DRAFT

Lecture Notes in:

MATRIX STRUCTURAL ANALYSIS

with an

Introduction to Finite Elements

CVEN4525/5525

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NOTATION

$\mathbf{a}$	Vector of coefficcients in assumed displacement field
A	Area
$\mathcal{A}$	Kinematics Matrix
$\mathbf{b}$	Body force vector
$\mathcal{B}$	Statics Matrix, relating external nodal forces to internal forces
$[\mathbf{B}']$	Statics Matrix relating nodal load to internal forces $\mathbf{p} = [\mathbf{B}']\mathbf{P}$
$[\mathbf{B}]$	Matrix relating assumed displacement fields parameters to joint displacements
C	Cosine
$[C1\|C2]$	Matrices derived from the statics matrix
$\{\mathbf{d}\}$	Element flexibility matrix (lc)
$\{d_c\}$	
$[\mathbf{D}]$	Structure flexibility matrix (GC)
E	Elastic Modulus
$[\mathbf{E}]$	Matrix of elastic constants (Constitutive Matrix)
$\{\mathbf{F}\}$	Unknown element forces and unknown support reactions
$\{\mathbf{F}_0\}$	Nonredundant element forces (lc)
$\{\mathbf{F}_x\}$	Redundant element forces (lc)
$\{\mathbf{F}_e\}$	Element forces (lc)
FEA	Fixed end actions of a restrained member
G	Shear modulus
I	Moment of inertia
$[\mathcal{L}]$	Matrix relating the assumed displacement field parameters to joint displacements
$[\mathbf{I}]$	Idendity matrix
$[\mathbf{ID}]$	Matrix relating nodal dof to structure dof
J	St Venant's torsional constant
$[\mathbf{k}]$	Element stiffness matrix (lc)
$[\mathbf{p}]$	Matrix of coefficients of a polynomial series
$[\mathbf{k}_g]$	Geometric element stiffness matrix (lc)
$[\mathbf{k}_r]$	Rotational stiffness matrix ($[\mathbf{d}]$ inverse)
$[\mathbf{K}]$	Structure stiffness matrix (GC)
$[\mathbf{K}_g]$	Structure's geometric stiffness matrix (GC)
L	Length
$\mathbf{L}$	Linear differential operator relating displacement to strains
l_{ij}	Direction cosine of rotated axis i with respect to original axis j
$\{LM\}$	structure dof of nodes connected to a given element
$\{\mathbf{N}\}$	Shape functions
$\{\mathbf{p}\}$	Element nodal forces = F (lc)
$\{\mathbf{P}\}$	Structure nodal forces (GC)
P, V, M, T	Internal forces acting on a beam column (axial, shear, moment, torsion)
$\mathbf{R}$	Structure reactions (GC)

S	Sine
$\mathbf{t}$	Traction vector
$\widehat{\mathbf{t}}$	Specified tractions along Γ_t
$\mathbf{u}$	Displacement vector
$\tilde{u}$	Neighbour function to $u(x)$
$\widehat{\mathbf{u}}(x)$	Specified displacements along Γ_u
u, v, w	Translational displacements along the x, y, and z directions
U	Strain energy
U^*	Complementary strain energy
x, y	loacal coordinate system (lc)
X, Y	Global coordinate system (GC)
W	Work
α	Coefficient of thermal expansion
$[\mathbf{\Gamma}]$	Transformation matrix
$\{\boldsymbol{\delta}\}$	Element nodal displacements (lc)
$\{\overline{\mathbf{\Delta}}\}$	Nodal displacements in a continuous system
$\{\mathbf{\Delta}\}$	Structure nodal displacements (GC)
$\boldsymbol{\epsilon}$	Strain vector
$\boldsymbol{\epsilon}_0$	Initial strain vector
$\{\mathbf{\Upsilon}\}$	Element relative displacement (lc)
$\{\mathbf{\Upsilon}_0\}$	Nonredundant element relative displacement (lc)
$\{\mathbf{\Upsilon}_x\}$	Redundant element relative displacement (lc)
θ	rotational displacement with respect to z direction (for 2D structures)
δ	Variational operator
δM	Virtual moment
δP	Virtual force
$\delta\theta$	Virtual rotation
δu	Virtual displacement
$\delta\phi$	Virtual curvature
δU	Virtual internal strain energy
δW	Virtual external work
$\delta\boldsymbol{\epsilon}$	Virtual strain vector
$\delta\boldsymbol{\sigma}$	Virtual stress vector
Γ	Surface
Γ_t	Surface subjected to surface tractions
Γ_u	Surface associated with known displacements
$\boldsymbol{\sigma}$	Stress vector
$\boldsymbol{\sigma}_0$	Initial stress vector
Ω	Volume of body

