
these large deformations and critical states may not actually be attained. Nevertheless the critical state line is still important as a line towards which tests are heading.

Parry (1958), for example, subjected the results of his triaxial tests on Weald clay to close scrutiny and concluded that when the quoted end points of tests did not seem to lie on the previously determined critical state lines, the states of the soils were in fact still moving towards critical states at the end of the tests. Schematically, certain drained tests ended to the right of the critical state line (at points such as $W$ in Fig. 6.12a). The value of mean effective stress $p'_t$ recorded at failure (or at the end of the test) was greater than the value of mean stress $p'_e$, on the critical state line at the same value of specific volume that the sample had at the end of the test. In such tests, the volume of the sample was observed to be decreasing at the end of the test and the state of the soil to be moving downwards towards the critical state line in the compression plane. That is, for $p'_e/p'_t < 1$, $(\partial u/\partial e_q)_L < 0$.

Conversely, when drained tests ended to the left of the critical state line ($X$ in Fig. 6.12a), so that $p'_e/p'_t > 1$, then the volume of the sample was observed to be increasing at the end of the test, $(\partial u/\partial e_q)_L > 0$, and the state of the soil to be moving upwards towards the critical state line in the compression plane.

When undrained tests ended to the right of the critical state line ($Y$ in Fig. 6.12b), so that $p'_e/p'_t < 1$, then the pore pressure was observed to be increasing at the end of the test, $(\partial p'/\partial e_q)_L > 0$, and hence the mean effective stress to be falling, $(\partial p'/\partial e_q)_L < 0$, and the state of the soil to be moving leftwards towards the critical state line in the compression plane. Conversely, when undrained tests ended to the left of the critical state line ($Z$ in Fig. 6.12b), so that $p'_e/p'_t > 1$, then the pore pressure was observed to be decreasing at the end of the test, and hence the mean effective stress

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**Fig. 6.13** End points of conventional triaxial tests on Weald clay: (a) rate of volume change in drained tests; (b) rate of change of pore pressure in undrained tests (after Parry, 1958).
to be rising, \( \frac{\partial p'}{\partial e_{qh}} > 0 \), and the state of the soil to be moving rightwards towards the critical state line in the compression plane.

The data assembled by Parry (1958) are shown in Fig. 6.13 in terms of values of \( \frac{\partial e_p}{\partial e_{qh}} \) for drained tests (Fig. 6.13a) and of \( \frac{\partial u' p'}{\partial e_{qh}} \) for undrained tests (Fig. 6.13b). The way in which the critical state line separates these contrasting modes of behaviour at the ends of the tests is clear.

### 6.4 Critical state line and qualitative soil response

The critical state line emerged in Section 6.1 as a consequence of the elastic–plastic model of soil behaviour which was constructed in preceding chapters. Now experimental evidence has been produced for such a line of states towards which triaxial compression tests tend to head. The acceptance of a critical state line permits an assessment to be made of the expected qualitative response in any triaxial compression test on a soil with any consolidation history. Although this section may appear repetitive when read in conjunction with Sections 5.3 and 5.4, we make this assessment here without reference to (or acknowledgement of) any detailed elastic–plastic model, building only on an acceptance of the existence of critical state lines and not on an understanding of all the details of Cam clay. Demonstration of how this can be achieved illustrates the importance of considering soil response in the effective stress plane and in the compression plane concurrently.

Two pairs of tests are considered: firstly, a pair of tests on samples which have been isotropically normally compressed (A in Fig. 6.14), and secondly, a pair of tests on samples which have been isotropically overconsolidated but have the same initial mean stress as samples A (B in Fig. 6.14). All four tests thus start at the same point on the \( p' \) axis, in the \( p':q \) effective stress plane (Figs. 6.14a, b), but their initial positions A and B in the \( p':v \) compression plane are on opposite sides of the critical state line (Fig. 6.14c).

