10

Stress paths and soil tests

10.1 Introduction

A complete description of the stress: strain behaviour of an elastic material can be embodied in a rather small number of parameters. For example, an isotropic linear elastic material (Section 2.1) requires only two independent parameters: Young's modulus and Poisson's ratio, or shear modulus and bulk modulus. A cross-anisotropic or transversely isotropic linear elastic material (Section 2.3) requires five parameters: which might be two Young's moduli, two Poisson's ratios, and a shear modulus. Once these elastic parameters have been determined, the response of the material to any changes in stress can be predicted. For the simple case of an isotropic elastic soil, the elastic parameters could be obtained from a uniaxial compression test, but the use of these parameters would in no way be confined to the prediction of the response in such tests. The testing programme required to determine the five elastic parameters for a cross-anisotropic elastic soil is rather more complex: simple compression tests in a triaxial apparatus are no longer sufficient, and tests which include rotation of principal axes are required (Graham and Houlsby, 1983). Nevertheless, once the values of the five parameters have been established, there is no limit to the range of stress paths to which the model can legitimately be applied.

A distinction has to be drawn between the description of the behaviour of inelastic soils and the description of the model which may be used to represent certain aspects of their behaviour. For it is certain that although simple elastic–plastic models of soil behaviour such as Cam clay may be very good at describing certain limited areas of soil response – those which are reasonably close to the data base of triaxial compression tests on which such models have typically been founded – their general success in
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matching the response of such a non-ideal material as soil is likely to be more doubtful.

Most of the data of soil behaviour that have been presented in previous chapters have been obtained from triaxial tests, primarily triaxial compression tests. In this chapter the stress paths that are actually followed by soil elements in certain typical geotechnical structures are considered. When such paths can be followed in a triaxial apparatus, then prediction of deformations, made with soil models based on triaxial test data, may be satisfactory. The further such paths deviate from the possibilities of the triaxial apparatus, the less satisfactory the predictions will be. The need to consider stress paths arises because real soils cannot be treated as isotropic or elastic materials: they must be treated as inelastic, anisotropic, and history-dependent materials whose behaviour cannot be described by a small number of elastic parameters. The general framework of elastic–plastic models of soil behaviour presented in Chapter 4 treats soils as just such non-ideal materials.

There are, of course, apparatus other than the triaxial apparatus and other tests which can be used to investigate soil response; some of these may match more closely the stress paths that are important in the ground. Each apparatus tests soil in its own way, and the relationships between the strengths of soils as measured by various tests are considered in Section 10.6.

The term stress path is familiar to many practising geotechnical engineers: It is linked with the stress path method of Lambe (1967). The stress path approach to predicting settlements of foundations requires triaxial tests on representative samples of soil which follow the stress changes estimated, usually from elastic analyses, to be relevant for selected elements of soil beneath the foundation. From the response observed in these tests, pseudo-elastic parameters can be deduced which are then applied to give numerical estimates of settlements.

This stress path method can in principle be extended to a wider range of geotechnical problems (Wood, 1984a). The necessary steps are

i. to identify critical soil elements around a structure,

ii. to estimate the stress paths followed by these elements as the structure is constructed or loaded,

iii. to perform laboratory tests on soil samples along these paths, and

iv. to estimate the deformations of the geotechnical structure from the results of these tests.

The problems of attempting to use this method are, however, that the actual stress paths followed are not independent of the very soil behaviour that is being investigated, that it may not be possible to follow the relevant
stress paths in laboratory tests, and that the method of working from the results of the tests to the estimated performance of the structure may not be obvious. Nevertheless, consideration of stress paths does impose a discipline on engineers which should help them to identify the shortcomings of the tests that they are able to perform, and to assess the relevance of the numerical models of soil behaviour which they may wish to use to make their predictions.

10.2 Display of stress paths

Most of the data on soil behaviour that have been discussed here were obtained from triaxial tests in which conditions of axial symmetry were necessarily imposed. Axially symmetric stress paths have just two degrees of freedom: an axial principal stress \( \sigma_a \) and a radial principal stress \( \sigma_r \) (Fig. 10.1), and it has been convenient to display triaxial stress paths in the \( p'q \) plane to emphasise the distinction between volumetric and distortional components of stress.

Conditions of axial symmetry occur infrequently in prototype situations; conditions of plane strain are more likely. As discussed in Section 1.5, plane strain effective stress paths can be displayed in terms of stress quantities \( s' \) and \( t \),

\[
s' = \frac{\sigma_1' + \sigma_3'}{2} \quad (10.1) \text{(cf. 1.58)}
\]

\[
t = \frac{\sigma_1' - \sigma_3'}{2} \quad (10.2) \text{(cf. 1.60)}
\]

which are calculated in terms of the principal effective stresses \( \sigma_1' \) and \( \sigma_3' \) in the plane of shearing. The stress quantities \( s' \) and \( t \) are particularly useful for describing stress changes which are not associated with significant rotation of principal axes, and most of the stress paths presented in Section 10.4 fall into this category.

The stress quantities \( s' \) and \( t \) cannot convey information about the orientation of principal stresses \( \sigma_1' \) and \( \sigma_3' \) or about the magnitude of the

Fig. 10.1 Stresses on cylindrical triaxial sample.
10.2 Display of stress paths

Fig. 10.2  General stress state specified by (a) normal and shear stresses referred to fixed axes and (b) principal stresses and principal axes.

Fig. 10.3  (a) General mean stress $p'$ and deviator stress $q$ in principal effective stress space; (b) cube with sides defined by principal stress axes $\sigma_1': \sigma_2': \sigma_3'$; (c) assignment of major, intermediate, and minor principal stresses in sectors of deviatoric view of principal stress space.
third principal stress \( \sigma'_2 \), orthogonal to the plane of shearing. Plane strain is a special case though it represents a situation which is met in practice rather more frequently than axial symmetry. A completely general stress state requires six quantities to describe it: these may be three normal stresses and three associated shear stresses (Fig. 10.2a) or three principal stresses and the directions of their three mutually orthogonal principal axes (Fig. 10.2b). In general, at a soil element in the ground, all six stress quantities are changing simultaneously. Six quantities are needed to describe the resulting strain increment, and there is no reason to suppose that principal axes of strain increment and principal axes of stress or of stress increment necessarily coincide.

Possibilities for controlled laboratory investigation of such general stress paths are, not surprisingly, limited. It is also unclear how such paths should best be displayed. This last problem is largely avoided here by restricting attention to paths in which no rotation of principal axes occurs.

The set of principal stresses \((\sigma'_1, \sigma'_2, \sigma'_3)\), where the subscripts \((1, 2, 3)\) do not signify any particular sequence of relative magnitudes, can be used as orthogonal cartesian axes to define a three-dimensional space (Fig. 10.3a) which can be viewed through two-dimensional projections. The most frequently used projection is the deviatoric or \(n\)-plane view (Fig. 10.3c), a view of principal stress space down the line \(\sigma'_1 = \sigma'_2 = \sigma'_3\) (Fig. 10.3b). This view only shows differences of principal stresses, so it is important to remember that this is a view of projected information and that the paths shown do not in general lie in the planes depicted.

10.3 Axially symmetric stress paths
10.3.1 One-dimensional compression of soil

Some soils have been deposited rather uniformly over an area of large lateral extent, for example, in marine or lacustrine conditions. For such soils, symmetry dictates that soil particles can only have moved downwards during the process of deposition (Fig. 1.21); lateral movements would violate the symmetry. The deformation of such soils during deposition is entirely one-dimensional, and the effective stress state can be reproduced in a conventional triaxial apparatus.

The ratio of horizontal to vertical effective stresses in soils which have a history of one-dimensional deformation (which may include some overconsolidation) is called the coefficient of earth pressure at rest \(K_o\):

\[
\frac{\sigma'_h}{\sigma'_v} = K_o
\]  \hspace{1cm} (10.3)

During monotonic one-dimensional normal compression, each state of
10.3 Axially symmetric stress paths

dehdration of a soil is essentially similar to all the preceding states, and the effective stress states have the same similarity. The value of $K_0$ is then found to be a constant $K_0 = K_{one}$ (Fig. 10.4a).

From the definitions of $p'$ and $q$, the slope $\eta_{Kne}$ of the one-dimensional normal compression line in the $p':q$ plane (Fig. 10.4b) is related to $K_{one}$:

$$\eta_{Kne} = \frac{3(1 - K_{one})}{1 + 2K_{one}}$$  \hspace{1cm} (10.4a)

and

$$K_{one} = \frac{3 - \eta_{Kne}}{3 + 2\eta_{Kne}}$$  \hspace{1cm} (10.4b)

The Cam clay model can be used to calculate values of $\eta_{Kne}$. The condition for one-dimensional compression is that

$$\frac{\delta e_p}{\delta e_q} = \frac{\delta e_p + \delta e_p'}{\delta e_q + \delta e_q'} = \frac{3}{2}$$  \hspace{1cm} (10.5)

The ratio of all stress components remains constant during such one-dimensional normal compression, and hence

$$\frac{\delta q}{\delta p'} = \frac{q}{p'} = \eta_{Kne}$$  \hspace{1cm} (10.6)

and

$$\frac{\delta q}{q} = \frac{\delta p'}{p'}$$  \hspace{1cm} (10.7)

The elastic and plastic components of strain during one-dimensional

Fig. 10.4 One-dimensional loading in (a) $\sigma'_b:\sigma'_h$ and (b) $p':q$ effective stress planes

(a)  \hspace{1cm} (b)
normal compression are, from the expressions in Section 5.2, with (10.7),

\[
\delta \varepsilon_p = \frac{\kappa \delta p'}{v p'} \tag{10.8}
\]

\[
\delta \varepsilon_q = \frac{2(1 + \nu')}{9(1 - 2\nu')} \frac{\eta_{Knc} \kappa \delta p'}{v p'} \tag{10.9}
\]

\[
\delta \varepsilon_p = \frac{\lambda - \kappa \delta p'}{v p'} \tag{10.10}
\]

\[
\delta \varepsilon_q = \frac{2 \eta_{Knc}}{M^2 - \eta_{Knc}^2} \frac{\lambda - \kappa \delta p'}{v p'} \tag{10.11}
\]

Two elastic parameters were used in the Cam clay model in Section 5.2: the slope \( \kappa \) of the unloading-reloading line and the shear modulus \( G' \). It is more convenient here to use Poisson's ratio \( \nu' \) because this immediately expresses a ratio of components of elastic strains. The links between the various elastic parameters were given in Section 2.1.

