2

Elasticity

2.1 Isotropic elasticity

Elastic, recoverable, material response is easier to describe and comprehend than plastic, irrecoverable response. Some essential elements of the theory of elasticity will be presented in this chapter.

A familiar introduction to the elastic properties of materials is obtained by the simple experimental procedure of hanging weights on a wire and measuring the elongation (Fig. 2.1a). For many materials, there is a range of loads for which the elongation varies linearly with the applied load $P$ (Fig. 2.1b) and is recovered when the load is removed. There is no permanent deformation of the wire.

Such a procedure provides a direct indication of the validity of Hooke's law as originally stated (1675); *ut tensio sic vis* (as the extension so the force). Hooke's law was originally published as an anagram *ceiinosssttuw* (Fig. 2.2) to fill space in his book on helioscopes, an enigmatic way of attempting to guarantee scientific precedence (Heyman, 1972).

The slope of the linear relationship is related to the unconfined uniaxial stiffness of the material of the wire. Young's modulus $E$ is expressed as

$$E = \frac{P/A}{\delta l/l}$$

(2.1)

where $P$ is the load on the wire of area $A$ and length $l$, and $\delta l$ is the extension of the wire.

The experimental introduction to elasticity is usually restricted to observation of the load:extension properties of tensile specimens. However, if a micrometer were available to measure the changing diameter $d$ of the wire, then one would observe that as the wire becomes longer, its diameter becomes smaller (Figs. 2.1c and 2.3a). The ratio of the magnitude of the induced diametral strain to the imposed longitudinal
strain is Poisson's ratio \( v \):

\[
v = \frac{-\delta d/d}{\delta l/l}
\]  

(2.2)

It is well known that the pair of constants \( E \) and \( v \) is sufficient to describe the elastic response of isotropic materials. These two constants are probably the most easily understood elastic constants because direct experimental observation of them is so straightforward. However, in many ways it is more fundamental to use an alternative pair of elastic constants: the bulk modulus \( K \) and shear modulus \( G \), which divide the elastic deformation into a volumetric part (change of size at constant shape, Fig. 2.3b) and a distortional part (change of shape at constant volume, Fig. 2.3c), respectively.

The benefit of using \( K \) and \( G \) is particularly great when elasticity of soils is considered. Undrained deformation of soils is specifically concerned with deformation of soil at constant volume, that is, pure distortion of soil, a change of shape without change in size. The distinction between undrained and drained processes is only relevant because there are some processes (in general, inelastic ones) during which soils express a desire to change in size as well as shape as they are sheared.

The relationships between the two sets of constants can be deduced by considering the change in volume of the wire as it is extended, which shows that the bulk modulus \( K \) is

\[
K = \frac{E}{3(1-2v)}
\]  

(2.3)

Fig. 2.1 Tensile test on metal wire: (a) test arrangement; (b) load, extension relationship; (c) changes in diameter and length.
A DESCRIPTION
OF
HELIOSCOPES,
And some other
INSTRUMENTS.
MADE BY:
ROBERT HOOKE,
Fellow of the Royal Society.

Hoc ego; &c.
Sic voc non voca—

LONDON,
Printed by T. R. for John Martyn Printer to the Royal Society,
at the Black St. Pauls Church-yard, 1676.

To fill the vacancy of the ensuing page, I have here added a decimate of the centesime of the Inventions I intend to publish, though possibly not in the same order, but as I can get opportunity and leisure; most of which, I hope, will be as useful to Mankind, as they are yet unknown and new.

1. A way of Regulating all sorts of Watches or Timekeepers, so as to make any way to equalize, if not exceed the Pendulum-Clocks now used.

2. The true Mathematical and Mechanical form of all manner of Arches for Building, with the true lengths necessary to each of them. A Problem which no Architectonic Writer hath ever yet attempted, much less performed.