In an undrained test on a normally compressed sample, starting from point A, the end point U on the critical state line (Fig. 6.14c) is dictated by the constant volume condition in the \( p':v \) plane. Hence, the end point U in the stress plane \( p':q \) can be deduced on the critical state line at the corresponding value of mean stress \( p' \) (Fig. 6.14a). The route in the effective stress plane from A to U is not known, but a simple curve can be sketched. In a conventional triaxial compression test with constant cell pressure, the total stress path has gradient \( \delta q/\delta p = 3 \). This line \( AW \) lies to the right of the effective stress path, and hence positive pore pressures are expected in the undrained test.
The total stress path of the undrained test, from $A$ towards $W$, becomes the effective (and total) stress path of a drained test on the other normally compressed sample. The end point of this test is governed by the intersection $W$ of this stress path with the critical state line in the $p':q$ stress plane (Fig. 6.14a). The end point $W$ in the $p':v$ compression plane.
is then fixed on the critical state line at the same value of mean stress $p'$ (Fig. 6.14c): volumetric compression is expected. Again, the shape of the path in the compression plane from $A$ to $W$ is not known, but a simple curve can be sketched.

The overconsolidated samples starting at point $B$ (Figs. 6.14b, c) have the same initial effective stresses as the samples starting at-point $A$. In a drained test, therefore, the effective (and total) stress path is the same, and consequently the critical state reached in this test is at $W$ just as for the normally compressed sample. Point $B$ lies well below point $W$ in the compression plane (Fig. 6.14c), so this drained shearing must be associated with volumetric expansion. That much can be stated with knowledge only of the position of the critical state line. Other experience of soil behaviour could be incorporated to suggest that the stress path might overshoot $W$.

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Fig. 6.15 Triaxial compression tests on normally compressed soil: applied total stress path such that drained strength is lower than undrained strength; (a) paths in $p'$-$q$ effective stress plane; (b) paths in $v$-$p'$ compression plane.
6.4 Soil response

in the $p':q$ plane (Fig. 6.14b) (so that the stress:strain curve would show a peak), and that the sample might compress initially before expanding.

The end point reached on the critical state line in the undrained test on the overconsolidated sample is quite different from that reached in the undrained test on the normally compressed sample. The history of overconsolidation has left the soil at B with a much lower specific volume (Fig. 6.14c) and hence a much higher undrained strength. The position of the critical state point V relative to the starting point B in the compression plane (Fig. 6.14c) and the corresponding points in the effective stress plane (Fig. 6.14b) shows that an increase in mean effective stress $p'$ has occurred during the test; in other words, a negative pore pressure is to be expected when the critical state is reached.

Fig. 6.16 X, tests on lightly overconsolidated samples with drained strength greater than undrained strength; Y, tests on heavily overconsolidated samples with drained strength lower than undrained strength; (a), (b) $p':q$ effective stress plane; (c) $v:p'$ compression plane.
It is important to note that the critical state line must not be considered as a rigid dividing line in the compression plane between a region in which positive pore pressures and volumetric compression occur (for samples such as A in Fig. 6.14) and a region in which negative pore pressures and volumetric expansion occur (for samples such as B in Fig. 6.14). The precise response always depends on the total stress path which is applied. For example, an initially normally compressed sample subjected to a total stress path $AT$ (Fig. 6.15), which has a considerable component of unloading of mean normal stress, shows negative pore pressures in an undrained test ($AU$) and volumetric expansion in a drained test ($AT$). The drained or long-term strength in this case is lower than the undrained strength.

However, it is correct to say that if normally compressed or lightly overconsolidated clays, which start above the critical state line in the compression plane, are subjected to total stress paths in which the mean stress $p$ is kept constant (or increased) ($X$ in Figs. 6.16a, c), then the undrained strength is lower than the drained strength. Short-term stability of geotechnical structures for which such stress paths are relevant would be a prime consideration since dissipation of the positive pore pressure set up during undrained shearing leads to volumetric compression and hardening of the soil. If heavily overconsolidated clays, which start below the critical state line in the compression plane, are subjected to total stress paths in which the mean stress $p$ is kept constant (or reduced) ($Y$ in Figs. 6.16b, c), then the undrained strength is higher than the drained strength. Long-term stability of geotechnical structures for which such stress paths are relevant would be a prime consideration since dissipation of the negative pore pressures set up during undrained shearing leads to volumetric expansion and softening of the soil. Actual stress paths associated with some typical geotechnical constructions are discussed in Chapter 10.