Expression (10.5) is a condition on total strain increments into which (10.8)–(10.11) can be substituted to give the expression

\[
\frac{\eta_{Knc}(1 + \nu')(1 - \Lambda)}{3(1 - 2\nu')} + \frac{3 \eta_{Knc} \Lambda}{M^2 - \eta_{Knc}^2} = 1 \tag{10.12}
\]

from which values of \( \eta_{Knc} \) can be determined numerically for any set of values of \( \nu' \), \( \Lambda = (\lambda - \kappa)/\lambda \), and \( M \). The first term on the left-hand side broadly expresses the elastic contribution, which is the entire contribution for \( \Lambda = 0 \). The second term expresses the plastic contribution.

Values of \( \eta_{Knc} \) can be converted to values of \( K_{one} \) using (10.4b), and the effect of different combinations of \( \nu' \) and \( \Lambda \) on the variation of \( K_{one} \) with angle of shearing resistance \( \phi' \) [which can be linked to \( M \) through (7.9)] is shown in Fig. 10.5a.

A soil that is composed of rigid, greatly interlocked particles can support its own weight without needing to push sideways very much to prevent lateral movement: such a soil would be expected to have a low value of \( K_{one} \). Such a soil would also be able to mobilise a high angle of shearing resistance before shear deformations occurred. A material which has no frictional strength, such as water, produces a lateral push equal to the overburden pressure: in other words, \( K_{one} = 1 \). Some link between \( K_{one} \)

Fig. 10.5 Dependence of coefficient of earth pressure at rest for normally compressed soil, \( K_{one} \), on angle of shearing resistance \( \phi' \) (a) according to Cam clay model, (b) according to Jaky (1944), and (c) experimental data (•, clays; +, sands) (after Wroth, 1972; Ladd, Foott, Ishihara, Schlosser, and Poulos, 1977).
and angle of shearing resistance \( \phi' \) might be anticipated. Jáky (1944) managed to arrive at the expression

\[
K_{\text{one}} = \left(1 + \frac{2}{3} \sin \phi'\right) \left(\frac{1 - \sin \phi'}{1 + \sin \phi'}\right)
\]  

(10.13)

by thinking of conditions at the centre of the base of a heap of granular material. Jáky noted that this elaborate expression could be approximated satisfactorily by the expression

\[
K_{\text{one}} \approx 0.9(1 - \sin \phi')
\]  

(10.14)

for values of \( \phi' \) between about 20° and 45°, but also suggested that the expression

\[
K_{\text{one}} = 1 - \sin \phi'
\]  

(10.15)

would be close enough for engineering purposes. This final expression has become enshrined in the soil mechanics canon with Jáky's name attached to it. The three expressions (10.13), (10.14), and (10.15) are plotted in Fig. 10.5b, and expression (10.15) is also plotted in Fig. 10.5a.

The experimental conditions necessary for the accurate measurement of \( K_0 \) are not easy to realise in practice (Bishop, 1958); even very small amounts of lateral movement can produce a significant change in the apparent value of \( K_0 \). Data of values of \( K_{\text{one}} \) and \( \phi' \) for normally compressed clays and sands collected by Wroth (1972) and by Ladd et al. (1977) are shown in Fig. 10.5c, and (10.15) provides a reasonable fit to these points. Cam clay tends to predict values of \( K_{\text{one}} \) which lie above the empirical expression (Fig. 10.5a) unless simultaneous low values of both \( \nu \) and \( \Lambda \) are assumed (implying dominance of the deformation by low Poisson's ratio elastic response).

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Fig. 10.6 Dependence of Poisson's ratio \( \nu' \) (●) and exponent \( a \) in (10.23) (x) on plasticity index \( I_p \) (data of \( \nu \) from Wroth, 1975).
10.3 Axially symmetric stress paths

The nature of the angle $\phi'$ that should be used in (10.15) is somewhat uncertain. It is sometimes found that the angles of shearing resistance that can be mobilised in conditions of plane strain are rather higher than those that can be mobilised in triaxial compression. (Wroth, 1984, suggests that plane strain angles are typically 10 per cent higher.) Perhaps on grounds of the axial symmetry of one-dimensional deposition, a logical argument could be made in favour of triaxial compression. However, theoretical justifications of (10.15) cannot proceed very far, and it should rather be regarded as a convenient experimental finding. Angles of shearing resistance are most commonly measured in triaxial compression, and (10.15) with Fig. 10.5c can be regarded as establishing a simple correlation between these angles and $K_{one}$.

Many of the data in Fig. 10.5c relate to values of $K_{one}$ and $\phi'$ for sands tested at a range of initial densities. It was noted in earlier chapters that it is difficult to change the structure of sand in proportional loading tests (in which all components of stress increase by the same proportion), of which one-dimensional compression is one example. When a sand is sheared, for example in triaxial compression, its void ratio changes until eventually the sand reaches a critical state condition, mobilising the critical state angle of shearing resistance. However, the value of the earth pressure coefficient at rest, $K_{one}$, depends on the in situ structure of the sand and can be expected to correlate with the peak angle of shearing resistance measured in triaxial compression, associated with the initial density and structure of the sand. It would not be appropriate to use the critical state angle in (10.15) to estimate the value of $K_{one}$ for a dense sand.

In the elastic-plastic models of soil behaviour discussed in Chapters 4 and 5, volume changes occurring on a normal compression line were described by an expression

$$v = v'_1 - \lambda \ln p'$$  \hspace{1cm} (10.16)(4.2bis)

where $v'_1$ is a reference value of specific volume $v$. During one-dimensional normal compression, mean effective stress $p'$ and vertical effective stress $\sigma'_v$ increase in a constant ratio:

$$\frac{p'}{\sigma'_v} = \frac{1 + 2K_{one}}{3}$$  \hspace{1cm} (10.17)

Consequently, changes in volume during one-dimensional normal compression can be described by

$$v = v'_1 - \lambda \ln \sigma'_v$$  \hspace{1cm} (10.18)

(where $v'_1$ is a different reference value of $v$). The compressibility $\lambda$ that is determined is the same whether these normal compression data are plotted in terms of $\sigma'_v$ or of $p'$. 

10 Stress paths and soil tests

10.3.2 One-dimensional unloading of soil

One-dimensional unloading of soil produces a more rapid drop of vertical effective stress than of horizontal effective stress. If it is supposed that soil behaves isotropically and elastically immediately on unloading, then the slope of this overconsolidation unloading stress path in the \( p' \):\( q \) plane can be deduced from (10.12), setting \( \Lambda = 0 \) (which implies that there are no plastic deformations):

\[
\frac{\delta q}{\delta p'} = \frac{3(1 - 2\nu')}{1 + \nu'}
\]

which implies

\[
\frac{\delta \sigma_h'}{\delta \sigma_v'} = \frac{\nu'}{1 - \nu'}
\]

(10.20)

Wroth (1975) has deduced values of \( \nu' \) from data of one-dimensional unloading of a number of soils; these are plotted against the plasticity of the soils in Fig. 10.6. The value of apparent Poisson's ratio is typically around 0.3 but rises slightly with increasing soil plasticity. A value \( \nu' = 0.3 \) in (10.20) implies a stress decrement ratio \( \delta \sigma_h'/\delta \sigma_v' = 0.43 \), whereas for most of the soils (particularly the plastic soils) shown in Fig. 10.5c, \( K_{onc} \) is above 0.5: the stress path followed on one-dimensional unloading does not retrace the stress path followed on one-dimensional normal compression. Typical stress paths are shown in Figs. 10.7a, b in \( \sigma_h' \):\( \sigma_v' \) and \( p' \):\( q \) effective stress planes: \( OA \) for one-dimensional normal compression, \( AB \) for initial one-dimensional unloading.

Expression (10.20) can be converted into a relationship between \( K_o \) and

![Fig. 10.7 Path followed on one-dimensional unloading and reloading in (a) \( \sigma_h', \sigma_v' \) and (b) \( p', q \) effective stress planes.](image-url)
10.3 Axially symmetric stress paths

overconsolidation ratio \( n \):

\[
K_0 = nK_{0_{nc}} - (n - 1) \frac{\nu'}{1 - \nu'}
\]  
(10.21)

where

\[
\nu = \frac{\sigma'_{\text{vmax}}}{\sigma'_{\nu}}
\]  
(10.22)

and \( \sigma'_{\text{vmax}} \) is the maximum value of \( \sigma'_{\nu} \) reached in one-dimensional normal compression.