3. The true Theory of Elasticity or Springing, and a particular Explication thereof in several Subjects in which it is to be found: And the way of computing the velocity of Bodies moved by them.
and by considering the change of the right angle between rays 'drawn' in the material of the wire at $\pm \pi/4$ to the axis of the wire (Fig. 2.3c), which shows that the shear modulus $G$ is

$$G = \frac{E}{2(1 + v)}$$  \hspace{1cm} (2.4)

### 2.2 Soil elasticity

Unconfined tensile tests on soils are not usually feasible. Compression tests are more commonly performed using the triaxial apparatus and with some lateral confinement provided by the cell pressure. The results of a typical drained test on a soil sample might resemble those shown in Fig. 2.4. This is a conventional triaxial compression test in which the axial stress (or deviator stress) is increased while the lateral stress (or cell pressure) is held constant. The initial linear sections of the stress-strain curve (Fig. 2.4a) (deviator stress $q$ plotted against triaxial shear strain $\varepsilon_s$) and of the volume-change curve (Fig. 2.4b) (volumetric strain $\varepsilon_v$ plotted

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**Fig. 2.3** (a) Young's modulus describing change in length and Poisson's ratio describing change in width; (b) bulk modulus describing change in size at constant shape; (c) shear modulus describing change in shape at constant volume.
2.2 Soil elasticity

against triaxial shear strain \( \varepsilon_s \) might be interpreted as the elastic response of the soil to the imposed changes of stress. Thus, the values of the elastic constants could be deduced.

It is a presumption throughout this book that soil response is governed by changes in effective stresses and that elastic response is no exception. Elastic properties of the soil skeleton will be written here with a prime to emphasise this point.

Working in terms of Young's modulus and Poisson's ratio, one can describe the response of a soil specimen to a general triaxial change of effective stress by these equations:

\[
\begin{bmatrix}
\delta \varepsilon_s \\
\delta \varepsilon_t
\end{bmatrix} = \frac{1}{E'} \begin{bmatrix}
1 & -2\nu' \\
-\nu' & 1 - \nu'
\end{bmatrix} \begin{bmatrix}
\delta \sigma_s' \\
\delta \sigma_t'
\end{bmatrix}
\]

(2.5)

Matrices are introduced here simply as a convenient shorthand means of writing sets of equations. Thus, (2.5) implies the following pair of equations:

\[
\delta \varepsilon_s = \frac{1}{E'} (\delta \sigma_s' - 2\nu' \delta \sigma_t')\]  

(2.5a)

\[
\delta \varepsilon_t = \frac{1}{E'} [ -\nu' \delta \sigma_s' - (1 - \nu') \delta \sigma_t']
\]

(2.5b)

Fig. 2.4 Elastic constants deduced from conventional drained triaxial compression test: (a) deviator stress \( q \) and triaxial shear strain \( \varepsilon_s \); (b) volumetric strain \( \varepsilon_v \) and triaxial shear strain \( \varepsilon_s \); (c) deviator stress \( q \) and axial strain \( \varepsilon_a \).
Writing equations in terms of matrix products is extremely helpful when trying to write compact, efficient computer programs. However, no aptitude for manipulation of combinations of matrices is either expected or demanded in this book.

The preferred strain increment and effective stress variables that were introduced in Section 1.4.1 for description and analysis of triaxial tests were the mean effective stress \( p' \) and deviator stress \( q \),

\[
\begin{bmatrix}
  p' \\
  q
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{3} & \frac{2}{3} \\
  1 & -1
\end{bmatrix}
\begin{bmatrix}
  \sigma'_s \\
  \sigma'_r
\end{bmatrix}
\]  
(2.6)

and the increments of volumetric strain \( \delta e_p \) and triaxial shear strain \( \delta e_q \),

\[
\begin{bmatrix}
  \delta e_p \\
  \delta e_q
\end{bmatrix}
= \begin{bmatrix}
  1 & 2 \\
  \frac{2}{3} & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
  \delta e_s \\
  \delta e_r
\end{bmatrix}
\]  
(2.7)

The elastic response in (2.5) can then be written more elegantly using bulk modulus and shear modulus to separate effects of changing size and changing shape:

\[
\begin{bmatrix}
  \delta e_p \\
  \delta e_q
\end{bmatrix}
= \begin{bmatrix}
  1/K' & 0 \\
  0 & 1/3G'
\end{bmatrix}
\begin{bmatrix}
  \delta p' \\
  \delta q
\end{bmatrix}
\]  
(2.8)

The off-diagonal zeroes in (2.8) indicate the absence of coupling between volumetric and distortional effects for this isotropic elastic material. Change in mean stress \( p' \) produces no distortion \( \delta e_q \), and change in the distortional deviator stress \( q \) produces no change in volume.

