Workspace analysis of fully restrained cable-driven manipulators

Cong Bang Pham a,*, Song Huat Yeo b,1, Guilin Yang c,2, I-Ming Chen b,3

a School of Mechanical Engineering, Ho-Chi-Minh University of Technology, Viet Nam
b School of Mechanical & Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore
c Mechatronics Group, Singapore Institute of Manufacturing Technology, 71 Nanyang Drive, Singapore 638075, Singapore

ARTICLE INFO

Article history:
Received 14 December 2006
Received in revised form 16 June 2009
Accepted 17 June 2009
Available online 26 June 2009

Keywords:
Tension conditions
Feasible-wrench workspace
Tension factor
Cable-driven manipulator

ABSTRACT

For Cable-Driven Parallel Manipulators (CDPMs), employing redundant driving cables is necessary to obtain the full manipulation of the moving platform because of the unilateral driving property of the cables. Unlike rigid-link manipulators, the workspace of CDPMs is always determined and characterized by positive tension status of driving cables. In addition, it has been realized that the Tension Factor (TF) reflecting the relative tension distribution among the driving cables is an appropriate measure to evaluate the quality of tension restraint for CDPMs. However, since redundant cables are employed to drive the moving platform, the TF values are not unique for a particular moving platform pose. Therefore, how to determine the workspace and obtain the optimal TF value so as to generate a workspace with optimized performance become the major subjects of this paper. It is shown that the workspace can be generally formed from tension conditions verified by a recursive dimension-reduction approach and that the optimal TF value at every pose can be efficiently determined through a linear optimization approach, although it is essentially a nonlinear optimization problem. Computational examples are provided to demonstrate the effectiveness of the proposed algorithms.

1. Introduction

As shown in Fig. 1, a cable-driven parallel manipulator (CDPM) is a closed-loop mechanism in which the moving platform is connected to the base by several cables. The CDPMs are commonly known for simple and light-weight structure, large reachable workspace, low moment inertia and easy reconfigurability. These advantages make the CDPM a promising alternative to the rigid-link mechanisms in many industrial applications, such as load lifting and positioning [1], coordinate measurement [2,3], aircraft testing [4], haptic devices [5,6], and robot rehabilitation [7]. However, due to the unilateral driving property of the cables, maintaining positive cable tension is essential to maneuver the moving platform. As a result, the number of driving cables must be more than the number of degrees of freedom (DOF) of the moving platform to obtain the full manipulation [8], e.g. 8 cables are used to manipulate a 6-DOF moving platform as illustrated in Fig. 1. To design a CDPM, the workspace analyses including both quantitative and qualitative evaluation are always issues concerned. It is easily realized in Fig. 1 that the eight cables themselves cannot fully constrain the moving platform outside the polyhedron defined by eight suspending points of the base. Preliminary observation reveals that workspace of this manipulator can be approximately determined by the volume formed by the eight suspending points. Moreover, because the tension of the cables is practically limited, this makes the actual workspace of the manipulator smaller, e.g. a subset of the polyhedron defined by the eight suspending points.

Currently, with different approaches to the study of cable tension, e.g. null space approach and geometrical approach, and depending on the purpose of applications, several terminologies of workspace have been proposed. For instance, with the null space approach, a statically reachable workspace which is defined as a set of end-effector poses that can be reached statically was addressed by Fattah [9]. A controllable workspace which is defined as a set of postures where forces and torques at the end-effector can be controlled was studied by Verhoeven [10,11], Takadokoro [12], Zheng [13]. For the geometrical approach, Bosscher [15] and Ebert-Uphoff [16] summarized the basic workspace terminologies...
extensively generalized to another type of workspace, e.g. feasible-wrench workspace, in which the cable tension is practically limited. Subsequently, the tension factor, i.e. the minimum tension over the maximum tension, of the cables is used to evaluate the property of force closure of a CDPM at particular configurations. It has been proved that the optimal tension factor can be efficiently computed through a linear optimization approach, which is transformed from an originally nonlinear optimization problem. By integrating the local maximum tension factor over the workspace, a global optimal tension index can be obtained.

The organization of this paper is as follows: Section 2 presents three tension-related issues and the technique of dimension reduction in checking tension conditions. Based on the tension conditions, two types of workspace are defined, generated, and quantified for fully restrained CDPMs in Section 3. Section 4 outlines the approach to evaluate the workspace quality of CDPMs. Finally, the paper is summarized in Section 5.