6.5 Critical states for sands and other granular materials

The qualitative picture that emerges from the discussion around Fig. 6.14 in the previous section, based only on experimental observation of the existence of a critical state line for clays, is that in drained tests

Fig. 6.17 (a) Loose packing of spherical particles compressing as it is sheared; (b) dense packing of spherical particles expanding as it is sheared.

(a)  (b)
6.5 Sands

conducted at a given effective stress level, normally compressed clays, which have an initially high specific volume, contract when they are sheared whereas overconsolidated clays, which have an initially low specific volume, expand when they are sheared. It is a truth universally acknowledged that loose sands contract and dense sands expand when they are sheared. The similarity between these two statements suggests that the existence of critical states may be a rather general feature of soil behaviour.

The significance of the general phenomenon of dilatancy (the change in volume associated with distortion of granular materials) emerges in the writings of Osborne Reynolds (1885, 1886), but he himself notes (1886) that 'dilatancy has long been known to those who buy and sell corn'. The need for volume changes to occur when regular packings of spherical particles are deformed is clear. The loose packing of spheres in Fig. 6.17a is clearly unstable and will collapse as soon as any shear deformation is imposed; the dense packing in Fig. 6.17b can deform (neglecting the elastic stiffness of the spheres) only if spheres in each layer rise up over the spheres in the layer below. The arrangements of particles in a real granular material are much more irregular than the packings suggested in Fig. 6.17, but the modes of deformation are essentially the same.

Reynolds performed many experiments with granular materials and was able to ascribe the phenomena that occurred to the dilatancy of the materials, but he did not make any measurements of the strengths of materials with various densities of packing. Casagrande (1936), however, does describe the stress:strain curves which are expected when dense and loose samples of sand are sheared: for example, in a shear box (Fig. 6.18).

During the shearing test on the dense sand, 'the shearing stress reaches a maximum \( S_D \) (point \( B \) on the curve) and if the deformation is continued, the shearing stress drops again to a smaller value, at which value it remains constant for all further displacement. During this drop in shearing stress, the sand continues to expand (curve \( EG \)), finally reaching a critical {void ratio} at which continuous deformation is possible at the constant shearing stress \( S_L \).

When a loose sample of sand is subjected to a shearing test under constant normal pressure, however, 'the shearing stress simply increases until it reaches the shearing strength \( S_L \), and if the displacement is continued beyond this point the resistance remains unchanged. Obviously, the volume of the sand in this state must correspond to the critical {void ratio} which we had finally reached when performing a test on the same material in the dense state. Therefore the curves representing the volume changes during shearing tests on material in the dense and the loose state
must meet at the critical \{void ratio\} when the stationary condition is established'.

Casagrande concludes that 'every cohesionless soil has a certain critical \{void ratio\}, in which state it can undergo any amount of deformation or actual flow without volume change'. (Casagrande actually writes critical density but notes in an addendum that critical void ratio gives a more correct meaning.)