The initial gradient of the stress: strain curve in Fig. 2.4a is then \( 3G' \). The initial gradient of the volume change curve in Fig. 2.4b is

\[
\frac{\delta e_p}{\delta e_q} = \frac{3G'}{K'} \frac{\delta p'}{\delta q}
\]  
(2.9)

For a conventional drained triaxial compression test on a soil specimen, in which

\[
\delta q = 3 \delta p'
\]  
(2.10)

this becomes

\[
\frac{\delta e_p}{\delta e_q} = \frac{G'}{K'}
\]  
(2.11)

and the elastic properties have been recovered. Evidently, values of Young's modulus and Poisson's ratio could be deduced using (2.3) and (2.4).

If, alternatively, drainage from the triaxial sample is prevented, then undrained constant volume response is observed. The initial response of the soil specimen may still be elastic, but now, with volume change prevented, pore pressures develop. The response of the soil can be depicted
2.2 Soil elasticity

in plots of deviator stress and pore pressure against triaxial shear strain (Fig. 2.5).

The imposition of a condition of constant volume on (2.8) implies that

\[
\frac{\delta p'}{K'} = 0 \tag{2.12}
\]

which requires either that

\[ K' = \infty \]

or that

\[ \delta p' = 0 \]

There is no reason why the bulk modulus of the soil skeleton should be infinite; certainly the act of closing the drainage tap on the triaxial apparatus can have no influence on the elastic properties of the soil skeleton. Consequently, the only reasonable solution to (2.12) is

\[ \delta p' = 0 \tag{2.13} \]

The pore pressure changes reflect directly the imposed changes in total mean stress

\[ \delta u = \delta p \tag{2.14} \]

Fig. 2.5 Elastic constants deduced from conventional undrained triaxial compression test: (a) deviator stress \( q \) and triaxial shear strain \( \varepsilon_q \); (b) pore pressure \( u \) and triaxial shear strain \( \varepsilon_q \); (c) deviator stress \( q \) and axial strain \( \varepsilon_a \).
and, irrespective of the total stress path, the effective stress path is vertical in the $p':q$ plane ($AB$ in Fig. 2.6a). Evidently, any total stress path could be imposed; if one of these paths had no applied change in total mean stress, $\delta p = 0$, then the soil would have no desire to change in volume under the purely distortional stress changes and hence no tendency to generate any pore pressure. This result implies that for this isotropic elastic material the pore pressure parameter $a$ in (1.65) is zero, which is just another way of saying that there is no coupling between volumetric and distortional effects, as illustrated in (2.8).

The constant volume condition imposes no constraint on the change in shape of the soil sample, and (2.8) makes it clear that the slope of the deviator stress:triaxial shear strain plot (Fig. 2.5a) will again be $3G'$, as in the drained test. For a conventional undrained triaxial compression test in which the cell pressure is held constant while the axial stress is increased,

$$\delta p = \frac{\delta q}{3} = \delta u$$

(2.15)

and the slope of the plot of pore pressure against triaxial shear strain (Fig. 2.5b) is $G'$.

If at some stage of the undrained loading the drainage tap is opened and the pore pressure is allowed to dissipate at constant total stress, then some deformation of the soil will occur. With the total stresses constant, the deviator stress $q = \sigma' - \sigma' = \sigma_s - \sigma_r$ cannot change, and the effective stress path for this dissipation process is parallel to the $p'$ axis ($BC$ in Fig. 2.6a). It is then apparent from (2.8) that the accompanying deformation will involve only change of size ($\delta \varepsilon_p > 0$) with no change in shape ($\delta \varepsilon_q = 0$).

Fig. 2.6 Undrained shearing $AB$ and subsequent pore pressure dissipation $BC$ in conventional triaxial compression test: (a) total and effective stress paths; (b) deviator stress $q$ and axial strain $\varepsilon_a$.  