2. Tension conditions

For CDPMs, the cable tension is found to be critical in determining workspace. As seen in Fig. 2, in order to resist any external wrench \( \mathbf{f}_p, \mathbf{m}_p \) applied on a moving platform, \( m \) cables must create tension forces \( \mathbf{t}_i (i = 1, 2, \ldots, m) \) to achieve equilibrium of the \( n \)-DOF platform. Equilibrium conditions at the moving platform are obtained as follows:

\[
\sum_{i=1}^{m} \mathbf{t}_i + \mathbf{f}_p = 0
\]

(1)

\[
\sum_{i=1}^{m} \mathbf{r}_i \times \mathbf{t}_i + \mathbf{m}_p = 0
\]

(2)

where \( \mathbf{t}_i = t_i \mathbf{u}_i \) represents the tension force along the cables. Substituting \( \mathbf{t}_i \) into Eqs. (1) and (2) leads to the following equation:

\[
\mathbf{A} \mathbf{T} = \mathbf{B}
\]

(3)

where

\[
\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ \mathbf{r}_1 \times \mathbf{u}_1 & \mathbf{r}_2 \times \mathbf{u}_2 & \cdots & \mathbf{r}_m \times \mathbf{u}_m \end{bmatrix} \in \mathbb{R}^{n \times m}
\]

\[
\mathbf{T} = [t_1 \ t_2 \ \cdots \ t_m]^T \in \mathbb{R}^m \text{ with } t_1, t_2, \ldots, t_m > 0
\]

\[
\mathbf{B} = -\mathbf{F}_p = -\begin{bmatrix} \mathbf{f}_p \\ \mathbf{m}_p \end{bmatrix} \in \mathbb{R}^n : \text{ external wrench.}
\]

Eqs. (1) and (2) can also be expressed in vector form:

\[
\sum_{i=1}^{m} \mathbf{s}_i \mathbf{t}_i = -\mathbf{F}_p
\]

(4)

where

\[
\mathbf{s}_i = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{r}_i \times \mathbf{u}_i \end{bmatrix} : \text{ the } i\text{th column vector of } \mathbf{A}
\]

\[
\mathbf{F}_p = \begin{bmatrix} \mathbf{f}_p \\ \mathbf{m}_p \end{bmatrix} \in \mathbb{R}^n : \text{ external wrench.}
\]

The directional unit vector \( \mathbf{u}_i (i = 1, 2, \ldots, m) \) in the structure matrix \( \mathbf{A} \) can be obtained from the vector loop-closure equation for each cable.

Studying about the cable tension at a specific pose of the moving platform leads to three common questions:

- Q1: Can the platform be restrained properly by the tensions of cables as illustrated in Fig. 3(a)?
- Q2: Can the platform be sustained by the cables with finite tensions when subjected to a directed external wrench as shown in Fig. 3(b)? and
Q3: Can the platform sustain a constant wrench acting from any direction as seen in Fig. 3(c)?

The three tension-related questions can be defined and formulated formally in terms of the following three tension conditions: force-closure condition, feasible-wrench condition and wrench-set condition.

* Force-closure condition: The cable mechanism is able to resist any arbitrary wrench (in magnitude and direction) acting on the moving platform (see Fig. 3(a)) provided that positive cable tensions can always be maintained. There is no upper limit on the cable tensions. This condition is mathematically described as:

\[ \forall F_p \in \mathbb{R}^n : \exists t_1, t_2, \ldots, t_m \in [0, \infty) : \sum_{i=1}^{m} t_i s_i = -F_p. \tag{5} \]

* Feasible-wrench condition: Here the external wrench is considered as a constant static wrench acting on the moving platform, e.g. the gravity of the platform. The tension range of the ith cable is limited from \( t_{i, \text{min}} \) to \( t_{i, \text{max}} \). The value \( t_{i, \text{min}} \) refers to the tension required to keep the cables taut, whereas the value \( t_{i, \text{max}} \) is limited by the torque of actuators or by the maximum tension without breaking the cables. Hence, the cable mechanism is able to sustain by the cables with finite tensions when subjected to a directed external wrench (see Fig. 3(b)) provided that there exists a set of appropriate tensions:

\[ \exists t_{i, \text{min}} \leq t_i \leq t_{i, \text{max}} (i = 1, 2, \ldots, m) : \sum_{i=1}^{m} t_i s_i = -F_p \in \mathbb{R}^n. \tag{6} \]