The simple assumption of elastic unloading leading to the straight unloading stress path \( AB \) in Fig. 10.7 cannot be sustained beyond overconsolidation ratios of about 2; experimental measurements during one-dimensional unloading show significant curvature of the effective stress path \( BC \) (Fig. 10.7). A simple expression has been found for this unloading stress path. Schmidt (1966) plotted \( K_0 \) against overconsolidation ratio \( n \) using logarithmic axes for both. He found linear relationships such as that shown in Fig. 10.8, implying a relationship between \( K_0 \) and \( n \) of the form

\[
K_0 = K_{0_{nc}} n^a
\]  
(10.23)

The parameter \( a \) shows a small decrease with increasing plasticity (Fig. 10.6), but a value \( a = 0.5 \), as suggested by Meyerhof (1976), would be a reasonable round number to use for most soils, implying

\[
\left( \frac{K_0}{K_{0_{nc}}} \right)^2 = n
\]  
(10.24)

An example of the application of (10.15), (10.21), and (10.24) is shown

Fig. 10.8 Relationship between coefficient of earth pressure at rest \( K_0 \) and overconsolidation ratio \( n \) for Weald clay (after Schmidt, 1966).
in Fig. 10.9. Ladd (1965) gives data of $K_0$ and overconsolidation ratio $n$ for undisturbed Boston blue clay. This clay has plasticity $I_p = 0.15$ and an angle of shearing resistance (at maximum stress ratio) $\phi' = 32.75^\circ$, which is the same value that would be predicted from expression (9.23). From (10.15), then, $K_{onc} = 0.46$. For $I_p = 0.15$ in Fig. 10.6, $\nu' = 0.29$. With these values, the values calculated from expressions (10.21) and (10.24) are shown in Fig. 10.9 together with Ladd’s data. Equation (10.24) provides a good fit, though of course with $\nu'$ set to an alternative value of 0.22 (deduced from the experimental observations for the Boston blue clay itself) a good initial fit could be obtained with expression (10.21), as also shown in Fig. 10.9.

An overconsolidation ratio $n_p$ can be defined in terms of mean effective
10.3 Axially symmetric stress paths

stress $p'$:

$$ n_p = \frac{p'_\text{max}}{p'} \quad (10.25) $$

where $p'_\text{max}$ is the maximum value of $p'$ reached in normal compression. Then, by comparison with (10.17) and (10.22),

$$ \frac{n_p}{n} = \frac{1 + 2K_{\text{one}}}{1 + 2K_0} \quad (10.26) $$

Since $K_0$ is a function of $n$, this ratio $n_p/n$ is not a constant, even for a particular soil. However, if the link between $K_0$ and $n$ is provided by (10.24), then the variation of $n_p/n$ with $n$ can be deduced (Fig. 10.10). It is found to be rather insensitive to the value of $K_{\text{one}}$; the range for $n = 32$ in Fig. 10.10 corresponds to the range of values of $K_{\text{one}}$ from 0.4 to 0.7 [$\phi' = 36.9^\circ$ to $17.5^\circ$ from (10.15)].

However, if it is assumed (as in the elastic–plastic models of Chapters 4 and 5) that recoverable volume changes occurring for stress paths lying within a current yield locus can be described by

$$ v = v_c - \kappa \ln p' \quad (10.27)(4.3\text{bis}) $$

then a simultaneous linear relationship between $v$ and $\ln \sigma'_v$ with the same slope $\kappa$ is not possible.

Expression (10.27) can be written as

$$ v = v_c + \kappa \ln n_p \quad (10.28) $$

or

$$ v = v_c + \kappa \ln n + \kappa \ln \left( \frac{n_p}{n} \right) \quad (10.29) $$

or

$$ v = v_d - \kappa \ln \sigma'_v + \kappa \ln \left( \frac{n_p}{n} \right) \quad (10.30) $$

where

$$ v_c = v_c - \kappa \ln p'_\text{max} \quad (10.31) $$

and

$$ v_d = v_c + \kappa \ln \sigma'_v \text{max} \quad (10.32) $$

The effect of the changing ratio $n_p/n$ on the volume changes occurring during one-dimensional unloading can best be illustrated by plotting (10.29) as $(v - v_c)/\kappa$ against $n$ (Fig. 10.11). In (10.29), $(v - v_c)$ is the change in volume that has occurred since the vertical stress was reduced from its
maximum normally compressed value $\sigma'_{\text{max}}$. The line

$$\frac{v - v_e}{\kappa} = \ln n$$

(10.33)

is shown in Fig. 10.11 for comparison. The curvature is not great, but because $n$ increases more rapidly than $n_p$, as $\sigma'_s$ falls more rapidly than $p'$ (which depends also on $\sigma'_{\text{pc}}$), the value of $\kappa$ that would be deduced from one-dimensional unloading in an oedometer (in which only the vertical stress is usually measured) is only about 0.7 of the value of $\kappa$ properly calculated in terms of change in mean effective stress $p'$. This must be taken into account if the only data of unloading that are available for a soil have been obtained from oedometer tests.

In Section 7.2, the form of experimentally observed critical state lines was used to predict the variation with overconsolidation of the ratio of undrained strength to in situ vertical effective stress. The expression that was deduced,

$$\frac{(c_u/\sigma'_{\text{vi}})}{(c_u/\sigma'_{\text{vi}})_{nc}} = \left(\frac{n}{n_p}\right)^{n_p^\Lambda}$$

(10.34)(7.31bis)

involved both $n$ and $n_p$. The ratio $(n_p/n)^{1-\Lambda}$ is plotted in Fig. 10.10 for $\Lambda = 0.8$; it falls gently from 1 to $\sim 0.8$ as $n$ rises from 1 to 32. If it is assumed that

$$\left(\frac{n_p}{n}\right)^{1-\Lambda} \sim 1$$

(10.35)

Fig. 10.11  Predicted volume change on one-dimensional unloading of soil.
then, from (10.34),
\[
\frac{\left(c_u/\sigma'_{in}\right)}{(c_u/\sigma'_{in})_{nc}} \sim n^a
\]  
(10.36)

and, as noted in Section 7.2, this is the form of relationship that has been successfully fitted to published data.

The earth pressure coefficient at rest \( K_0 \) increases as a soil becomes progressively more overconsolidated. The horizontal effective stress falls more slowly than the vertical effective stress; and at some overconsolidation ratio, \( K_0 \) becomes greater than 1, and the horizontal stress becomes the major principal stress. The overconsolidation ratio \( n \) (or \( n_p \)) at which this occurs depends on the value of \( K_{one} \) according to (10.24); these values are plotted in Fig. 10.12.

The Mohr–Coulomb failure criterion discussed in Section 7.1 sets an upper limit to the ratio of major and minor principal effective stresses, \( \sigma'_1 \) and \( \sigma'_3 \), respectively. For a soil with angle of shearing resistance \( \phi' \), the maximum ratio occurs when the Mohr circle touches the failure line (Fig. 7.1a) and
\[
\frac{\sigma'_1}{\sigma'_3} = \frac{1 + \sin \phi'}{1 - \sin \phi'}
\]  
(10.37)

In the one-dimensionally unloaded ground, this ratio is expected to set a limit to the value of \( K_0 \) that can be attained. When this limit is reached, then the soil is in a state of incipient passive failure, a mode of failure

Fig. 10.12 Values of overconsolidation ratio predicted to give earth pressure coefficient at rest \( K_0 \) equal to 1 and equal to passive pressure coefficient \( K_p \) as function of angle of shearing resistance \( \phi' \).
being driven by the high horizontal stresses against the weight of the soil. The values of \( n \) and \( n_p \) at which (10.37) and (10.24) coincide are also plotted in Fig. 10.12. For a soil with a high value of \( K_{\text{one}} = 0.7 \) [corresponding to \( \phi' = 17.5^\circ \) from (10.15)] and probable high plasticity, the required overconsolidation ratio \( n \) is only about 7. For a soil with a low value of \( K_{\text{one}} = 0.4 \) (\( \phi' = 36.9^\circ \)) and probable low or zero plasticity, the required overconsolidation ratio is about 100.

Of course, the continuing one-dimensional symmetry of the loading condition in principle prevents a passive failure from taking place. However, in practice, if a heavily overconsolidated soil deposit is sitting in such a state of incipient passive failure, then any attempt to upset the symmetry releases the failure and large deformations may occur. For example, a deep excavation (Fig. 10.13a) may remove the vertical effective stress locally and lead to ground movements into the base of the hole. Nature, too, can upset the symmetry by eroding a river and valley. This again removes the vertical effective stress locally and allows the underlying strata to bulge in the bottom of the valley; a typical example in which such valley bottom bulge can be detected in the deformed strata is shown in Fig. 10.13b. Once such a passive failure has been released, there is a

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**Fig. 10.13** (a) Heave of base of excavation in heavily overconsolidated clay; (b) valley bottom bulge at site of Empingham Dam, Rutland (after Horswill and Horton, 1976).
danger that the remaining soils may contain failure planes on which movements sufficient to reduce the strength of the soil to a residual value (Section 7.7) have occurred. Such a possibility must be taken into account in assessing the stability of any proposed structure, for example, an embankment dam at the site shown in Fig. 10.13b, which is to be founded on the soil.