(a)  

(b)
2.2 Soil elasticity

Although the behaviour of soil elements is controlled by changes in effective stresses, it is often useful to describe the elastic response of soil in terms of changes in total stresses. Equilibrium equations for a soil continuum can be written in terms of total stresses without having to introduce pore pressures, and analytical procedures may often lead more readily to distributions of total stresses than to distributions of effective stresses. The observed response of a soil element must, however, be identical whether it is treated in terms of total stresses or effective stresses.

The total stress equivalent of (2.8) is

\[
\begin{bmatrix}
\delta \varepsilon_p \\
\delta \varepsilon_q
\end{bmatrix} =
\begin{bmatrix}
1/K_u & 0 \\
0 & 1/3G_u
\end{bmatrix}
\begin{bmatrix}
\delta p \\
\delta q
\end{bmatrix}
\]  

(2.16)

The distinction between elastic properties in terms of total or effective stresses is only helpful for constant volume undrained conditions; this is the reason for the subscript u on the bulk modulus and shear modulus in (2.16).

For an undrained constant volume condition, \( \delta \varepsilon_p = 0 \) implies that

\[
\frac{\delta p}{K_u} = 0
\]  

(2.17)

Whereas the effective stress description of (2.8) and (2.12) looks at the triaxial soil sample from inside the membrane, the total stress description of (2.16) and (2.17) looks at the sample from outside the membrane. There can now be no constraint on the total stress path that is imposed. The condition of no volume change must emerge whatever the externally applied changes in total stress. Hence, the total stress undrained bulk modulus \( K_u \) must be infinite,

\[
K_u = \infty
\]  

(2.18)

which, from the general equation (2.3), implies that the undrained Poisson's ratio is

\[
\nu_u = \frac{1}{2}
\]  

(2.19)

The deviator stress \( q \) is not affected by drainage conditions because, as a difference of two stresses, it is independent of pore pressure. The shearing, or change of shape, of the soil \( \delta \varepsilon_q \) calculated from (2.8) and (2.16) must be identical and hence

\[
G_u = G'
\]  

(2.20)

and the shear modulus is independent of the drainage conditions.

Given the link between shear modulus, Young's modulus, and Poisson's ratio implied by (2.4), the undrained and drained values of Young's modulus, \( E_u \) and \( E' \), respectively, are not independent. For from (2.4)
and (2.20),
\[ \frac{E_u}{2(1 + v_u)} = \frac{E'}{2(1 + v')} \]
and then with (2.19),
\[ E_u = \frac{3E'}{2(1 + v')} \]  \hspace{1cm} (2.21)

Young's modulus describes the slope of the axial stress:axial strain relationship. In conventional triaxial compression tests \( \delta \sigma_u = \delta q \), and Young's modulus is the slope of the deviator stress:axial strain relationship. Different slopes, in the ratio given by (2.21), will be seen in drained and undrained tests (Figs. 2.4c and 2.5c). The effect of allowing drainage to occur after some increments of undrained loading (effective stress path \( BC \) after \( AB \) in Fig. 2.6a) is to take the deviator stress: axial strain state from the undrained to the drained line, as shown in Fig. 2.6b.

In summary, for isotropic elastic soil there are only two independent elastic soil constants. Elastic constants to describe the behaviour of soil under special conditions (e.g. in terms of total stresses for undrained conditions) can be deduced from the more fundamental effective stress constants and cannot be chosen independently.

2.3 Anisotropic elasticity

The discussion in previous sections has been restricted to the ideal case of isotropic elasticity. Real soil may not fit into this simple picture. Deviations from this picture may result from inelasticity, but they can also occur if the soil is elastic but anisotropic.

A completely general description of an anisotropic elastic material requires the specification of 21 elastic constants (e.g. see Heyman, 1982; Love, 1927), but analyses using such general material characteristics are rarely practicable. Besides, the depositional history of many soils introduces symmetries which may reduce considerably the number of independent elastic constants.