* Wrench-set condition: This condition stems from a practical point of view that the mechanism must possess certain stiffness in translation as well as in torsion. Similar to the feasible wrench condition, the ith cable tension is limited from \( t_{i, \text{min}} \) to \( t_{i, \text{max}} \). Here, the cable mechanism is able to sustain a certain minimum force \( f_{p,r} \) and a certain minimum moment \( m_{p,r} \) from any direction (see Fig. 3(c)) provided that:

\[ \forall F_p \mid \| F_p \| = f_{p,r}, \| m_p \| = m_{p,r} : \exists t_{i, \text{min}} \leq t_i \leq t_{i, \text{max}} \]  

\[ (i = 1, 2, \ldots, m) : \sum_{i=1}^{m} t_i s_i = -F_p. \tag{7} \]

The tension conditions expressed by Eqs. (5)–(7) are transformed into the problem of finding the solution of the linear equation system in Eq. (4) with bounded variables. For fully restrained CDPMs, this is an under-determined system in which there are many sets of tension solutions. However to check the three tension conditions, it does not need to obtain all the solutions but important to identify the existence of appropriate solutions. The authors have proposed a dimension-reduction method to verify the force-closure condition, in which an original linear system can be equivalently decomposed to a set of 1-dimensional systems; and if there exists a tension solution in each of the 1-dimensional systems, then the original system definitely has appropriate solutions (please refer to [24] for more details). The following will show that the method can also be extensively generalized to the other two conditions, i.e. the feasible-wrench condition and the wrench-set condition.

### 2.1. Tension conditions in 1-dimensional systems

Generally, when the original system \( \mathbf{A} \mathbf{T} = \mathbf{B} \) is decomposed step by step until 1-dimensional systems through Gaussian eliminations, it can be expressed as:

\[
\begin{bmatrix}
    a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,n}^{(n)} & \cdots & a_{1,m}^{(n)} \\
    0 & a_{2,2}^{(n-1)} & \cdots & a_{2,n}^{(n-1)} & \cdots & a_{2,m}^{(n-1)} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a_{n,n}^{(1)} & \cdots & a_{n,m}^{(1)}
\end{bmatrix}
\begin{bmatrix}
    t_1 \\
    t_2 \\
    \vdots \\
    t_m
\end{bmatrix} = \begin{bmatrix}
    b_1^{(n)} \\
    b_2^{(n-1)} \\
    \vdots \\
    b_n^{(1)}
\end{bmatrix}.
\tag{8}
\]

In Eq. (8), the last row composing components with the superscript (1) is the 1-dimensional system, which can be rewritten as follows:

\[ \sum_{i=n}^{m} a_{n,i}^{(1)} t_i = b_n^{(1)} \tag{9} \]

where \( a_{n,i}^{(1)} \) is a coefficient calculated from Gaussian eliminations; \( t_i \) varies in a range according to the respective tension bound;
and $b_n^{(1)}$ is an equivalent wrench in 1-dimensional systems. If the term $b_n^{(1)}$ is known, checking the existence of a tension solution of Eq. (9) is found simple and straightforward using scalar calculation and comparison. The term $b_n^{(1)}$ is constant for the force-closure condition and the feasible-wrench condition whereas it is a function of force components and moment components for the wrench-set condition as follows:

$$b_n^{(1)} = (c_{i,x}fp_x + c_{i,y}fp_y + c_{i,z}fp_z) + \left( cm_{x}mp_x + cm_{y}mp_y + cm_{z}mp_z \right)$$  \hspace{1cm} (10)

where $c_{i,x}, c_{i,y}, c_{i,z}, cm_{x}, cm_{y}$ and $cm_{z}$ are coefficients calculated from Gaussian eliminations; $fp_x, mp_x, fp_y, mp_y$ and $fp_z$ are values obtained from the required wrench set ($fp_r, mp_r$). The maximum absolute value of $b_n^{(1)}$ which linearly depends on the required force set and the required moment set needs to be calculated.