lc: Local Coordinate system
GC: Global Coordinate System

Chapter 1

INTRODUCTION

1.1 Why Matrix Structural Analysis?

1 In most Civil engineering curriculum, students are required to take courses in: Statics, Strength of Materials, Basic Structural Analysis. This last course is a fundamental one which introduces basic structural analysis (determination of reactions, deflections, and internal forces) of both statically determinate and indeterminate structures.

2 Also Energy methods are introduced, and most if not all examples are two dimensional. Since the emphasis is on hand solution, very seldom are three dimensional structures analyzed. The methods covered, for the most part lend themselves for "back of the envelope" solutions and not necessarily for computer implementation.

3 Those students who want to pursue a specialization in structural engineering/mechanics, do take more advanced courses such as Matrix Structural Analysis and/or Finite Element Analysis.

4 Matrix Structural Analysis, or Advanced Structural Analysis, or Introduction to Structural Engineering Finite Element, builds on the introductory analysis course to focus on those methods which lend themselves to computer implementation. In doing so, we will place equal emphasis on both two and three dimensional structures, and develop a thorough understanding of computer aided analysis of structures.

5 This is essential, as in practice most, if not all, structural analysis are done by the computer and it is imperative that as structural engineers you understand what is inside those "black boxes", develop enough self assurance to be capable of opening them and modify them to perform certain specific tasks, and most importantly to understand their limitations.

6 With the recently placed emphasis on the finite element method in most graduate schools, many students have been tempted to skip a course such as this one and rush into a finite element one. Hence it is important that you understand the connection and role of those two courses. The Finite Element Method addresses the analysis of two or three dimensional continuum. As such, the primary unknowns is $\mathbf{u}$ the nodal displacements, and internal "forces" are usually

restricted to stress $\boldsymbol{\sigma}$. The only analogous one dimensional structure is the truss.

7 Whereas two and three dimensional continuum are essential in civil engineering to model structures such as dams, shells, and foundation, the majority of Civil engineering structures are constituted by "rod" one-dimensional elements such as beams, girders, or columns. For those elements, "displacements" and internal "forces" are somehow more complex than those encountered in continuum finite elements.

8 Hence, contrarily to continuum finite element where displacement is mostly synonymous with translation, in one dimensional elements, and depending on the type of structure, generalized displacements may include translation, and/or flexural and/or torsional rotation. Similarly, "internal forces" are not stresses, but rather axial and shear forces, and/or flexural or torsional moments. Those concepts are far more relevant in the analysis/design of most civil engineering structures.

9 Hence, Matrix Structural Analysis, is truly a bridge course between introductory analysis and finite element courses. The element stiffness matrix $[\mathbf{k}]$ will first be derived using methods introduced in basic structural analysis, and later using energy based concepts. This later approach is the one exclusively used in the finite element method.

10 An important component of this course is computer programing. Once the theory and the algorithms are thoroughly explained, you will be expected to program them in either Fortran (preferably 90) or C (sorry, but no Basic) on the computer of your choice. The program (typically about 3,500 lines) will perform the analysis of 2 and 3 dimensional truss and frame structures, and many students have subsequently used it in their professional activities.

11 There will be one computer assignment in which you will be expected to perform simple symbolic manipulations using *Mathematica*. For those of you unfamiliar with the *Bechtel Laboratory*, there will be a special session to introduce you to the operation of Unix on Sun workstations.