Many soils have been deposited over areas of large lateral extent, and the deformations they have experienced during and after deposition have been essentially one-dimensional. Soil particles have moved vertically downwards (and possibly also upwards) with, from symmetry, no tendency to move laterally (Fig. 1.21). The anisotropic elastic properties of the soil reflect this history. The soil may respond differently if it is pushed in
vertical or horizontal directions, but it will respond in the same way if it is pushed in any horizontal direction. For example, cylindrical sample \( A \) in Fig. 2.7, taken from the ground with its axis vertical, behaves differently from samples \( B, C, D, \) and \( E \), which have been taken from the ground with their axes in various horizontal directions; but samples \( B, C, D, \) and \( E \) all behave identically.

This special form of anisotropy, known as transverse isotropy or cross anisotropy, requires only five elastic constants for its specification. The form of the relationship between stress increments and strain increments takes the form

\[
\begin{bmatrix}
\delta \varepsilon_{xx} \\
\delta \varepsilon_{yy} \\
\delta \varepsilon_{zz} \\
\delta \gamma_{yz} \\
\delta \gamma_{zx} \\
\delta \gamma_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
1/E_h & -v_{hh}/E_h & -v_{vh}/E_v & 0 & 0 & 0 \\
-v_{hh}/E_h & 1/E_h & -v_{vh}/E_v & 0 & 0 & 0 \\
-v_{vh}/E_v & v_{vh}/E_v & 1/E_v & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2G_{vh} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2G_{vh} & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + v_{hh}/E_h)
\end{bmatrix}
\begin{bmatrix}
\delta \sigma'_{xx} \\
\delta \sigma'_{yy} \\
\delta \sigma'_{zz} \\
\delta \tau_{yz} \\
\delta \tau_{zx} \\
\delta \tau_{xy}
\end{bmatrix}
\]

where the stress and strain increments are referred to rectangular Cartesian axes \( x, y, \) and \( z \) with the \( z \) axis vertical (Fig. 2.7).

Most of the routine soil tests that are performed in practice are triaxial compression tests on samples such as \( A \) in Fig. 2.7, taken out of the ground with their axes vertical, for example, from some sort of borehole. Graham and Houlsby (1983) show that it is not possible from such tests to recover more than three elastic constants for the soil; since two constants are needed for the description of isotropic elastic response, that leaves only one constant through which some anisotropy can be incorporated. They propose a particular form of one-parameter anisotropy which allows certain analytical advantages and leads to a particular form of stiffness.

![Fig. 2.7 Cylindrical soil samples taken out of the ground with their axes vertical (\( A \)) and horizontal (\( B, C, D, E \)).](image-url)
matrix relating stress increments and strain increments:

\[
\begin{bmatrix}
\delta e_{xx}' \\
\delta e_{yy}' \\
\delta e_{zz}' \\
\delta e_{tx}' \\
\delta e_{ty}' \\
\delta e_{tz}'
\end{bmatrix} = \frac{E^*}{(1 + v^*)(1 - 2v^*)} \begin{bmatrix}
a^2(1 - v^*) & a^2v^* & a^2v^* & 0 & 0 & 0 \\
a^2v^* & a^2(1 - v^*) & a^2v^* & 0 & 0 & 0 \\
a^2v^* & a^2v^* & a^2(1 - v^*) & 0 & 0 & 0 \\
0 & 0 & 0 & a(1 - 2v^*)/2 & 0 & 0 \\
0 & 0 & 0 & 0 & a(1 - 2v^*)/2 & 0 \\
0 & 0 & 0 & 0 & 0 & a^2(1 - 2v^*)/2 \\
\end{bmatrix} \begin{bmatrix}
\delta e_{xx} \\
\delta e_{yy} \\
\delta e_{zz} \\
\delta e_{tx} \\
\delta e_{ty} \\
\delta e_{tz}
\end{bmatrix}
\]

(2.23)