For instance, Fig. 4 illustrates the required force set as a sphere of a radius $fp_r$. The first three components in Eq. (10) defines a plane and the maximum value $fp_{max}$ is reached when the plane is tangential to the sphere. In other words, when the force components are limited within the sphere, the product of the first three components in Eq. (10) varies from $-fp_{max}$ to $+fp_{max}$. Similarly, the product of the last three components in Eq. (10) is also expected to be in a range from $-mp_{max}$ to $+mp_{max}$. Generally, for the wrench-set condition, the equivalent wrench value $b_n^{(1)}$ is within a range from $(-fp_{max} + mp_{max})$ to $(+fp_{max} + mp_{max})$. Subsequently, the tension components on the left-hand side of Eq. (9) just need to satisfy the two extreme values of the equivalent wrench, e.g. $-fp_{max} + mp_{max}$ and $fp_{max} + mp_{max}$.

Since Eq. (10) is linear in terms of force components and moment components, the equivalent wrench value $b_n^{(1)}$ can be readily computed for different geometries of the required wrench set.

2.2. Recursive algorithms

For the dimension-reduction method, every column of the structure matrix $A \in \mathbb{R}^{n \times m}$ needs to be decomposed, which results in $m \times (m-1)/2$ 1-dimensional systems (please refer to [24] for more details). Procedures of decomposition and Gaussian elimination can be effectively implemented in a recursive form. Three algorithms called FCC, FWC and WSC are formulated and coded in Matlab for the force-closure condition, the feasible-wrench condition and the wrench-set condition, respectively. The major difference among the three algorithms is the 1-dimensional tension conditions. Therefore, algorithms of FWC and WSC can be readily extended from that of FCC which has been presented in studying force-closure workspace of cable-driven manipulators (please refer to [24] for more details).

3. Workspace determination

3.1. Workspace definitions

Based on the three tension conditions mentioned above, two types of workspaces namely force-closure workspace and feasible-wrench workspace are defined for fully restrained CDPMs.

• Force-closure workspace (FCWS): a set of postures where the force-closure condition is satisfied. This workspace is generated by the force-closure condition. The force-closure workspaces only depend on the geometrical parameters of CDPMs.

• Feasible-wrench workspace (FWWS): a set of postures that satisfies both the constant wrench feasibility and the wrench set requirement. This workspace depends on not only the geometrical parameters of the manipulator, but also the tension ranges, gravitational effect, dynamic load and the required wrench set acting on the platform.

For planar manipulators, the workspace can be visualized with a three-dimensional coordinate system. However, for a spatial manipulator with six DOFs, it is common to use the subsets of the six independent parameters for representing the workspace. In addition, CDPMs are well known for large reachable workspaces but limited orientation workspace. Hence, a constant orientation workspace and a total orientation workspace (which is a set of orientations that can be reached with all defined orientations) are commonly employed [25] for visualization. When the posture of the moving platform is known, Eq. (3) is determined and decomposed into 1-D systems. Fig. 5 shows two 1-D systems used for generating FCWS and FWWS, respectively.

With the definition of the two types of workspace, Fig. 5(a) is the 1-dimensional vectorial diagram used for checking whether a specific pose of the moving platform belongs to the force-closure workspace and Fig. 5(b) is the 1-dimensional vectorial diagram used for verifying whether a specific pose of the moving platform belongs to the feasible-wrench workspace. In these diagrams, $f_p$, $f_{p,c}$, and $f_{p,s}$ are equivalent external wrenches in 1-dimensional workspaces.

Fig. 4. Required force set and equivalent required force.

Fig. 5. Vectorial diagrams for two types of workspace.
systems, which their directions and magnitudes are according to the value of \( b^{(1)}_n \) in Eq. (9). The other vectors represent tension components, which their directions and magnitudes depend on the product of \( a^{(1)}_n \), \( t_i \) in Eq. (9). Generally, there are \( u \) tension components in positive direction \( \{p_1, p_2, \ldots, p_u\} \) and \( v \) tension components in negative direction \( \{n_1, n_2, \ldots, n_v\} \). Therefore, in Fig. 5(a), the specific pose belongs to the force-closure workspace if there must be at least two opposite components such that:

\[
\sum_{k=1}^{v} n_{k, \max} + \sum_{k=1}^{u} p_{k, \min} < 0.
\]  

As shown in Fig. 5(b), the resultant external wrench is ranged from \((f_{p, c} - f_{p, s})\) to \((f_{p, c} + f_{p, s})\). Hence, the following inequalities must be satisfied so that the specific pose belongs to the feasible-wrench workspace.