1.2 Overview of Structural Analysis

12 To put things into perspective, it may be helpful to consider classes of Structural Analysis which are distinguished by:

1. Excitation model
   (a) Static
   (b) Dynamic
2. Structure model
   (a) Global geometry
      - small deformation ($\boldsymbol{\epsilon} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$)

   - large deformation ($\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$, P-$\Delta$ effects), chapter 13

   (b) Structural elements element types:
   - 1D framework (truss, beam, columns)
   - 2D finite element (plane stress, plane strain, axisymmetric, plate or shell elements), chapter 12
   - 3D finite element (solid elements)

   (c) Material Properties:
   - Linear
   - Nonlinear

   (d) Sectional properties:
   - Constant
   - Variable

   (e) Structural connections:
   - Rigid
   - Semi-flexible (linear and non-linear)

   (f) Structural supports:
   - Rigid
   - Elastic

3. Type of solution:

   (a) Continuum, analytical, Partial Differential Equation

   (b) Discrete, numerical, Finite ELement, Finite Difference, Boundary Element

13 Structural design must satisfy:

1. Strength ($\sigma < \sigma_f$)
2. Stiffness (“small” deformations)
3. Stability (buckling, cracking)

14 Structural analysis must satisfy

1. Statics (equilibrium)
2. Mechanics (stress-strain or force displacement relations)
3. Kinematics (compatibility of displacement)

1.3 Structural Idealization

15 Prior to analysis, a structure must be idealized for a suitable mathematical representation. Since it is practically impossible (and most often unnecessary) to model every single detail, assumptions must be made. Hence, structural idealization is as much an art as a science. *Some* of the questions confronting the analyst include:

1. Two dimensional versus three dimensional; Should we model a single bay of a building, or the entire structure?
2. Frame or truss, can we neglect flexural stiffness?
3. Rigid or semi-rigid connections (most important in steel structures)
4. Rigid supports or elastic foundations (are the foundations over solid rock, or over clay which may consolidate over time)
5. Include or not secondary members (such as diagonal braces in a three dimensional analysis).
6. Include or not axial deformation (can we neglect the axial stiffness of a beam in a building?)
7. Cross sectional properties (what is the moment of inertia of a reinforced concrete beam?)
8. Neglect or not haunches (those are usually present in zones of high negative moments)
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations (In the analysis of a high rise building subjected to wind load, the moments should be amplified by the product of the axial load times the lateral deformation, $P-\Delta$ effects).
11. Time dependent effects (such as creep, which is extremely important in prestressed concrete, or cable stayed concrete bridges).
12. Partial collapse or local yielding (would the failure of a single element trigger the failure of the entire structure?).
13. Load static or dynamic (when should a dynamic analysis be performed?).
14. Wind load (the lateral drift of a high rise building subjected to wind load, is often the major limitation to higher structures).
15. Thermal load (can induce large displacements, specially when a thermal gradient is present.).
16. Secondary stresses (caused by welding. Present in most statically indeterminate structures).

1.3.1 Structural Discretization

16 Once a structure has been idealized, it must be discretized to lend itself for a mathematical representation which will be analyzed by a computer program. This discretization should uniquely define each node, and member.

17 The node is characterized by its nodal id (node number), coordinates, boundary conditions, and load (this one is often defined separately), Table 1.1. Note that in this case we have two

Node No.	Coor.		B. C.		
	X	Y	X	Y	Z
1	0.	0.	1	1	0
2	5.	5.	0	0	0
3	20.	5.	0	0	0
4	25.	2.5	1	1	1

Table 1.1: Example of Nodal Definition

nodal coordinates, and three degrees of freedom (to be defined later) per node. Furthermore, a 0 and a 1 indicate unknown or known displacement. Known displacements can be zero (restrained) or non-zero (as caused by foundation settlement).

18 The element is characterized by the nodes which it connects, and its group number, Table 1.2.

Element No.	From Node	To Node	Group Number
1	1	2	1
2	3	2	2
3	3	4	2

Table 1.2: Example of Element Definition

19 Group number will then define both element type, and elastic/geometric properties. The last one is a pointer to a separate array, Table 1.3. In this example element 1 has element code 1 (such as beam element), while element 2 has a code 2 (such as a truss element). Material group 1 would have different elastic/geometric properties than material group 2.