This expression can be inverted so that the stiffness matrix becomes a compliance matrix:

\[
\begin{bmatrix}
\delta e_{xx} \\
\delta e_{yy} \\
\delta e_{zz} \\
\delta e_{tx} \\
\delta e_{ty} \\
\delta e_{tz}
\end{bmatrix} = \frac{1}{E^*} \begin{bmatrix}
1/a^2 & -v^*/a^2 & -v^*/a & 0 & 0 & 0 \\
-v^*/a^2 & 1/a^2 & -v^*/a & 0 & 0 & 0 \\
-v^*/a & -v^*/a & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1 + v^*)/a & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1 + v^*)/a & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + v^*)/a^2 \\
\end{bmatrix} \begin{bmatrix}
\delta e_{xx}' \\
\delta e_{yy}' \\
\delta e_{zz}' \\
\delta e_{tx}' \\
\delta e_{ty}' \\
\delta e_{tz}'
\end{bmatrix}
\]

(2.24)

In these expressions, \( E^* \) and \( v^* \) represent modified values of Young's modulus and Poisson's ratio for the soil, and \( a \) is the anisotropy parameter. Equation (2.24) can be compared with the completely general five-constant description of transverse isotropy (2.22).

Expressions (2.22)–(2.24) give the complete stiffness and compliance matrices which are necessary for any analysis of transversely isotropic elastic soil. However, to interpret the results of triaxial tests, it is helpful once again to look at description of changes in size and in shape. Still following Graham and Houlsby (1983), one finds the stiffness equation to be

\[
\begin{bmatrix}
\delta p' \\
\delta q
\end{bmatrix} = \begin{bmatrix} K^* & J \\ J & 3G^* \end{bmatrix} \begin{bmatrix}
\delta e_p \\
\delta e_q
\end{bmatrix}
\]

(2.25)

where \( K^* \) and \( G^* \) are modified values of bulk modulus and shear modulus, and the presence of the two off-diagonal terms \( J \) shows that there is now some cross-coupling between volumetric and distortional effects. The quantities \( K^*, G^*, \) and \( J \) can be expressed in terms of the quantities \( E^*, v^*, \)
and $\alpha$ from (2.23):
\[
K^* = \frac{E^*(1 - \nu^* + 4\alpha \nu^* + 2\alpha^2)}{9(1 + \nu^*)(1 - 2\nu^*)}
\] (2.26)
\[
G^* = \frac{E^*(2 - 2\nu^* - 4\alpha \nu^* + \alpha^2)}{6(1 + \nu^*)(1 - 2\nu^*)}
\] (2.27)
\[
J = \frac{E^*(1 - \nu^* + \alpha \nu^* - \alpha^2)}{3(1 + \nu^*)(1 - 2\nu^*)}
\] (2.28)

It may be confirmed that for $\alpha = 1$, these expressions for $K^*$ and $G^*$ reduce to (2.3) and (2.4) and that $J = 0$. The compliance form of (2.25) is
\[
\begin{bmatrix}
\delta \varepsilon_p \\
\delta \varepsilon_q
\end{bmatrix} = \frac{1}{D} \begin{bmatrix}
3G^* & - J \\
- J & K^*
\end{bmatrix}
\begin{bmatrix}
\delta p' \\
\delta q
\end{bmatrix}
\] (2.29)
with
\[
D = 3K^*G^* - J^2
\] (2.30)

The coupling between volumetric and distortional effects implies that constant volume effective stress paths are no longer vertical constant $p'$ paths in the $p'q$ plane [(2.13) and Fig. 2.6a]. The direction of the path will depend on the value of $\alpha$. With $\alpha > 1$ the soil is stiffer horizontally than vertically, and the undrained effective stress path shows a decrease in $p'$ (Fig. 2.8). With $\alpha < 1$ the soil is stiffer vertically, and the undrained effective stress path shows an increase in $p'$ (Fig. 2.8). The pore pressure that is observed in an undrained test will be different from the change in total mean stress [see (2.14)], with the difference depending on $\alpha$ and $\nu^*$. The direction of the effective stress path is, from (2.29):
\[
\frac{\delta q}{\delta p'} = \frac{3G^*}{J}
\] (2.31)
or
\[ \frac{\delta q}{\delta p'} = \frac{3(2 - 2v^* - 4\alpha v^* + \alpha^2)}{2(1 - v^* + \alpha v^* - \alpha^2)} \]  
(2.32)

This ratio has limiting values \(-\frac{3}{2}\) and \(+3\) for \(\alpha\) very large and very small, which imply effective stress paths with constant axial stress and constant radial stress, respectively. Comparison with (1.65) shows that for this cross-anisotropic elastic soil, the pore pressure parameter \(a\) is given by
\[ a = -\frac{J}{3G^*} \]  
(2.33)

Fig. 2.9 Changes in effective stress in undrained triaxial compression of Winnipeg clay (data from Graham and Houlsby, 1983).