\[
\begin{align*}
\sum_{k=1}^{u} p_{k, \max} &\geq \sum_{k=1}^{u} p_{k, \min} + (f_{p, c} + f_{p, s}) \\
\sum_{k=1}^{v} n_{k, \min} &\geq \sum_{k=1}^{v} n_{k, \max} + (f_{p, c} + f_{p, s}) \\
\sum_{k=1}^{u} p_{k, \max} &\leq \sum_{k=1}^{u} p_{k, \min} + (f_{p, c} - f_{p, s}) \\
\sum_{k=1}^{v} n_{k, \min} &\leq \sum_{k=1}^{v} n_{k, \max} + (f_{p, c} - f_{p, s})
\end{align*}
\]  

(12)

3.2. Workspace generation and quantification

A general numerical workspace generation approach is employed here. With two types of workspace defined above, the
illustrate the workspaces with constant orientation at $\alpha = 2^\circ$ and $\alpha = 2^\circ$ respectively whereas Fig. 9(e) is the workspace that the platform is able to change its orientation $\alpha$ in the whole range from $-2^\circ$ to $+2^\circ$. Similarly, the right three figures are sets of postures where the platform can sustain an arbitrary moment up to 2 Nm.

Generally, the feasible-wrench workspace is often a subset of the force-closure workspace where the tension is not limited unless there is a certain constant wrench large enough to partly shift the feasible-wrench workspace out of the force-closure workspace. Such poses are undesirable and should be avoided because the manipulator will be out of control if a certain disturbance appears in the direction against the constant wrench.

4. Workspace quality evaluation

Apart from workspace quantification, it is also important to determine the quality of the workspace as a mechanism designed for a maximum workspace may have undesirable kinematic characteristics. Various performance indices such as the velocity/force ellipsoid, the condition number and the manipulability measure have been devised for assessing kinematic performance of robotics manipulators. Among them, the condition number is commonly used as a measure of dexterity. However, it cannot be applied directly for CDPMs. This is illustrated by the examples in Fig. 10. It is seen that the mechanism shown in Fig. 10(a) cannot resist the force $f_x$, because this posture is at the limits of the workspace. Likewise, the mechanism shown in Fig. 10(c) is not able to counter the applied moment $m_p$. Hence, they are poorly conditioned configurations compared to the one in Fig. 10(b) in terms of tension distribution. However, this may not be reflected by the condition number ($k_c$), which is the ratio of the largest singular value to the smallest singular value of the Jacobian matrix [26]. From the values of the condition number, the configuration in Fig. 10(c) is better conditioned than the one in Fig. 10(b).

4.1. Tension factor (TF)

For CDPMs, a tension factor is proposed to be used as a performance index to evaluate the quality of force closure at a specific configuration. The tension factor is defined as the minimum tension over the maximum tension of the cables. If $N$ is the homogeneous solution of the cable tensions, then:

$$TF = \frac{\min(N)}{\max(N)}$$

(14)

The tension factor (TF) is a measure of the positive tension condition of the structure matrix. It reflects the relative tension distribution among the cables for a specific platform pose inside the force-closure workspace. The range of the TF is from zero to one. When the TF approaches to zero, one of the cable tension is close to zero, i.e. the platform is located near to the workspace boundary. Hence if the TF approaches one, the platform is positioned far from the workspace boundary. The CDPM is called isotropic if its TF always attains one. A larger TF is more favourable because there is a better tension balance among the cables. The TF can be geometrically expressed by the vector of homogeneous solution. This concept can be illustrated using a 3-2-CDPM, i.e. a point-mass on a plane driven by three cables, as an example. From Eq. (3), to achieve a unity force in all directions, i.e. $B^T B = 1$, the cable tensions must satisfy:

$$(AT)^T A T = 1 \quad \text{or} \quad T^T A^T A T = 1.$$  

(15)
Since the structure matrix $A$ is of rank two, the quantity $(A^T A)$ is a symmetric $3 \times 3$ matrix of rank two. Eq. (15) therefore describes a cylinder with its elements oblique to the coordinate axes in the cable tension space as shown in Fig. 11. The principal axes of this tension ellipsoid, i.e., the cylinder, coincide with the eigenvectors of $(A^T A)$ and the length of its principal axes are equal to the reciprocals of the square roots of the eigenvalues. Since $(A^T A)$ is a symmetric $3 \times 3$ matrix of rank two, this quantity always has an eigenvalue of zero magnitude, resulting in a principal axis of infinite length. This axis defines the direction of the homogeneous solution $N$.