20 From the analysis, we first obtain the nodal displacements, and then the element internal forces. Those internal forces vary according to the element type. For a two dimensional frame, those are the axial and shear forces, and moment at each node.

21 Hence, the need to define two coordinate systems (one for the entire structure, and one for

Group No.	Element Type	Material Group
1	1	1
2	2	1
3	1	2

Table 1.3: Example of Group Number

each element), and a sign convention become apparent.

1.3.2 Coordinate Systems

22 We should differentiate between 2 coordinate systems:

Global: to describe the structure nodal coordinates. This system can be arbitrarily selected provided it is a Right Hand Side (RHS) one, and we will associate with it upper case axis labels, X, Y, Z, Fig. 1.1 or 1,2,3 (running indeces within a computer program).

Local: system is associated with each element and is used to describe the element internal forces. We will associate with it lower case axis labels, x, y, z (or 1,2,3), Fig. 1.2.

23 The x-axis is assumed to be along the member, and the direction is chosen such that it points from the 1st node to the 2nd node, Fig. 1.2.

24 Two dimensional structures will be defined in the X-Y plane.

1.3.3 Sign Convention

25 The sign convention in structural analysis is completely different than the one previously adopted in structural analysis/design, Fig. 1.3 (where we focused mostly on flexure and defined a positive moment as one causing "tension below". This would be awkward to program!).

26 In matrix structural analysis the sign convention adopted is consistent with the prevailing coordinate system. Hence, we define a positive moment as one which is counter-clockwise, Fig. 1.3

27 Fig. 1.4 illustrates the sign convention associated with each type of element.

28 Fig. 1.4 also shows the geometric (upper left) and elastic material (upper right) properties associated with each type of element.