Some quantitative results in support of Casagrande's notion of critical void ratios are shown for the shearing of 1-mm diameter steel balls in Fig. 6.19. Samples of steel balls were prepared by Wroth (1958) at various initial void ratios and then sheared in an early simple shear apparatus. Just as for the Weald clay (Figs. 5.9 and 5.10), the results of shearing tests can be shown in terms of the change of volume or volumetric strain which occurs with deformation of the samples, indicated by relative displacement $x$. Such diagrams, Fig. 6.19a, show the pattern of changing volumetric

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Fig. 6.18 Effect of shearing on the volume of dense and loose sands (after Casagrande, 1936).
Fig. 6.19  Simple shear tests on 1-mm diameter steel balls with normal stress 138 kPa (20 lbf/in²): (a) volume change $\Delta v$ and shear displacement $x$; (b) specific volume $v$ and shear displacement $x$; (c) end points of tests in $\sigma_{xx}$, $\tau_{xy}$ effective stress plane; (d) end points of tests in $v$, $\sigma_{pp}$ compression plane (after Wroth, 1958).
Fig. 6.20 Drained triaxial compression tests on Chattahoochee River sand with constant mean effective stress. Dense samples: A (o), \(p' = 98\, \text{kPa}, \quad \nu_a = 1.69\); C (•), \(p' = 2.07\, \text{MPa}, \quad \nu_a = 1.72\); D (.), \(p' = 34.4\, \text{MPa}, \quad \nu_a = 1.69\). Loose sample: B (x), \(p' = 98\, \text{kPa}, \quad \nu_a = 2.03\). (a) Stress ratio \(q/p'\) and triaxial shear strain \(\varepsilon_q\). (b) Volumetric strain \(\varepsilon_v\) and triaxial shear strain \(\varepsilon_q\). (c) Specific volume \(v\) and triaxial shear strain \(\varepsilon_q\) (data from Vesić and Clough, 1968).
response as the initial void ratio of the sample changes: the initially dense samples with low void ratios expand, and the initially loose samples contract, as they are sheared. However, such diagrams do not help to demonstrate the existence of a critical void ratio. For this, absolute values of void ratio are required (Fig. 6.19b). In this figure, it is apparent that irrespective of the initial void ratio of the samples, at large shear displacement all samples are deforming at the same critical void ratio. All the tests shown in Figs. 6.19a, b were performed with a normal stress of 20lb/in² (138 kPa), and for this stress level a critical void ratio of about 0.64 would be deduced.

A similar pattern of response can be observed in the results of triaxial tests on samples of sand. The triaxial apparatus produces more complete information concerning the stress state in a soil sample than does the simple shear apparatus (or shear box); but as is seen in Section 7.4, it can be difficult to follow reliably the post-peak softening which is a feature of the response of dense samples. The results in Fig. 6.20 for Chattahoochee River sand (from Vesic and Clough, 1968) are typical. Note that these are not standard triaxial tests performed with constant cell pressure but are tests with constant mean effective stress \( p' \). In Figs. 6.20a, b, c (curves A and B), results from tests on initially dense and initially loose samples with a mean effective stress of 98 kPa are shown. It is clear that testing difficulties have prevented the dense sample from attaining an ultimate condition with shearing continuing at constant volume (curve A in Fig. 6.20c); however, from the test on the loose sand, a critical void ratio of about 0.9 can be estimated. (The attainment of this critical void ratio would imply a volumetric expansion of about 17 per cent for the initially dense sample.)

The results of triaxial tests performed by Vesic and Clough at other constant mean effective stresses are also shown in Fig. 6.20. The effect of increasing the stress level on the volumetric strains which develop in samples prepared at essentially the same dense initial void ratio can be seen in Figs. 6.20a, b (curves A, C, and D). The effect of increasing the stress level is to eliminate the peak observed in the conventional response, seen in the test at the lowest stress level. The pattern which is observed is essentially the same as that in Fig. 6.19a. There, however, the response changed as the initial void ratio was varied at constant stress level; now, the response is changing as the stress level is varied at approximately constant void ratio.

Casagrande (1936) remarks that 'static pressure is relatively ineffective in reducing the volume of a sand; for example, it is not possible to change a loose sand into a dense sand by static pressure alone'. What is shown
here is that it is possible to change a dense sand into a 'loose' sand by static pressure alone. Accepted notions of what constitutes a loose sand or a dense sand are rendered valueless unless accompanied by a statement about the stress level of interest. (The mean stress level at which the dense Chattahoochee River sand in Fig. 6.20 ceases to expand on shearing is about 2000 kPa. If a ratio of horizontal to vertical effective stress of about 0.4 is assumed, then this stress corresponds to a depth of about 200 m in dry sand or 330 m in submerged sand.)