The effective stress paths followed on one-dimensional unloading might be expected to become tangential to the passive failure line at high overconsolidation ratios (Fig. 10.7), though the curve plotted in Fig. 10.12 shows that these overconsolidation ratios are not unreasonably high for plastic soils. When soils are again re-loaded one-dimensionally, the value of $K_0$ again decreases, and the effective stress path tends towards the original stress path for the one-dimensionally normally compressed soil ($CD$ in Fig. 10.7). Evidently, the value of $K_0$ for a soil deposit which is known to have been cyclically unloaded and reloaded can only be estimated within the bounds set by the normal compression line $OA$ and the primary unloading curve $ABC$, starting from the preconsolidation state at $A$.

10.3.3 Fluctuation of water table

Parry (1970) notes that most natural deposits of soft clays are lightly overconsolidated, which is to say, with the background of Chapter 3, that the current effective stress state lies inside the current yield locus. Such overconsolidation may result from actual changes in effective

Fig. 10.14 Overconsolidation of soil resulting from fluctuation of water table: (a) variations of stress with depth with water table at ground surface; (b) variations of stress with depth with water table at depth $d$; (c) profile of overconsolidation ratio with depth.
stresses, or it may appear as a result of other effects such as cementation between soil particles or time effects (which will be discussed in Section 12.2). Changes in effective stresses obviously occur where overburden has been eroded by ice or by water, but in soft soils it is very likely that effective stress changes have occurred as a result of fluctuations in water table, even without any erosion.

The effect of water table fluctuation is illustrated in Fig. 10.14. Suppose that the water table fluctuates between the ground surface (Fig. 10.14a) and a depth \( d \) (Fig. 10.14b). Suppose also that the soil remains saturated above the water table. Then with the water table at depth \( d \), the effective vertical stress at depth \( z \) is

\[
\sigma'_{v1} = \gamma z - \gamma_w (z - d) = \gamma' z + \gamma_w d
\]  
(10.38)

With the water table at the ground surface, the vertical effective stress at depth \( z \) is

\[
\sigma'_{v2} = \gamma z - \gamma w z = \gamma' z
\]  
(10.39)

and \( \sigma'_{v2} < \sigma'_{v1} \) because more of the soil is buoyant in the groundwater. The overconsolidation ratio of the soil, supposing that the water table is currently at the ground surface, is

\[
n = \frac{\sigma'_{v1}}{\sigma'_{v2}} = 1 + \frac{\gamma_w d}{\gamma' z}
\]  
(10.40)

and the soil becomes more nearly normally compressed \( (n \to 1) \) at depths much greater than \( d \) (Fig. 10.14c).

In discussing typical stress paths for geotechnical structures in subsequent sections, we assume that the soil starts from an initially lightly overconsolidated state – as a result, for example, of some past variation in the level of the water table.

10.3.4 Elements on centrelines beneath circular load

Elements of soil within a soil deposit of large lateral extent and on the centreline beneath a circular load will be subjected to axially symmetric changes of stresses. This is the only engineering situation for which the stress path can be followed precisely in the conventional triaxial apparatus. Such situations are unusual but do occur; one can think of oil storage tanks as an example (Fig. 10.15a). Oil storage tanks are usually founded on soft soils near river estuaries or coastal sites, so an initial lightly overconsolidated effective stress state due to fluctuation of water table, as discussed in Section 10.3.3, is clearly appropriate.

The stress path followed by an element such as \( X \) beneath the oil tank in Fig. 10.15a is shown qualitatively in Figs. 10.15b, c. It has been assumed
that the soil beneath the tank is initially one-dimensionally lightly overconsolidated with past maximum effective stress state $A'$, present effective stress state $B'$ and total stress state $B$. The separation $BB'$ in the $p':q$ plane (Fig. 10.15b) is equal to the in situ pore pressure. The paths followed beneath a circular load will be considered in more detail in Section 11.2; here it suffices to note that the total stress changes $BCD$ resulting from loading the tank are likely to involve an increase in total mean stress $p$ and an increase in deviator stress $q$. If the tank is filled rapidly, then the response of the soil is likely to be undrained, and the effective stress path will show no change in mean effective stress $p'$ until the current yield locus is reached ($B'C'$). The effective stress path for a lightly overconsolidated clay then turns to the left $C'D'$ as the soil yields and extra pore pressures are generated. If the tank is left and the excess pore pressures allowed to dissipate back to the in situ values existing at $B$ and $B'$, then the effective stress path will show an increase in mean effective stress $D'E'$, with perhaps little change in total stresses, so that dissipation can occur at approximately constant deviator stress $q$.

Fig. 10.15 (a) Element on centreline beneath circular tank on soft clay; (b) stress paths in $p':q$ and $p:q$ planes; (c) stress paths in deviatoric plane.
10 Stress paths and soil tests

The stress changes are shown again in Fig. 10.15c in the deviatoric view of principal stress space with principal stresses $\sigma'_1$, $\sigma'_2$, and $\sigma'_3$ assigned to the z, x, and y directions, respectively. For these axially symmetric changes in stress, the stresses in the horizontal x and y directions (Fig. 10.15a) are always equal, $\sigma'_2 = \sigma'_3$. Hence the stress paths followed during one-dimensional compression and unloading ($OAB$), undrained loading beneath the tank ($BCD$), and subsequent dissipation of excess pore pressures, ($DE$), lie entirely on the vertical diameter of this view of stress space (on the $\sigma_1$ axis or its extension as seen in this view). The presence of pore pressure merely displaces the effective stress state from the total stress state in a direction parallel to the line $\sigma'_1 = \sigma'_2 = \sigma'_3$, and there is no distinction between total and effective stress paths in Fig. 10.15c.

10.4 Plane strain stress paths

10.4.1 One-dimensional compression and unloading

One-dimensional deformation is both an axially symmetric and a plane strain process, and the path followed during one-dimensional compression and unloading can be displayed in the $s':t$ effective stress diagram (Fig. 10.16). The principal effective stresses $\sigma'_1$ and $\sigma'_3$ are the vertical and horizontal effective stresses $\sigma'$ and $\sigma'_h$, so combining (10.3) with (10.1) and (10.2) yields

$$\frac{t}{s'} = \frac{1 - K_0}{1 + K_0} \tag{10.41}$$

During one-dimensional normal compression,

$$K_0 = K_{one} = 1 - \sin \phi' \tag{10.15bis}$$

and

$$\frac{t}{s'} = \frac{1 - K_{one}}{1 + K_{one}} = \frac{\sin \phi'}{2 - \sin \phi'} \tag{10.42}$$

Fig. 10.16 One-dimensional loading and unloading in $s':t$ effective stress plane.
This is line $OA$ in Fig. 10.16. A typical effective stress path $ABC$ followed during one-dimensional unloading is also shown. The assignment of $\sigma'_1$ and $\sigma'_3$ to vertical and horizontal effective stresses does not imply any relative magnitude, so negative values of $t$ are possible when the horizontal stress exceeds the vertical stress, $\sigma'_2 > \sigma'_1, K_0 > 1$.

The Mohr–Coulomb failure criterion (10.37) can also be converted to a relationship between $s'$ and $t$. From (10.1) and (10.2),

$$ \frac{t}{s'} = \pm \sin \phi' $$

(10.43)

where the positive sign refers to failure with the vertical stress greater than the horizontal stress: that is, *active* failure, with the weight of the soil assisting deformation. The negative sign refers to failure with the horizontal stress greater than the vertical stress: that is, *passive* failure, with the weight of the soil resisting deformation. These two failure lines are also plotted in Fig. 10.16; their slope $\beta$ is given by

$$ \tan \beta = \pm \sin \phi' $$

(10.44)

The definitions of $s'$ and $t$, like the Mohr–Coulomb failure criterion, involve only the major and minor principal effective stresses and leave out of consideration the value of the intermediate principal effective stress $\sigma'_2$, which acts normal to the plane of shearing. Consequently, the Mohr–Coulomb failure criterion has a symmetry in the $s':t$ plane (Fig. 10.16) which is absent in the $p':q$ plane (see Section 7.1).

### 10.4.2 Elements beneath long embankment

The total and effective stress paths in the $s:t$ and $s':t$ stress planes for an element of soil $X$ on the centreline beneath a long embankment (Fig. 10.17a) are qualitatively similar to the total and effective stress paths in the $p:q$ and $p':q$ stress planes for an element of soil on the centreline beneath a circular load (Section 10.3.4 and Fig. 10.15). Probable paths are shown in Fig. 10.17b, and the effect of dissipation of construction pore pressures again tends to move the effective stress state away from the active failure line ($D'E'$ in Fig. 10.17b).

The stress paths for this plane strain problem have been presented only in the $s':t$ and $s:t$ effective and total stress planes. It is instructive to consider, at least qualitatively, what the stress paths might look like in the deviatoric view of principal stress space that was introduced in Section 10.2 and used in Section 10.3.4 and Fig. 10.15c. The initial stress state $B$ results from a one-dimensional history, so $\sigma'_2 = \sigma'_3$ and point $B$ lies on the projection of the $\sigma_1$ axis (Fig. 10.18, where again principal stresses $\sigma'_1, \sigma'_2,$ and $\sigma'_3$ are assigned to the $z, x$, and $y$ directions,
respectively). The precise details of path $BD$, as the difference between the vertical and horizontal stresses increases, depend on the details of the assumed soil model; the path shown in Fig. 10.18 is a qualitative estimate. As soon as plane strain deformation begins, the stresses $\sigma'_2$ and $\sigma'_3$ are no longer equal, and the stress path leaves the projection of the $\sigma'_1$ axis. This is just a reflection of the fact that plane strain stress paths cannot be followed in a conventional triaxial apparatus, in which two principal stresses are always equal, $\sigma'_2 = \sigma'_3$.