Fig. 2.10 Volumetric strain-triaxial shear strain paths for compression of cross-anisotropic elastic soil under isotropic stresses.
2.3 Anisotropic elasticity

Some typical data for a natural clay from Winnipeg, Manitoba are shown in Fig. 2.9 (after Graham and Housby, 1983). For this clay, Graham and Housby estimate a value of \( v^* = 0.2 \); they quote a range of slopes of effective stress paths \( \delta q/\delta p' \) between \(-15.8 \) and \(-4.45 \), and they deduce an average value for \( \alpha^2 \) of 1.52, which is the ratio of horizontal to vertical stiffness, \( E_h/E_v \), in (2.22).

The coupling between change of size and change of shape also implies that shear strains will occur along stress paths in which \( q = 0 \) (Fig. 2.10). For such a path, from (2.25),

\[
\frac{\delta e_q}{\delta e_p} = \frac{-J}{3G^*} \tag{2.34}
\]

or

\[
\frac{\delta e_q}{\delta e_p} = \frac{-2(1 - v^* + \alpha v^* - \alpha^2)}{3(2 - 2v^* - 4\alpha v^* + \alpha^2)} \tag{2.35}
\]

and the path of this isotropic compression will not in general lie along the \( \delta e_p \) axis \( (e_q = 0) \) in the \( e_p, e_q \) strain plane. This ratio (2.35) has limiting values \( \frac{2}{3} \) and \( -\frac{1}{3} \) for \( \alpha \) very large and very small. These limiting values correspond to compression at constant radial strain \( (\delta e_r = 0) \) and constant axial strain \( (\delta e_a = 0) \), respectively.

Typical data for the Winnipeg clay are shown in Fig. 2.11 (after Graham and Housby, 1983). Graham and Housby quote a range of ratios \( \delta e_q/\delta e_p \) between 0.1 and 0.32 and deduce an average value of \( \alpha^2 \) between 1.8 and 1.9.

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Fig. 2.11  Volumetric strain: triaxial shear strain paths for compression of Winnipeg clay under isotropic stresses (data from Graham and Housby, 1983).
2.4 The role of elasticity in soil mechanics

Because many subsequent sections of this book are devoted to discussion of the inelastic or plastic behaviour of soils, it may be wondered what real role elasticity plays in soil mechanics. Two applications are briefly presented here: calculation of deformations of geotechnical structures under working loads and selection of stress paths to guide appropriate laboratory testing.

Soil mechanics has traditionally been concerned firstly with ensuring that geotechnical structures do not actually collapse and secondly with ensuring that the working deformations of these structures are acceptable. If it can be assumed that the soil will respond elastically to applied loads, then the whole body of elastic theory becomes available to analyse the deformations of any particular problem. Settlements of foundations and deformations of piles are frequently estimated by using charts computed from elastic analyses of more or less standard situations. Much site investigation, whether with triaxial tests in the laboratory or with pressuremeter tests or plate loading tests in the field, is devoted to determining accurate values of moduli for soils for subsequent use in deformation analyses. The results of such analyses will of course be as good as the quality of the determination of the moduli and the quality of the assumption of elasticity.

The behaviour of an isotropic elastic soil is encapsulated in just two elastic constants, which can be obtained from a simple programme of soil testing. Most soils cannot satisfactorily be described as isotropic and elastic, so a more elaborate model will be required to describe soil response. Such models are used to extrapolate from available experimental data (typically obtained under the rather restrictive stress conditions imposed in conventional laboratory tests) to the complex states of stress and strain which develop around a prototype structure. The quality of the prediction of soil response will depend on the extent of this extrapolation. If the soil response is very stress path dependent, then it is helpful if the laboratory testing can bear some relation to the stress paths to which soil elements in the ground around a geotechnical structure may be subjected.