A confirmation of this feature is provided by the test results shown in Fig. 6.20. The expected contrast appears between the responses of the initially dense and the initially loose samples when tested at the same low mean effective pressure ($p' = 98$ kPa) (curves A and B in Figs. 6.20a, b). However, the responses of the initially dense sample tested at a high mean effective stress ($p' = 34,433$ kPa) and of the initially loose sample tested at the low pressure are qualitatively the same (curves B and D in Figs. 6.20a, b).

Initially dense samples of sand sheared at low stress levels expand. Initially dense samples of sand sheared at high stress levels contract. The act of increasing the stress level reduces the volume or void ratio of the sand; consequently, the critical void ratio attained when the sand is sheared

Fig. 6.21  (a) Isotropic compression of dense and loose Chattahoochee River sand; paths of triaxial compression tests with constant mean effective stress (end points: • dense, o loose); and possible location of critical state line; (b) end points and critical state line in $p':q$ effective stress plane; (c) end points and critical state line in $v:p'$ compression plane (data from Vesić and Clough, 1968).
at a high stress level must be significantly lower than the critical void ratio attained when the sand is sheared at a low stress level. Casagrande's (1936) conclusion that 'Every cohesionless soil has a certain critical void ratio, in which state it can undergo any amount of deformation or actual flow without volume change' is seen to be too simple. Just as for clays, critical void ratios for sands are stress level dependent.

The paths followed in the tests on Chattahoochee River sand conducted by Vesic and Clough (1968) can be plotted in the compression plane $e: \ln p'$ (Fig. 6.21a), with the mean effective stress plotted on a logarithmic scale.
to accommodate large ranges of pressure. The paths of tests on loose and dense samples head towards a somewhat diffuse, but clearly pressure dependent, zone of critical void ratios.

Mathematical description of such a diffuse band is not particularly helpful. Upper and lower constraints on critical void ratios may be noted. The absolute minimum value of void ratio must be zero, implying a minimum specific volume of unity. For sand to shear at constant volume at a near-zero void ratio, a mechanism other than the rolling and sliding of grains, which is associated with low-pressure shearing, has to be available. Grading curves are shown by Vesić and Clough for Chattahoochee River sand (Fig. 6.22) in its initial condition, after isotropic compression to 62.1 megapascals (MPa), and after shearing in a conventional triaxial compression test at that constant cell pressure. A considerable increase in the proportion of material finer than 0.1 mm occurs both during isotropic compression and during shearing, indicating that particle crushing is a primary mechanism of deformation at high stress levels. Thus, the material that emerges from the test at a high stress level is very different from the material that went into the tests at low stress levels.

At the other extreme, at extremely low pressures, reliable data are difficult to obtain because of the inevitable gravitational variation of stress in a sample, though this could be avoided by performing experiments on board an orbiting space laboratory. Indirect measurements made as part of hopper flow experiments by Crewdson, Ormond, and Nedderman (1977) appear to indicate a levelling off of the zone of critical void ratios to a roughly constant void ratio at very low pressures. This region of the compression plane may be of importance in studies of liquefaction of soils, in which flow of soil under effective stresses which are near zero, because of the presence of high pore pressures, is of importance.

Also plotted in the compression plane (Fig. 6.21a) are the average isotropic compression curves for dense and loose Chattahoochee River sand. These curves illustrate the difficulty of exploring the compression plane using only isotropic stresses: the loose structure of loose sand is not disturbed by isotropic compression. Considerable particle rearrangement is required for the initial structures of loose and dense sands to become the same, and this rearrangement can occur readily only in the presence of shear stresses. It is only at high isotropic pressures that the compression curves for loose and dense samples start to converge. Isotropic compression is, at these high stresses, associated with particle crushing, but still the original structure of the sand is not completely eliminated.