On the centreline, the directions of the principal stresses remain horizontal and vertical as the embankment is constructed. Away from the centreline (e.g. $Y$ in Fig. 10.17a), there is no longer the same symmetry,

Fig. 10.17 (a) Elements beneath centre and edge of long embankment; (b) stress paths in $s':t$ and $s:t$ planes.

Fig. 10.18 Stress paths for elements deforming actively or passively in plane strain seen in deviatoric view of principal stress space.
and vertical and horizontal planes experience a development of shear stresses as the embankment is built. The principal axes rotate, and the stress state can only be incompletely depicted in a stress plane whose axes are defined as functions of principal stresses, such as the $s':t$ plane of Fig. 10.17b or the deviatoric view of principal stress space of Fig. 10.18. There are still no shear stresses on planes normal to the length of the embankment, so there are still two principal effective stresses $\sigma'_1$ and $\sigma'_3$ in the $xz$ plane. An $s':t$ or $s:t$ stress path could therefore still be generated for the $xz$ plane, but the rotation of principal axes cannot be shown in such a diagram.

10.4.3 Elements adjacent to long excavation

A section through a long trench excavation is shown in Fig. 10.19a with elements of soil $X$ and $Y$ adjacent to the wall of the excavation and beneath the base of the excavation, respectively. The total and effective stress paths for element $X$ (Fig. 10.19b) are essentially the same as those for an element behind a retaining wall which is moving forward, so the soil deforms actively with the vertical stress driving the deformation.

The directions of the principal stresses remain essentially unchanged during the movement of the wall, so the vertical and horizontal effective stresses $\sigma'_v$ and $\sigma'_h$ can be associated with the principal effective stresses $\sigma'_1$ and $\sigma'_3$. The vertical total stress at $X$ remains essentially constant as
the wall deforms, so
\[
\delta t = -\frac{\delta \sigma_b}{2}
\]
and the total stress path has a slope of \(-1\) in the \(s:t\) total stress plane.

As the soil moves forwards towards a state of active failure, \(\delta \sigma_b < 0\) and, from (10.45a), \(\delta t > 0\). The corresponding total stress path \(BCD\) is shown in Fig. 10.19b. If the deformation of the wall occurs rapidly, then the soil deformation is undrained and the effective stress path, again assuming an initially lightly overconsolidated condition, is \(B'C'D'\) (Fig. 10.19b). The mean effective stress \(s'\) remains constant for plane strain constant volume deformation so long as the soil behaves elastically and isotropically \((B'C')\); but because the total and effective stress paths are converging, the pore pressure falls. When yielding occurs at \(C'\), the pore pressure may start to increase as the effective stress path turns towards the active failure line. What happens as pore pressure equilibrium is re-established in the soil behind the excavation depends on the drainage conditions that it imposes. If the water table is able to regain the position it occupied before the excavation was created, and if the total and effective stress states regain their in situ separation \(BB'\), then the effective stress state moves further towards the failure line \((D'E')\). In practice, however, it might be expected that the presence of the excavation will lead to a lower equilibrium water table in the adjacent soil, with lower pore pressures and improved stability.

The element \(Y\) beneath the base of the excavation is subjected to a large drop in total vertical stress with perhaps only a small change in total horizontal stress. Then
\[
\delta t \sim \frac{\delta \sigma_v}{2}
\]
and since \(\delta \sigma_v < 0, \delta t < 0\). The total stress path \(BFG\) is shown in Fig. 10.19b. For undrained deformation, the effective stress path initially shows no change in mean effective stress \(s'\) until yielding occurs \((B'F')\), followed by reduction in effective stress towards the passive failure lines as the soil deforms plastically \((F'G')\). Because the total stress path shows a large decrease in mean stress \(s\), the effect of dissipation of pore pressure may tend to move the effective stress state still closer to passive failure \((G'H')\),
though, of course, details of this path depend on the ultimate ground water conditions.

In the deviatoric view (Fig. 10.18), the actively deforming element $X$ again follows a stress path of the general shape $BD$, though in this case the deformation is being driven by a decrease in horizontal stress $\sigma'_2$ rather than an increase in vertical stress $\sigma'_1$. The passively deforming element $Y$ follows a path such as $BG$. In both cases, the plane strain constraint causes $\sigma'_2$ and $\sigma'_3$ to differ from each other as soon as deformation begins, and both paths leave the projection of the $\sigma'_1$ axis.

10.4.4 Element in long slope

The analysis of the stability of a slope of soil which is long in its direction of dip was considered in Section 7.6. If the slope is also of large lateral extent (in its direction of strike), then deformations can be assumed to occur in plane strain. At an element $X$ in such a slope (Fig. 10.20a), the principal axes of stress are certainly not vertical and horizontal, but the principal total stresses $\sigma_1$ and $\sigma_3$, and hence the values of $s$ and $t$, can be calculated following the analysis of Section 7.6.

It was shown in Section 7.6 that the stability of the slope was severely reduced when pore pressures associated with steady seepage parallel to the slope and down the slope were present in the soil. The presence of any positive pore pressure in the soil makes the effective mean stress $s'$ lower than the total mean stress $s$ and hence impairs the stability of the slope. Conversely, the presence of any negative pore pressure in the soil makes the effective mean stress $s'$ higher than the total mean stress $s$ and hence enhances the stability of the slope.

In tropical areas such as Hong Kong, where rainfall and temperature changes are seasonal, high negative pore pressures may be present in near surface soils for much of the year as a result of low ground-water levels and less than complete saturation of soil voids. Slopes in such soils may be able to stand stably at angles considerably in excess of their effective

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**Fig. 10.20** (a) Element in long slope with infiltration of rainfall; (b) stress paths in $s':t$ and $s:t$ planes.
angle of friction (see Section 7.6) provided the presence of these negative pore pressures can be guaranteed. When the rain does come, water may infiltrate from the ground surface (Fig. 10.20a), fill the pores in the soil, and remove some or all of the negative pore pressure. Whether or not this leads to failure of the slope depends on the strength that the soil can mobilise on the appropriate effective stress path. Brand (1981) suggests that the change in pore pressure occurs at essentially constant total stresses; in the \( s : t \) and \( s' : t \) stress planes, the total and effective stress paths are as shown in Fig. 10.20b. The initial negative pore pressure puts the initial effective stress state \( B' \) to the right of the total stress state \( B \) (\( s' > s \)). As the pore pressure increases (becomes less negative), the effective stress path moves from \( B' \) to \( C' \) at constant shear stress \( t \), and the total stress remains unchanged.

Previous examples of stress paths have demonstrated that plane strain conditions diverge from axially symmetric conditions, and this divergence is particularly evident when the deviatoric projection of Fig. 10.18 is considered. This example indicates that, even with the distinction between plane strain and axial symmetry temporarily set aside, the nature of the triaxial test which might be most relevant for assessing the performance of the soil in the slope is very different from the standard compression test that might automatically be performed. The standard drained test reaches failure after steady increase of stress ratio \( q/p' \), with deviator stress increasing at constant cell pressure. The test that is suggested by the effective stress path in Fig. 10.20b also induces failure by steady increase of stress ratio \( q/p' \) but with the deviator stress held constant as the mean effective stress is reduced.

### 10.5 General stress paths

It should be clear that the range of geotechnical situations for which stress paths can be qualitatively assessed with any confidence is rather limited. As the geotechnical structure becomes more complex, the stress paths also become more complex and more uncertain, and the possibility of reproducing them in a laboratory testing apparatus becomes more remote.

That is not intended as a cry of despair, however, because the whole object of developing numerical models for soil behaviour is precisely to provide a rational basis for the extrapolation from the known region of laboratory test data towards the unknown region of actual field response. Numerical analyses such as those presented in Section 11.3 will then give an indication of the stress paths expected by the computation and reveal the extent of the necessary extrapolation.
10.6 Undrained strength of soil in various tests

The behaviour of soils can be described in terms of effective stresses using numerical models of which Cam clay is only one example. There are many ways in which limiting conditions can be reached with these models, and each could be used to define a failure state and hence a strength for the soil. This section is concerned with a subset of these limiting conditions, failure states reached in undrained, constant volume tests which define values of undrained strength for the soil.

The undrained strength of soils can be measured in many different tests in the laboratory or in situ in the ground. Engineers often talk about the undrained strength of soil in a way which ignores not only the various modes of deformation to which soil elements can be subjected in these tests but also the differences between the stress and strain paths that may be followed in these tests, and the stress and strain paths that may be relevant for any particular problem of design or analysis. Stress paths in certain practical problems have been discussed in previous sections; here the differences arising from various testing configurations are considered.

Undrained strengths are typically used in traditional plastic collapse analyses for geotechnical structures which involve the rapid loading of clays. These analyses usually implicitly assume that the undrained failure of soils is governed by a Tresca failure criterion (Section 3.2), so that it is assumed that, irrespective of the mode of deformation, the maximum shear stress that the soil can support is the same. This makes it very easy to justify using undrained strengths obtained, for example, from triaxial tests for analysis of the collapse of plane structures but takes no account of the overall effective stress response of the soil, of which undrained testing explores only a small part. The behaviour of soils is fundamentally described in terms of effective stresses, and any other statement about soil response must be consistent with the underlying effective stress behaviour.