From the geometric point of view, each element on the cylindrical surface in the cable tension space maps onto a point in the force space. This implies that cable tensions can be proportionally increased along the direction of its homogeneous solution without affecting the resultant forces. If the direction of the homogeneous solution is not in the first quadrant, then increasing cable tensions along the longitudinal direction of the cylindrical surface will result in some negative tensions. In addition, if the structure matrix is not of full rank, the cylinder is degenerated to a plane in which at least one cable tension is always zero. However, as long as the force-closure condition is satisfied, the two situations mentioned above will not appear.
From the above discussion, it can be concluded that the direction of the longitudinal axis plays an important role to the distribution of cable tensions. The more the longitudinal axis is skewed towards any axis, the larger the differences in tensions among the cables. It is therefore desirable to have the longitudinal axis of the cylinder making equal angles with the coordinate axes. The vector which makes equal angles with all coordinate axes is called the isotropic vector. In an $m$-dimensional tension space, the unit isotropic vector is given by $\{\sqrt{1/m}, \sqrt{1/m}, \ldots, \sqrt{1/m}\}$.

For completely restrained CDPMs ($m = n + 1$), the homogeneous solution, i.e. the longitudinal axis of the cylinder, is obtained for a specific pose. Hence, the TF determined using Eq. (14) is unique. However, for redundantly restrained CDPMs ($m > n + 1$), the homogeneous solution spans the whole null space that has $(m - n)$ dimensions. In this case, the TF is determined by the homogeneous solution which is nearest to the isotropic axis. Geometrically, this solution is obtained by projecting the isotropic vector onto the space spanned by the null vectors ($N_1, N_2, \ldots, N_{m-n}$). Computationally, this is an optimization process of finding the homogeneous solution with the largest TF, i.e. the optimal null solution. Let $T_{iso}$ and $T_{opt}$ be the isotropic tension vector and the optimal null vector, respectively. Their relative angle is given by:

$$\cos(T_{iso}, T_{opt}) = \frac{T_{iso} \cdot T_{opt}}{||T_{iso}|| \cdot ||T_{opt}||}.$$
Necessity

An null vector is optimal if and only if the summation of all its components is minimum.

Proof. Necessity: This is to verify that an optimal null vector will have the smallest summation of all its components. Eq. (17) shows that if \( \mathbf{T}_{\text{opt}} \) satisfies the linear constraints, i.e. \( \mathbf{A} \mathbf{T}_{\text{opt}} = 0 \), then \( \lambda \mathbf{T}_{\text{opt}} \) also satisfies these constraints, i.e. \( \mathbf{A} (\lambda \mathbf{T}_{\text{opt}}) = 0 \), where \( \lambda \) is a scalar. If the cable tension is bounded with the positive minimum tension \( t_{\text{min}} > 0 \), every component of \( \mathbf{T}_{\text{opt}} \) must be greater than or equal to \( t_{\text{min}} \). If \( t_{\text{opt},i} \) is assumed the smallest among the components of \( \mathbf{T}_{\text{opt}} \), \( \lambda \) can be chosen as \( t_{\text{min}}/t_{\text{opt},i} \) such that the equivalent optimal vector \( \mathbf{T}_{\text{opt}} \) is obtained as follows:

\[
\mathbf{T}_{\text{opt}} = \frac{t_{\text{min}}}{t_{\text{opt},i}} \mathbf{T}_{\text{opt}} = \left\{ \frac{t_{\text{opt},1}}{t_{\text{opt},i}}, \ldots, \frac{t_{\text{opt},m}}{t_{\text{opt},i}} \right\}. 
\]

(18)

This is an equivalent optimal null vector because \( \mathbf{T}_{\text{opt}} \) results in the same value of the objective function in Eq. (17) as \( \mathbf{T}_{\text{opt}} \). It is seen that the \( i \)th component gets the value \( t_{\text{min}} \) while the others are greater than or equal to the minimum tension value. It is noted that any \( \lambda \) which is smaller than \( t_{\text{min}}/t_{\text{opt},i} \) will make the \( i \)th component of \( \mathbf{T}_{\text{opt}} \) smaller than the minimum tension \( t_{\text{min}} \). Subsequently, this leads to:

\[
\lambda \sum_{j=1}^{m} t_{\text{opt},j} \geq \frac{t_{\text{min}}}{t_{\text{opt},i}} \sum_{j=1}^{m} t_{\text{opt},j}.
\]

(19)

Eq. (19) means that the summation of components of \( \mathbf{T}_{\text{opt}} \) is minimum because any scalar which is greater than \( t_{\text{min}}/t_{\text{opt},i} \) will always proportionally increase all the components. Therefore, this leads to a larger summation of components.