Rockfill can be considered as an extreme granular material, but it is a
material which shows the same pattern of response that has been demonstrated for finer granular materials. Problems of obtaining good quality test data from tests on rockfill arise because of the large specimen which is needed in order that the complete grading of particle sizes will be included and that the particles will not be unduly restrained by the proximity of the boundaries. Some triaxial tests on rockfill materials are reported by Marachi, Chan, and Seed (1972), and paths in the compression

Fig. 6.22 Grading curves for Chattahoochee River sand (1) before testing, (2) after isotropic compression to 62.1 MPa, and (3) after conventional drained triaxial compression to failure with cell pressure \( \sigma_c = 62.1 \) MPa (after Vesić and Clough, 1968).

Fig. 6.23 Paths in compression space of conventional drained triaxial compression tests on Pyramid Dam rockfill material (data from Marachi, Chan, and Seed, 1972).
plane are shown in Fig. 6.23 for the tests on one of these materials: rockfill from the Pyramid Dam quarry. For the tests shown, the samples were 0.91 m (3 ft) in diameter and 2.29 m (7.5 ft) high. The change in the character of the response that occurs as the confining pressure of the triaxial tests is increased is apparent. Dilatation at low pressure becomes marked compression at higher pressures. For this material, the transition from 'dense' to 'loose' response occurs at a stress level of about 1000 kPa corresponding to a depth of the order of 100 m in a dry mass of the rockfill.

In presenting data of critical states for clays, parallel use was made of the \( p':q \) stress plane and the \( p':v \) compression plane. The critical state lines predicted by the Cam clay model (Fig. 6.3) and deduced for the Weald clay (Fig. 6.10) show an ultimate critical state deviator stress which is proportional to the ultimate critical state mean effective stress. The parameter \( M \) introduced into the Cam clay model implies an ultimate purely frictional strength.

The ultimate failure and flow of sands and other granular materials are governed almost entirely by frictional factors. If a critical void ratio is reached and shearing continues at constant volume and constant stress level, then the shearing resistance is linked to the stress level by a coefficient of friction. This frictional relationship can be illustrated by plotting results in a stress plane: shear stress: normal stress for simple shear or shear box tests and deviator stress \( q: \) mean normal stress \( p' \) for triaxial tests. The conditions associated with critical void ratios observed in simple shear tests on steel balls (Wroth, 1958) are shown in a stress plane (Fig. 6.19c) and a compression plane (Fig. 6.19d). For this ideal material, a fairly well-defined line of critical void ratios is obtained in both the stress plane and the compression plane. The data concerning the critical void ratio for a single normal stress level (Figs. 6.19a, b) provide one point on this line of critical void ratios.

Critical void ratios are not so well defined for the Chattahoochee River sand tested by Vesić and Clough (1968), but some of the relevant data are collected in similar plots in Figs. 6.21b, c.

The term critical state was used in Section 6.1 to describe a combination of effective stresses and specific volume at which shearing of soil could continue indefinitely, and the idea of a critical state line linking critical states was introduced. The lines of end points (Figs. 6.19 and 6.21) corresponding to attainment of critical void ratios for various granular materials can be thought of, similarly, as critical state lines. It is apparent that the pattern of critical states which is observed is equivalent to that which emerged from the Cam clay model in Section 6.1 and which was discovered for Weald clay in Section 6.3.
6.6 Conclusion

This chapter has shown that there emerges from the Cam clay model a set of combinations of effective stresses and specific volume $p':q:v$ at which indefinite shearing (perfect plasticity) occurs. These critical states emerge merely because the plastic potentials assumed in the model have a slope in the stress plane $\delta q/\delta p' = 0$, at a particular stress ratio $q/p' = \eta = M$. Critical states emerge automatically from this elastic-plastic soil model and do not require any further assumptions to be made.