Various soil tests will be compared in Section 10.6.1 through the modes of constant volume deformation that they impose; this gives a clear visual impression of the differences. In Section 10.6.2, the Cam clay model will be used to calculate the undrained strengths, thought of as the values of maximum shear stress, that would be measured in different soil tests.

10.6.1 Modes of undrained deformation

A series of constant volume deformations of an initially cubical soil element are shown to an exaggerated scale in Fig. 10.21. The soil elements are assumed to have been taken from or to be in the ground with the z axis vertical.

Figure 10.21a shows the deformation that is imposed in a triaxial
Fig. 10.21  Modes of constant volume deformation: (a) triaxial compression, (b) triaxial extension, (c) plane strain compression, (d) plane strain extension, (e) pressuremeter cylindrical cavity expansion, (f) simple shear on vertical sample, (g) field vane, and (h) cone penetration test.
10.6 Undrained strength of soil

compression test. The radial tensile strain of the triaxial sample has half
the magnitude of the axial compressive strain:

$$\varepsilon_z = -2\varepsilon_x = -2\varepsilon_y; \quad \varepsilon_z > 0$$  \hspace{1cm} (10.47)
(with compressive strains positive). The deformation imposed in a triaxial
extension test (Fig. 10.21b) is the opposite of this: whereas compression
makes the element short and fat, extension makes it tall and thin. Thus,

$$\varepsilon_x = \varepsilon_y = -\frac{\varepsilon_z}{2}; \quad \varepsilon_x > 0$$  \hspace{1cm} (10.48)

In plane strain tests on samples taken vertically from the ground, the
strain in one of the originally horizontal directions, for example, the y
direction, is zero. Plane strain compression (Fig. 10.21c) corresponds
to active loading of the soil with the sample becoming shorter ($\varepsilon_z$
compressive),

$$\varepsilon_z = -\varepsilon_x, \quad \varepsilon_y = 0, \quad \varepsilon_z > 0$$  \hspace{1cm} (10.49)

and plane strain extension (Fig. 10.21d) corresponds to passive loading of
the soil with the sample becoming thinner ($\varepsilon_x$ compressive),

$$\varepsilon_x = -\varepsilon_z, \quad \varepsilon_y = 0, \quad \varepsilon_x > 0$$  \hspace{1cm} (10.50)

Expansion of a long cylindrical cavity is used in the pressuremeter test
to determine in situ properties of soils (Baguelin, Jézéquel, and Shields,
1978; Mair and Wood, 1987). A schematic diagram of a pressuremeter is
shown in Fig. 10.22. A long cylindrical rubber membrane is expanded by
internal pressure, and the radial or volumetric expansion of the cavity is
measured. The soil is deformed in plane strain with the vertical z direction
as the direction in which no strain occurs. Soil elements are compressed
in the radial direction, and a balancing circumferential extension has to
occur to maintain the constant volume condition. Converting this to a
deformation mode for the cubical element (Fig. 10.21e), we find the strain
path to be specified by

$$\varepsilon_y = -\varepsilon_x, \quad \varepsilon_z = 0, \quad \varepsilon_y > 0$$  \hspace{1cm} (10.51)

Fig. 10.22 Expansion of cylindrical cavity in pressuremeter test.
Such a deformation could be imposed in a plane strain apparatus in the laboratory, but it is more common for plane strain tests to be performed with no strain in one of the horizontal $x$ or $y$ directions, as shown in Figs. 10.21c, d.

The strain paths associated with undrained triaxial compression (TC), triaxial extension (TE), plane strain compression (PSC), plane strain extension (PSE), and pressuremeter expansion (PM) are shown in Fig. 10.23a in the deviatoric projection of principal strain space. Since these are all constant volume paths, the deviatoric projection provides a true view of each path, with no foreshortening.

Some results of simple shear tests on soils were presented briefly in Sections 6.5, 8.3, and 8.6. After conventional triaxial apparatus, simple shear apparatus are probably the most frequently used pieces of equipment for laboratory investigations of the stress: strain behaviour of soils; simple shear tests provide the only readily available opportunity for studying the effects of rotation of principal axes on the stress: strain behaviour of soils. The mode of deformation in a simple shear test is shown in Fig. 10.21f. The sample is deformed in plane strain so that $\varepsilon_y = 0$. In the $xz$ plane, the deformation is such that the shape of the sample changes from a rectangle to a parallelogram. There is no direct strain in the $x$ direction, and because the deformation is assumed to be at constant volume, there is no vertical strain either. The only non-zero component of strain is the shear strain $\gamma_{xz}$:

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0; \quad \gamma_{xz} \neq 0$$

(10.52)

Fig. 10.23  (a) Deviatoric strain paths and (b) deviatoric stress paths according to Cam clay model with Mohr-Coulomb failure in constant volume triaxial compression TC, triaxial extension TE, plane-strain compression PSC, plane strain extension PSE, and pressuremeter cylindrical cavity expansion PM.
10.6 Undrained strength of soil

It is not easy to show the expected modes of deformation for in situ tests other than for the pressuremeter. Two tests commonly used for the in situ estimation of strengths of soil are the vane and the cone penetration test. The exaggerated deformation of an element $X$ of soil around the outer radius of a cruciform vane being rotated in the ground, before a failure surface has developed, is shown, tentatively, in Fig. 10.21g. This element of soil is subjected to a deformation rather like the simple shear deformation of a sample with its vertical axis in the apparatus corresponding to a horizontal direction in the ground (compare Figs. 10.21f, g):

$$
\varepsilon_x = \varepsilon_y = \varepsilon_z = 0; \quad \gamma_{xy} \neq 0
$$

On the other hand, an element $Y$ of soil which is going to form part of the top or bottom horizontal failure surface created by the rotating vane is, before the failure surface forms, subjected to a mode of deformation similar to the simple shear test on a sample with its vertical axis corresponding to the vertical direction in the ground (Fig. 10.21f).

A cone penetration test pushes elements of soil through a variety of modes of deformation. The penetration of the tip may subject soil elements ahead of the tip to deformations similar to those experienced in the expansion of a spherical cavity (Fig. 10.21h). Expansion of a spherical cavity in an isotropic soil subjects elements of soil to the constant volume deformation associated with conventional triaxial compression (Fig. 10.21a); there are now two circumferential directions which elongate equally as the radial deformation increases and radial compression occurs. However, only the elements of soil directly below the tip of the cone are correctly equivalent to vertical soil samples, that is, to conventional triaxial samples taken from the ground with their axes vertical, in the $z$ direction ($A$ in Fig. 10.24). Other elements are equivalent to inclined samples

![Inclined cylindrical samples taken out of the ground with their axes in a vertical plane.](image)
(B, C, D, and E in Fig. 10.24), and any anisotropy of the soil strongly influences their behaviour.

The penetration of the shaft of the cone is similar to the penetration of a driven pile which Randolph, Carter, and Wroth (1979) have suggested is equivalent to the undrained expansion (or rather creation) of a long cylindrical cavity. Soil elements away from the heavily disturbed zone adjacent to the cone might therefore be subjected to deformations similar to those imposed in the pressuremeter test (Fig. 10.21e). It is because the cone penetration test causes such a variety of modes of deformation that there is such an uncertainty concerning the correct factor to be used to convert cone penetration resistance to some shear strength parameter for a soil (e.g. see Meigh, 1987).

Each soil test subjects elements of soil to different modes of deformation. It is for this reason that it is inappropriate to talk about the undrained strength of a soil without stating by what means this particular strength was determined. Triaxial compression strengths and field vane strengths are perhaps the commonest strengths that are quoted in the literature. The various diagrams in Fig. 10.21 show that, quite apart from any other factors which may influence the strengths determined in any particular test, the modes of deformation are so different that similarity of values should be regarded as the exception rather than the rule.

10.6.2 Undrained strengths: Cam clay model

So far the discussion of the differences between various tests has been largely intuitive. It is possible to quantify these differences if assumptions are made about the way in which the soil behaves. When the strength of the soil was discussed in Chapter 7, it was really only the strength of soils in conventional triaxial compression (and extension) tests that was considered; stress and strain paths associated with common tests and with geotechnical constructions escape from the constraints imposed by the triaxial apparatus, and it is necessary to suggest what might be expected in other parts of the deviatoric projection of stress space, away from the diameter XOY in Fig. 10.18.

The Cam clay model described in Chapter 5 is a simple elastic–plastic model of soil behaviour which goes at least some way towards incorporating a rather more realistic description of the effective stress changes that occur in constant volume shearing. In Chapter 5 this model was described only in terms of stress changes that could be applied in the conventional triaxial apparatus. It is necessary to make some assumptions about how this observed response should be extrapolated to more general stress conditions. Experimental data for non-axially symmetric stress
conditions are few (and to some extent contradictory). The simplest possible assumption is that the response of the soil depends on the mean effective stress $p'$ and a general deviator stress $q$ as defined in Section 1.4.1. For a set of principal stresses, $\sigma'_1, \sigma'_2, \sigma'_3$, the mean effective stress $p'$ indicates the extent to which the stresses are all the same:

$$p' = \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3}$$

(10.54)(cf. 1.34)

The general deviator stress $q$ is a function of principal stress differences and indicates the extent to which the stresses are not the same:

$$q = \sqrt{\frac{(\sigma'_2 - \sigma'_3)^2 + (\sigma'_3 - \sigma'_1)^2 + (\sigma'_1 - \sigma'_2)^2}{2}}$$

(10.55)(cf. 1.35)

The values of $p'$ and $q$ represent (with factors of proportionality) the distance in principal stress space (Fig. 10.3a) along the mean stress axis $\sigma'_1 = \sigma'_2 = \sigma'_3$ and the orthogonal distance from the current effective stress state to this axis, respectively. It is convenient to assume that the stress:strain behaviour of soil depends only on $p'$ and $q$: the Cam clay yield surface is assumed to be formed by rotation about the mean stress axis (or stress space diagonal $\sigma'_1 = \sigma'_2 = \sigma'_3$) (Fig. 10.25), and any section through the yield surface at constant mean effective stress $p'$ is consequently a circle.