This argument is circular since the stress paths which are predicted to develop in the ground depend on the details of the stress-strain response which the laboratory testing is trying to evaluate. This circle has to be broken. Elastic stress distributions are available for many loading situations (e.g. see the comprehensive collection of Poulos and Davis, 1974); these are frequently a useful starting point in assessing plausible stress changes for soil elements. Stress paths for certain simple geotechnical problems will be considered in more detail in Chapter 10.
Exercises

Paradoxically, then, this major application of elasticity in soil mechanics is guiding the study of the inelastic stress-strain behaviour of soils.

Exercises

E2.1. Use Hooke’s law to deduce relationships between stress increments for samples of isotropic elastic soil which are being (i) deformed in plane strain and (ii) compressed one-dimensionally.

E2.2. A conventional undrained triaxial compression test, with the cell pressure \( \sigma_r \) held constant, is carried out on a sample of stiff overconsolidated clay. The stress-strain relationship is found to be linear up to failure, so it is deduced that the clay behaves as an isotropic perfectly elastic material.

i. After an axial strain \( \Delta \varepsilon_a = 0.9 \) per cent, the deviator stress is measured to be 90 kPa. For this stage of the test, calculate the values of \( \Delta u \), \( \Delta p \), \( \Delta \sigma_r \), \( \Delta \sigma_a \), \( \Delta \sigma_r \), \( \Delta \varepsilon_r \), and \( E_u \).

ii. At this time, the axial stress and cell pressure are kept constant, and the sample is allowed to drain so that the pore pressures dissipate and the sample undergoes a volumetric strain \( \Delta \varepsilon_p = 0.3 \) per cent. What are the values of \( \Delta \sigma_a \), \( \Delta \sigma_r \), \( \Delta u \), \( \Delta p \), \( \Delta q \), \( \Delta \varepsilon_a \), and \( \Delta \varepsilon_r \) for this stage of the test, and what are the values of the elastic constants \( K' \), \( G' \), \( E' \), and \( v' \)?

E2.3. The effective stress elastic behaviour of a cross-anisotropic soil is characterised by a Young’s modulus \( E^* \) for the vertical direction, a Young’s modulus \( \alpha^2 E^* \) for the horizontal direction, a Poisson’s ratio \( v^* \) for the strain in a horizontal direction due to a strain in a vertical direction, and a Poisson’s ratio \( v^* \) for the strain in a horizontal direction due to a strain in an orthogonal horizontal direction [see (2.24)].

A cuboidal specimen of this soil is compressed by a total normal stress increment \( \Delta \sigma_1 \) in the vertical direction, with no change in total stress in one horizontal direction, \( \Delta \sigma_3 = 0 \), and with no strain in the other horizontal direction, \( \Delta \varepsilon_2 = 0 \).

Find an expression for the pore pressure change \( \Delta u \) in the soil if drainage is not permitted. Check that your expression gives \( \Delta u = \Delta \sigma_1 / 2 \) for an isotropic elastic soil.

E2.4. Conventional drained and undrained triaxial compression tests are performed on samples of cross-anisotropic soil taken from the ground with their axes horizontal. Show that the cross section of the samples does not remain circular as they are compressed. Determine the slope of the effective stress path in the \( p' - q \) plane for the undrained test in terms of appropriate elastic constants.
2 Elasticity

Assuming that measurements are made only of axial and volumetric strain in the usual way, determine the slope of the strain path in the $\varepsilon_p:\varepsilon_q$ plane that would be deduced for the drained test.

Check that these expressions reduce to the correct values for isotropic elastic soil.

E2.5. Consider the strains that occur on the stress cycle $(p', q) = (p'_a, 0)$; $(p'_b, 0)$; $(p'_b, q_b)$; $(p'_a, q_b)$; and $(p'_a, 0)$ applied to a sample of elastic soil which has a bulk modulus dependent on mean stress ($K' = \alpha p'$). Show that if the shear modulus $G'$ is also dependent on $p'$, then energy can be created or lost on this closed cycle, and hence that it would not be thermodynamically admissible to assume a constant value of Poisson's ratio for this soil.