When data from shear tests on geotechnical materials are examined in terms of 'limited' quantities such as effective stresses and specific volume (as opposed to the 'unlimited' quantity shear strain), it is found that all tests tend towards such critical states. Critical states are a major feature of observed response.

In some cases, rather simple relationships exist between the combinations of effective stresses and specific volume at which critical states are attained: it is then possible to make powerful quantitative statements about expected patterns of soil behaviour without recourse to the complexities of Cam clay or other elastic-plastic models.

Exercises

E6.1. For a particular soil, the critical state line is defined by the following soil parameters: $M = 0.9$, $\lambda = 0.25$, $\Gamma = 3.0$.

Find the end states in terms of $p':q:v$ for conventional drained and undrained triaxial compression tests ($\Delta \sigma = 0$) on soils with the initial states tabulated. For the undrained tests, find also the pore pressure at failure.

<table>
<thead>
<tr>
<th></th>
<th>$p'_i$ (kPa)</th>
<th>$q_i$ (kPa)</th>
<th>$v_i$</th>
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<tbody>
<tr>
<td>a</td>
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<td>0</td>
<td>2.049</td>
</tr>
<tr>
<td>b</td>
<td>100</td>
<td>0</td>
<td>1.700</td>
</tr>
<tr>
<td>c</td>
<td>100</td>
<td>0</td>
<td>1.849</td>
</tr>
<tr>
<td>d</td>
<td>200</td>
<td>0</td>
<td>1.875</td>
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<tr>
<td>e</td>
<td>100</td>
<td>50</td>
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<td>f</td>
<td>50</td>
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E6.2. A stress-controlled compression test is carried out on a sample of normally compressed clay with a total confining stress of 490 kPa, which is held constant throughout the test. Each load increment is left on for sufficient time to allow the sample to reach a new state of equilibrium, as recorded in the table below. For the first three increments AB, BC, and CD, the sample is undrained. The
6 Critical states

drainage connection is then opened, and the sample reaches equilibrium again at E. Further load increments are applied, with drainage allowed, until the sample reaches failure at F, and the sample is then in a critical state.

Plot the progress of the test in the \( p':q \) effective stress plane and in the \( p':\nu \) compression plane and estimate the position of the critical state line for this clay.

Two additional samples of this clay (X and Y) are normally compressed under an isotropic pressure \( p' = 350 \) kPa. If a conventional drained triaxial compression test is performed on X, predict the values of \( p', q, \) and \( \nu \) at failure. If a conventional undrained triaxial compression test is performed on Y, predict the values of \( p', u, \) and \( q \) at failure.

<table>
<thead>
<tr>
<th>( q ) (kPa)</th>
<th>( u ) (kPa)</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 0</td>
<td>0</td>
<td>2.270</td>
</tr>
<tr>
<td>B 39.2</td>
<td>28.0</td>
<td>2.270</td>
</tr>
<tr>
<td>C 78.4</td>
<td>70.0</td>
<td>2.270</td>
</tr>
<tr>
<td>D 154.0</td>
<td>200.9</td>
<td>2.270</td>
</tr>
<tr>
<td>E 154.0</td>
<td>0</td>
<td>2.205</td>
</tr>
<tr>
<td>F 623.0</td>
<td>0</td>
<td>1.990</td>
</tr>
</tbody>
</table>

E6.3. Study the models described in Exercises E4.3, E4.4, E4.5, and E4.6 to discover whether or not each model predicts the existence of a unique critical state line.

E6.4. Obtain data for a natural clay from oedometer and triaxial tests performed as part of a site investigation. Plot the oedometer data in the compression plane, and estimate values for \( \lambda \) and \( \kappa \) for the soil. From the ends of the triaxial tests, estimate the slope \( M \) of the critical state line in the effective stress plane and its position \( \Gamma \) in the compression plane. From the initial stages of the triaxial tests, estimate a value for the shear modulus \( G' \).

Use the Cam Clay model to estimate the complete stress:strain response for one of the triaxial tests, and compare the estimated response with the observed response.