The Mohr–Coulomb failure criterion specifies a maximum value for the ratio between major and minor principal effective stresses:

$$\frac{\sigma'_1}{\sigma''_{\text{III}}} = \frac{1 + \sin \phi'}{1 - \sin \phi'}$$

(10.37bis)

This expression involves only two of the principal effective stresses and

Fig. 10.25  Ellipsoidal yield surface of Cam clay model in principal effective stress space (drawn for $M = 0.9$).
requires nothing of the remaining principal effective stress $\sigma''_h$ except that it should be intermediate between the other two. In different parts of principal effective stress space $\sigma'_1:\sigma'_2:\sigma'_3$ (Fig. 10.3), the major and minor principal stresses are associated with different pairs selected from the three axes 1:2:3. The assignment of relative magnitudes of principal stresses can be seen in the deviatoric view (Fig. 10.3c). In terms of $\sigma'_1:\sigma'_2:\sigma'_3$, (10.37) becomes six separate expressions defining six planes through the origin of principal effective stress space which intersect to form an irregular hexagonal pyramid centred on the axis $\sigma'_1 = \sigma'_2 = \sigma'_3$ (Fig. 10.26a) and having an irregular hexagonal section in the deviatoric view (Fig. 10.26b). This can be compared with the shapes of the Tresca and von Mises yield criteria proposed for metals, which produce prismatic yield surfaces (Fig. 3.7) with regular hexagonal and circular sections in the deviatoric view (Fig. 3.8).

The experimental evidence from laboratory tests with general stress

Fig. 10.26 (a) Mohr-Coulomb failure criterion as hexagonal pyramid in principal effective stress space; (b) section through Mohr-Coulomb failure criterion at constant mean effective stress $\mu'$ (drawn for $\phi' = 23^\circ$).
conditions (e.g. see Gens, 1982) suggests that the critical state failure of soils can be more accurately approximated by a Mohr–Coulomb failure criterion with a constant angle of shearing resistance, the irregular hexagonal pyramid of Fig. 10.26, than by the von Mises criterion that the generalisation of Cam clay with only \( p' \) and \( q \) implies (Fig. 10.25). It is assumed here, therefore, that although the prefailure stress: strain behaviour is described by Cam clay with expanding ellipsoidal yield surfaces as shown in Fig. 10.25, failure is governed by a Mohr–Coulomb failure criterion with angle of shearing resistance corresponding to the critical state reached with Cam clay in triaxial compression. The resulting model is then described by a sort of intersection of the Cam clay ellipsoid and the Mohr–Coulomb hexagonal pyramid (Fig. 10.27a). At any value of mean stress \( p' \), the value of general deviator stress \( q \) that can be

Fig. 10.27 (a) Ellipsoidal yield surface of Cam clay model combined with irregular hexagonal pyramid of Mohr–Coulomb failure criterion; (b) sections at constant mean effective stress \( p' \) (drawn for \( M = 0.9, \phi' = 23^\circ \)).
reached depends on the direction of loading in the deviatoric projection (Fig. 10.27b).

No attempt has been made to include the Hvorslev surface representation of peak strengths for heavily overconsolidated clays (Section 7.4.1) in this composite model, which is being applied here only to the estimation of undrained strengths of normally compressed clays. Tests on initially one-dimensionally normally compressed soil are considered. The initial effective stress state resulting from one-dimensional compression emerges from the model without further assumption, as described in Section 10.3.1.

Calculations of the undrained strengths expected in various tests have been made using $\nu = 0.3$ and $\Lambda = 0.8$ as a typical set of parameters, and the effect of various values of $M$ [related to angle of shearing resistance $\phi'$ in triaxial compression through (7.9)] has been studied. The resulting values of undrained strength $c_u$, which is usually the maximum shear stress measured in each particular test, are normalised with respect to the effective vertical consolidation pressure $\sigma_{zc}'$ that exists before the test is started. The resulting strength ratios $c_u/\sigma_{zc}'$ are shown in Fig. 10.28.

A typical set of stress paths in the deviatoric view of stress space is shown in Fig. 10.23b. So long as the soil is behaving isotropically and elastically inside the yield surface, constant volume deformation implies no change in mean effective stress $p'$, and the deviatoric stress paths have the same directions as the deviatoric strain increment paths shown in

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**Fig. 10.28** Dependence of strength ratio $c_u/\sigma_{zc}'$ on angle of shearing resistance $\phi'$ for normally compressed soil according to Cam clay model combined with Mohr–Coulomb failure criterion (TC, triaxial compression; TE, triaxial extension; PS, plane strain; PM, pressuremeter; DSS, direct simple shear; FV, field vane).
Fig. 10.23a. However, the stress paths bend round when yielding occurs. The occurrence of plastic volumetric strain in undrained deformation requires, following Section 4.3, changes in mean effective stress through development of pore pressures in order to produce equal and opposite elastic volumetric strains and an overall constant volume condition. Therefore, the stress paths shown in Fig. 10.23b do not actually lie in a single deviatoric plane at constant $p'$; the diagram represents projections of three-dimensional paths.

The symmetry contained in the model and in the Mohr–Coulomb failure criterion forces the strength ratios for plane strain compression and extension to be the same. According to this model, the strength ratio for plane strain is always higher than that for triaxial compression.

The pressuremeter tests run into an ambiguity. In the analysis of the pressuremeter test (Mair and Wood, 1987), the apparent undrained strength is calculated from the difference between the radial and circumferential effective stresses at failure, which implies, in terms of the stresses $\sigma'_x, \sigma'_y, \sigma'_z$,

$$c_u = \frac{\sigma'_r - \sigma'_s}{2} \quad (10.56)$$

There is a possibility, however, that for a low value of $K_{one}$ (which from Fig. 10.5 implies a high value of angle of shearing resistance $\phi'$ and a high initial value of stress ratio $q/p' = \eta_{kel}$), failure can occur with the vertical stress $\sigma'_s$ still the major principal stress ($\sigma'_1 > \sigma'_s > \sigma'_2$ in Fig. 10.3c, with principal stresses $\sigma'_1, \sigma'_2, \sigma'_3$ assigned to the $z, x, y$ directions, respectively) instead of the intermediate stress as assumed in the analysis ($\sigma'_3 > \sigma'_1 > \sigma'_2$ in Fig. 10.3c, with the same assignment of stresses). The undrained strength which is calculated from (10.56) is not then the maximum shear stress sustained by the soil. For the particular set of soil parameters used here, the two results diverge only for values of $\phi'$ above about 31°.

For the simple shear test, further assumptions are required. De Josselin de Jong (1971) shows that there are two alternative sets of failure planes on which deformation can occur, both compatible with an overall simple shear deformation. The most obvious set of planes is the horizontal set shown in Fig. 10.29b, and the corresponding Mohr circle of effective stress (Fig. 10.29a) shows that the horizontal planes are planes on which the largest angle of friction in the soil is mobilised. Alternatively, deformation can take place on a vertical set of planes (Fig. 10.29d) with accompanying rotation to give the overall simple shear deformation. In this case, the maximum angle of friction is mobilised on these vertical planes (Fig. 10.29c), and the angle of friction mobilised on the horizontal
boundary of the sample may be considerably lower. De Josselin de Jong maintains that this second mode of deformation is likely to be attained by many soils more easily than the first. Randolph and Wroth (1981) show some experimental evidence in support of this contention. In either case, however, the shear stress measured on the horizontal boundary of the sample is lower than the maximum shear stress experienced by the soil by a factor $\cos \phi'$ (see the Mohr circles in Figs. 10.29a,c). Simple shear deformation is a plane strain process, so it can be assumed in the Cam clay model that the changes of principal effective stresses that are experienced during the simple shear test are the same as those experienced during a plane strain compression test; the only difference, the rotation of the directions of the principal stresses, is assumed to have no effect per se. Then the shear strength in the simple shear test, which should more properly be called a measured maximum shear stress, is lower than the mobilised strength and can be calculated by multiplying the undrained

Fig. 10.29 Simple-shear deformation with sliding on (a), (b) horizontal planes and (c), (d) vertical planes combined with rotation (after de Josselin de Jong, 1971).
strength in plane strain compression by \( \cos \phi' \). The factor \( \cos \phi' \) decreases as \( \phi' \) increases, and the curve for the apparent simple shear strength in Fig. 10.28 diverges steadily from the plane strain compression curve and actually reaches a peak at an angle of friction of about 40°.