Sufficiency: On the other hand, it is needed to verify that if \( \mathbf{T}_{\text{opt}} \) is a positive null vector with the smallest summation of its components, this vector will possess the largest tension factor. Let the objective function in Eq. (17) be \( f \). It can then be expressed as follows:

\[
f = \frac{s}{\sqrt{s^2 - c}}
\]

(20)

where

\[
s = \sum_{i=1}^{m} t_i > 0
\]

\[
c = \sum_{i,j=1; i \neq j}^{m} t_i t_j > 0.
\]
Differentiating Eq. (20) with respect to \( s \) leads to:

\[
f' = -\frac{c}{\sqrt{(s^2 - c)^3}}.
\]

In Eq. (21), \( f' \) is always negative for positive null vectors because \( c > 0 \) and \( \sqrt{(s^2 - c)^3} > 0 \). It implies that \( f \) is a decreasing function with respect to \( s \) which means that increasing \( s \) decreases \( f \). Therefore, if \( T_{opt} \) has the smallest summation of its components, substituting its components into Eq. (20) results in the largest function value, i.e. the largest tension factor. □

Using the above proposition, the nonlinear optimization problem in Eq. (17) can be converted to a linear optimization problem given by:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{m} t_i \\
\text{subject to} & \quad A \mathbf{T} = 0 \\
& \quad t_i \geq t_{min} > 0 \quad (i = 1, 2, \ldots, m).
\end{align*}
\]

This optimization problem in Eq. (22) can be solved by using linear optimization algorithms and the optimal tension factor can be determined efficiently. In this study, a standard Matlab function ‘linprog’, which is based on the algorithm of ‘Linear Interior Point Solver’ [27], is employed because it is found to be efficient and stable in searching for the optimal tension factor. Using this approach, the TF in four different layers of the 8-6-CDPM within its force-closure workspace is shown in Fig. 12, with the moving platform fixed at an orientation of \( \alpha = \beta = \gamma = 0^\circ \).

The TF is plotted by making the size of the squares proportional to its values at discrete positions. The biggest square at the centre in Fig. 12(d) has the value ‘1’, corresponding to the case that the tensions in the cables are equal. This is only attainable when the platform is located at the position \((0.5, 0.5, 0.5)\) with zero orientation. The simulation results show that the nearer the platform is to the base centre, i.e. \( x = y = z = 0.5 \), the larger the TF becomes.

**4.2. Global tension index (GTI)**

In order to evaluate the quality of the whole workspace, a global tension index (GTI) is proposed. The TF is a local measure because it characterizes the tension distribution at a given posture of the moving platform. By integrating the local TF over the workspace, the GTI can be obtained as follows:

\[
\text{GTI} = \frac{\int_V TF(A) dV}{\int_V dV}
\]

where \( V \) denotes the workspace volume of the manipulator.

According to the finite workspace generation, the GTI value can be computed numerically. Eq. (23) can be written separately for both the constant orientation workspace and the total orientation workspace as:
\[
\begin{align*}
\text{GTI} &= \frac{\sum_{i=1}^{n_B} \text{TF}_i(\mathbf{A})}{n_B} \quad \text{for constant orientation WS} \\
\text{GTI} &= \frac{\sum_{i=1}^{n_B} \sum_{j=1}^{n_s} \text{TF}_{ij}(\mathbf{A})}{n_B n_s} \quad \text{for total orientation WS} 
\end{align*}
\]

where \( \text{TF}_i(\mathbf{A}) \) represents the TF of the structure matrix \( \mathbf{A} \) computed from every posture \( i \) in the constant orientation workspace.

\( \text{TF}_{ij}(\mathbf{A}) \) represents the TF of the structure matrix \( \mathbf{A} \) computed from every position \( i \) at every orientation \( j \) in the total orientation workspace.

\( n_s \) is the number of points covering the range of possible orientations.

For task-based optimization problems, the common objective is to design an optimal CDPM in terms of minimum base area/volume so that the resultant workspace can cover all postures of a desired trajectory of the moving platform. Multiple optimal configurations may exist in these problems. For such cases, the GTI can be adapted to evaluate the force-closure property of the trajectory in every configuration. The configuration that results in the best GTI will be considered as the optimal solution.