It was suggested in Fig. 10.21g that elements of soil around the boundary of the cylinder of soil that is brought to failure in the ground by a rotating field vane would, before a failure surface actually forms, be subjected to a mode of deformation approximating to the simple shear of a sample taken out of the ground with a horizontal axis destined to become the vertical axis in the apparatus. By the same argument as in the previous paragraph, the shear stress in the soil on that vertical failure plane is lower than the undrained strength of the soil by a factor \( \cos \phi' \). The undrained strength could be estimated from an appropriate plane strain test in which no rotation of principal axes occurs, which in this case is the pressuremeter test. Wroth (1984) presents experimental data and theoretical analyses which show that the contribution of the shear stress mobilised on the top and bottom surfaces of the failing cylinder of soil to the torque required to rotate the vane is often considerably smaller than the contribution from the vertical surface. The curve of vane strengths (FV in Fig. 10.28) is simply estimated from the pressuremeter strength (PM) multiplied by \( \cos \phi' \). There is again the problem that for angles of shearing resistance greater than about 31° (a high angle for normally compressed clay) failure occurs according to the Mohr–Coulomb criterion, with the vertical effective stress still the major principal effective stress. If the contribution to the torque from the horizontal shear surfaces is reckoned to be non-negligible, then the shear stress on these surfaces is similar to that measured in the simple shear test, and the field vane curve (FV in Fig. 10.28) would move nearer to the simple shear curve (DSS).

It is not really worthwhile to labour the assumptions behind this particular model. It was noted anyway in Section 9.6 that the trend of major increase of strength ratio \( c_u/\sigma_{sz} \) with increasing angle of shearing resistance \( \phi' \) (and hence by implication with decreasing plasticity \( I_P \)) is not supported by experimental evidence. Nevertheless, similar conclusions about the dependence of the measured strength on the mode of testing have been drawn by, for example, Prévost (1979), Levadoux and Baligh (1980), Borsetto, Imperato, Nova, and Peano (1983), using other more elaborate models of soil behaviour.

For a typical value of \( \phi' \) between 20° and 25°, the sequence of strength values shown in Fig. 10.28 is PS > PM > TC > DSS > FV > TE. Typical field data for an Italian soft clay site that are based on tests reported by Ghionna, Jamiołkowski, Lacasse, Ladd, Lancellotta, and Lunne (1983)
Table 10.1 Undrained strength ratios in different tests

<table>
<thead>
<tr>
<th>Clay</th>
<th>$I_p$</th>
<th>$c_u / \sigma'_{zc}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bangkok (1)</td>
<td>0.85</td>
<td>0.72 0.37 — — — —</td>
</tr>
<tr>
<td>Boston blue (2)</td>
<td>0.21</td>
<td>0.328 0.130 0.335 0.175 0.210 0.220</td>
</tr>
<tr>
<td>Drammen (1)</td>
<td>0.29</td>
<td>0.39 0.20 — — — —</td>
</tr>
<tr>
<td>Haney (3)</td>
<td>0.18</td>
<td>0.268 0.168 0.296 0.211 — —</td>
</tr>
<tr>
<td>AGS marine clay (4)</td>
<td>0.43</td>
<td>0.325 0.200 — — — —</td>
</tr>
<tr>
<td>James Bay intact (5)</td>
<td>0.16</td>
<td>0.45 0.235 — — — —</td>
</tr>
<tr>
<td>James Bay destructured (5)</td>
<td>0.335</td>
<td>0.20 — — — — 0.275</td>
</tr>
</tbody>
</table>

Notes: (1) Data from Prévost (1979)
(2) Data from Levadoux and Baligh (1980)
(3) Data from Vaid and Campanella (1974)
(4) Atlantic Generating Station marine clay; data from Jamiołkowski, Ladd, Germaine, and Lancellotta (1985)
(5) Data from Jamiołkowski et al. (1985)

*Measured shear stress on horizontal surfaces, increased by 10% to allow for non-uniform boundary stresses (e.g. see Wood and Budhu, 1980).

Fig. 10.30 Field data of undrained strength measured with different tests at Porto Tolle, Italy (after Ghionna, Jamiołkowski, Lacasse, Ladd, Lancellotta, and Lunne, 1983) (TC, triaxial compression; TE, triaxial extension; PM, pressuremeter; DSS, direct simple shear; FV, field vane).
are shown in Fig. 10.30. The variation of deduced strength with depth differs from test to test; the sequence of magnitudes is similar to that deduced from Fig. 10.28 (except that DSS and FV are interchanged), but the spread of magnitudes is rather greater.

A series of values of the strength ratio \( c_u / \sigma'_w \) determined from laboratory experiments on a number of one-dimensionally normally compressed clays are tabulated in Table 10.1. These data confirm the theoretical conclusion even though the numerical values may be different from those predicted with the Cam clay model. An additional moral can be drawn from the discussion about the simple shear and vane tests: in these the apparent undrained strength that is deduced is a shear stress mobilised on a particular plane or surface and is not necessarily the maximum shear stress experienced by the soil. There is even less reason to expect that this mobilised shear stress should be the same as an undrained strength measured as a maximum shear stress in another device.

10.7 Conclusion

In Sections 10.3, 10.4, and 10.5, qualitative stress paths for a number of field situations were discussed to show that different geotechnical constructions would load and deform soil elements in different ways. Lest it be thought that this comparison and distinction of stress paths was unnecessary, examination in Section 10.6 of the stress paths applied in different common soil tests showed that even something as apparently straightforward as undrained strength of soils is not independent of the stress path (or strain path) on which it is measured. In particular, undrained strengths (and other soil properties) determined under conditions of plane strain are not in general the same as undrained strengths (and other soil properties) determined under conditions of axial symmetry. However, most soil testing is actually done under conditions of axial symmetry, whereas plane strain conditions obtain more frequently in practice. It is necessary to be aware of the distinction when selecting soil parameters for design purposes.

**Exercises**

E10.1. Estimate the total and effective stress paths in \( s : t, s' : t \) and deviatoric stress planes for elements of soil behind a typical gravity wall retaining clay soil which moves rapidly towards a condition of (a) active, and (b) passive, failure.

Consider two long-term drainage conditions:

i. Drainage beneath the wall is so effective that the water table
in the neighbourhood of the wall is drawn down to the level of the base of the wall.

ii. Drains through the wall become blocked, and the water table remains at the original ground surface which is level with the top of the wall.

E10.2. An element of Cam clay, with soil parameters \( \lambda = 0.193, \kappa = 0.047, M = 0.97, N = 3.17, \) and \( G' = 2400 \text{kPa} \) is in a normally compressed state subjected to the following stresses, referred to right-handed cartesian axes:

\[
\begin{align*}
\sigma'_{xx} &= 110 \text{kPa} & \tau_{xy} &= 25 \text{kPa} & \sigma'_{yy} &= 54 \text{kPa} \\
\sigma'_{zz} &= 67 \text{kPa} & \tau_{yz} &= 0 & \tau_{zx} &= 0
\end{align*}
\]

The element of soil is subjected to the following drained changes in stress, referred to the same axes:

\[
\begin{align*}
\delta \sigma'_{xx} &= 2 \text{kPa} & \delta \tau_{xy} &= 3 \text{kPa} & \delta \sigma'_{yy} &= 1 \text{kPa} \\
\delta \sigma'_{zz} &= 4 \text{kPa} & \delta \tau_{yz} &= 0 & \delta \tau_{zx} &= 0
\end{align*}
\]

Making appropriate assumptions, calculate the increments of volumetric and shear strain. Calculate direction cosines for the principal axes of stress increment and strain increment and for the principal axes of stress before and after the increment.

E10.3. A sample of Cam clay is set up in a true triaxial apparatus in a normally compressed condition with effective stress \( \sigma'_1 = 200 \text{kPa} \) and \( \sigma'_2 = \sigma'_3 = 130 \text{kPa} \). What should the principal effective stresses be at yield and at the critical state in the following undrained tests?

i. Axisymmetric compression \( \delta \varepsilon_1 > 0, \delta \varepsilon_2 = \delta \varepsilon_3 \)

ii. Axisymmetric extension \( \delta \varepsilon_1 < 0, \delta \varepsilon_2 = \delta \varepsilon_3 \)

iii. Plane strain active \( \delta \varepsilon_1 > 0, \delta \varepsilon_2 = 0 \)

iv. Plane strain passive \( \delta \varepsilon_2 = 0, \delta \varepsilon_3 > 0 \)

v. Pressuremeter expansion \( \delta \varepsilon_1 = 0, \delta \varepsilon_2 > 0 \)

Assume that the behaviour of the clay can be described in terms of general stress quantities \( p' \) and \( q \), appropriately defined; that the mean effective stress at the critical state is independent of the preceding strain path (note that this is a different assumption from that in Section 10.6.2); and that the values of soil parameters for the Cam clay model are \( M = 1.05, \lambda = 0.213, \kappa = 0.036, N = 3.58, G' = 1780 \text{kPa} \).

E10.4. A thin-walled hollow cylindrical sample of soil of mean radius \( r \) and wall thickness \( t \) is compressed isotropically under an effective pressure \( P_1 \). It is then subjected to a drained test in which a
torque \( \alpha Q \) is applied, the axial stress is increased to \( P_1 + \alpha \sigma_a \), the internal pressure is reduced to \( P_1 - \alpha \sigma_b \), and the external pressure is increased to \( P_1 - \alpha \sigma_c \). Choosing suitable definitions of average stress components, find expressions for the principal stresses acting on a typical element of soil.

For a particular specimen, \( 20b = 20c = a = Q/(2\pi r^2 t) \) and \( r = 10t \). The behaviour of the soil can be described by the Cam clay model with yield loci \( \sigma' / \sigma'_0 = M^2 / (M^2 + \eta^2) \). If the sample has been previously isotropically normally compressed to a mean effective stress \( 2P_1 \), at what value of \( \alpha \) does the soil start to yield?