5. Conclusion

In this paper, two typical types of workspace, namely force-closure workspace and feasible-wrench workspace, for fully restrained cable-driven parallel manipulators \((m \geq n + 1)\) are addressed through tension conditions. Implementing recursive algorithms in checking the tension conditions is found straightforward and efficient. This facilitates the workspace generation and quantification of fully restrained CDPMs. In addition, a tension factor is proposed as a performance index to evaluate the quality of force closure for fully restrained CDPMs at a specific configuration. It is found to be an appropriate measure for CDPMs because it reflects a relative tension distribution among the cables for a specific platform pose inside the force-closure workspace. It has been proved that the nonlinear optimization of TF can be converted into a linear optimization problem. An efficient linear optimization approach is then employed to find the optimal TF. Subsequently, the global tension index was developed to evaluate the quality of tension distribution over the entire feasible-wrench workspace. Our future research will be focused on the workspace optimization issue using the proposed algorithms.

Acknowledgement

The authors would like to thank the School of Mechanical & Aerospace Engineering at Nanyang Technological University for sponsoring this research program.

References

Song Huat Yeo joined the School of Mechanical and Aerospace Engineering at NTU in 1992 and is currently an Associate Professor. He received both his B.Sc. and Ph.D. degrees from the University of Birmingham (UK) in 1983 and 1987 respectively. He has conducted research in mechanisms of gripping, mechanism synthesis, kinematics of parallel robots and expert systems. As a postdoctoral research fellow at the University of Birmingham, he worked on an industrial project on the design and development of computer-controlled high-speed confectionery packaging systems. His research interests are in cable-driven mechanisms, synthesis of mechanisms, wearable haptic devices, and kinematics of reconfigurable robots. Dr. Yeo has published more than 80 technical articles in refereed international journals and conferences. He was voted by students for MAE Teacher of the Year Award in 1998.

Guilin Yang received the B.Eng. degree and M.Eng. degree from Jilin University of Technology (now Jilin University), China, in 1985 and 1988 respectively, and Ph.D. degree from Nanyang Technological University in 1999. Since 1988, he had been with the School of Mechanical Engineering, Shijiazhuang Railway Institute, China, as a lecturer, a division head, and then the vice dean of the school, for nearly seven years. Currently, he is a research scientist and the deputy group manager of the Mechanics Group, Singapore Institute of Manufacturing Technology, Singapore. His current research interests and expertises include computational kinematics, multi-body dynamics, parallel-kinematics machines, modular robots, flexure-based precision mechanisms, electromagnetic actuators, and industrial robot systems. He has 20 years of working experience in the areas of precision engineering, mechatronics, robotics, and automation. He has published over 120 technical papers in refereed journals and conferences. He is a technical committee member of Robotics, IFToMM (International Federation for the Promotion of Mechanism and Machine Science); a co-chair of the technical committee of Rapid Prototyping for Robotics and Automation, IEEE Robotics and Automation Society; and the deputy chair of IEEE Singapore Robotics and Automation Chapter.

I-Ming Chen received the B.S. degree from National Taiwan University in 1986, and M.S. and Ph.D. degrees from California Institute of Technology, Pasadena, CA in 1989 and 1994 respectively. He has been with the School of Mechanical and Aerospace Engineering of Nanyang Technological University (NTU) in Singapore since 1995. He is currently Director of Intelligent Systems Center in NTU, a partnership between Singapore Technology Engineering Ltd. and NTU. He was JSPS Visiting Scholar in Kyoto University, Japan in 1999, Visiting Scholar in the Department of Mechanical Engineering of MIT in 2004, and currently Fellow of Singapore–MIT Alliance under Manufacturing Systems and Technology (MST) Program. He is also Adjunct Professor of Xian Jiao Tong University, China. His research interests are in wearable sensors, human–robot interaction, reconfigurable automation, parallel kinematics machines (PKM), biomorphic underwater robots, and smart-material-based actuators. Dr. Chen has published more than 150 technical articles in refereed international journals and conferences. He is now serving on the editorial boards of IEEE Transactions on Robotics, IEEE/ASME Transactions on Mechatronics and Robotics. He was General Chairman of 2006 IEEE Conferences on Cybernetics, Intelligent Systems, and Robotics (CIS-RAM) in Thailand, and will be General Chairman of 2009 IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM2009) in Singapore. He is a senior member of IEEE and member of ASME, and member of RoboCup Singapore National Committee.