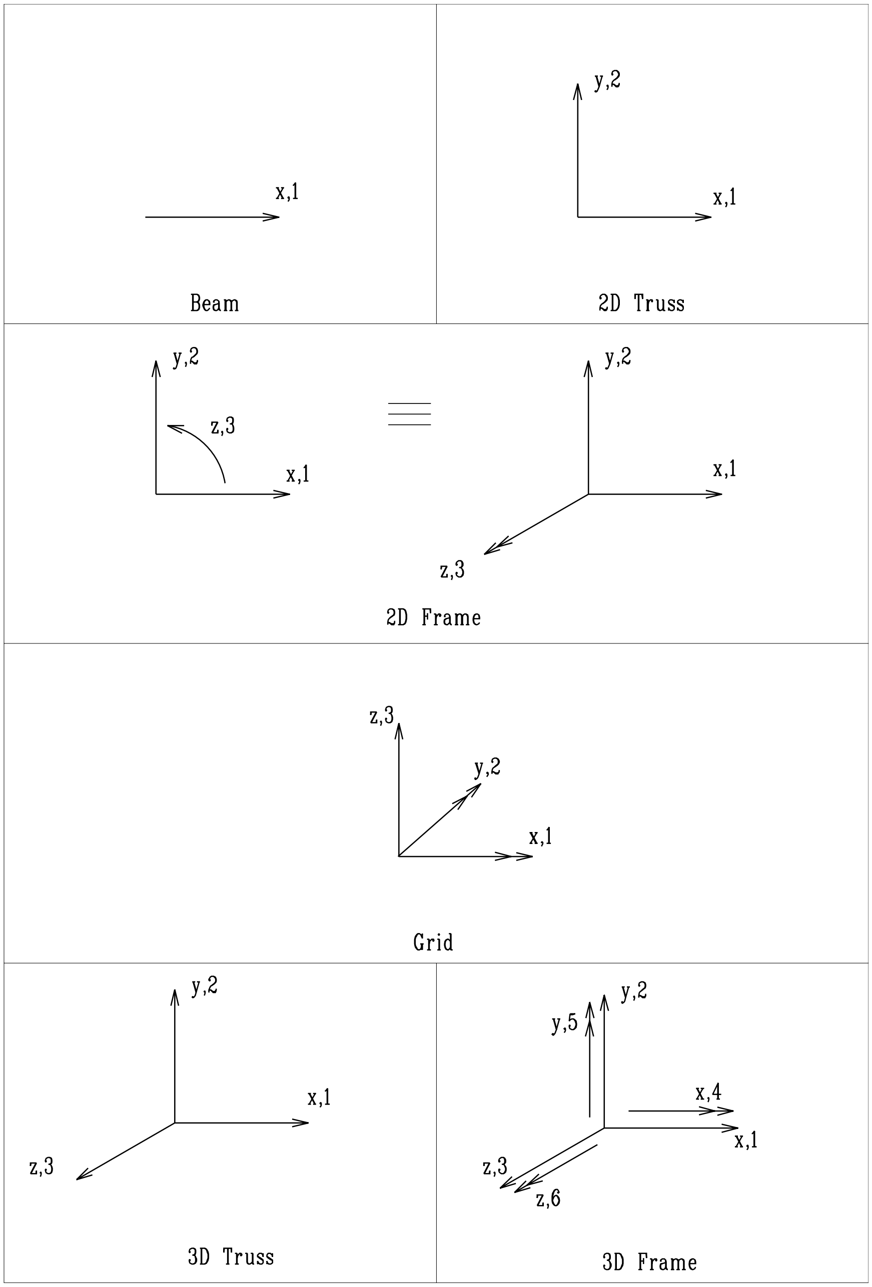


Figure 1.1: Global Coordinate System

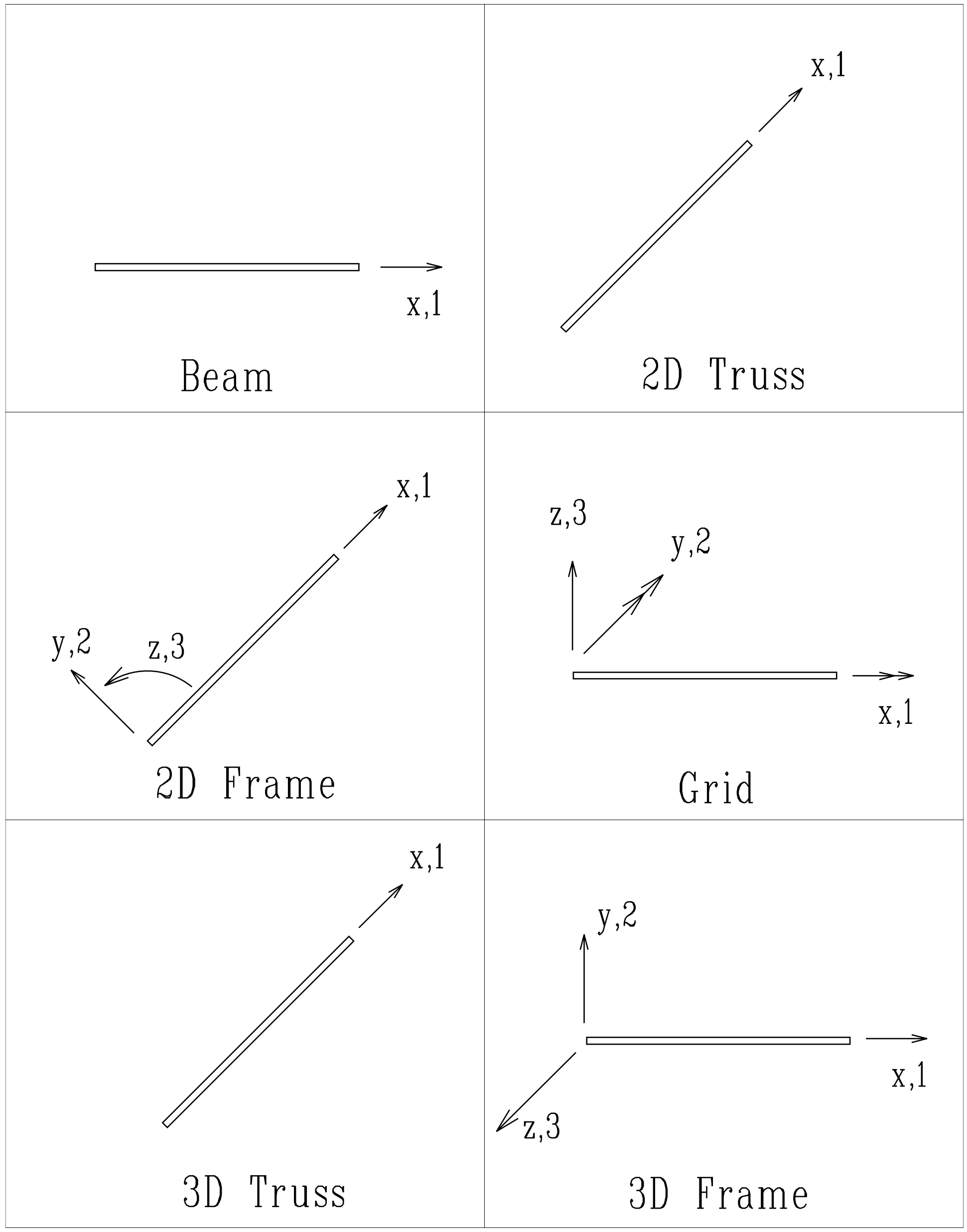


Figure 1.2: Local Coordinate Systems

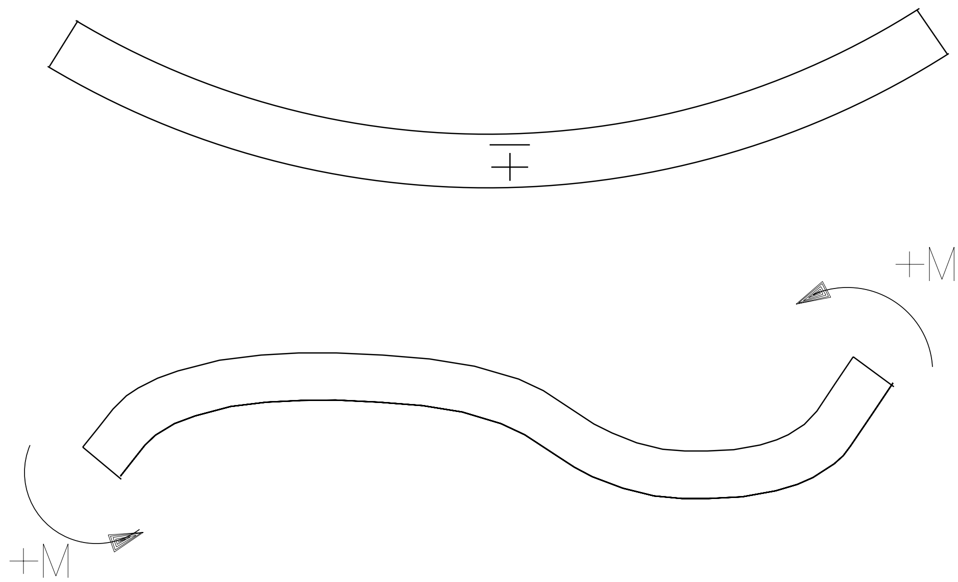


Figure 1.3: Sign Convention, Design and Analysis

1.4 Degrees of Freedom

29 A degree of freedom (d.o.f.) is an independent generalized nodal displacement of a node.

30 The displacements must be linearly independent and thus not related to each other. For example, a roller support on an inclined plane would have three displacements (rotation θ, and two translations u and v), however since the two displacements are kinematically constrained, we only have two independent displacements, Fig. 1.5.

31 We note that we have been referring to *generalized* displacements, because we want this term to include translations as well as rotations. Depending on the type of structure, there may be none, one or more than one such displacement. It is unfortunate that in most introductory courses in structural analysis, too much emphasis has been placed on two dimensional structures, and not enough on either three dimensional ones, or two dimensional ones with torsion.

32 In most cases, there is the same number of d.o.f in local coordinates as in the global coordinate system. One notable exception is the truss element. In local coordinate we can only have one axial deformation, whereas in global coordinates there are two or three translations in 2D and 3D respectively for each node.

33 Hence, it is essential that we understand the degrees of freedom which can be associated with the various types of structures made up of one dimensional rod elements, Table 1.4.

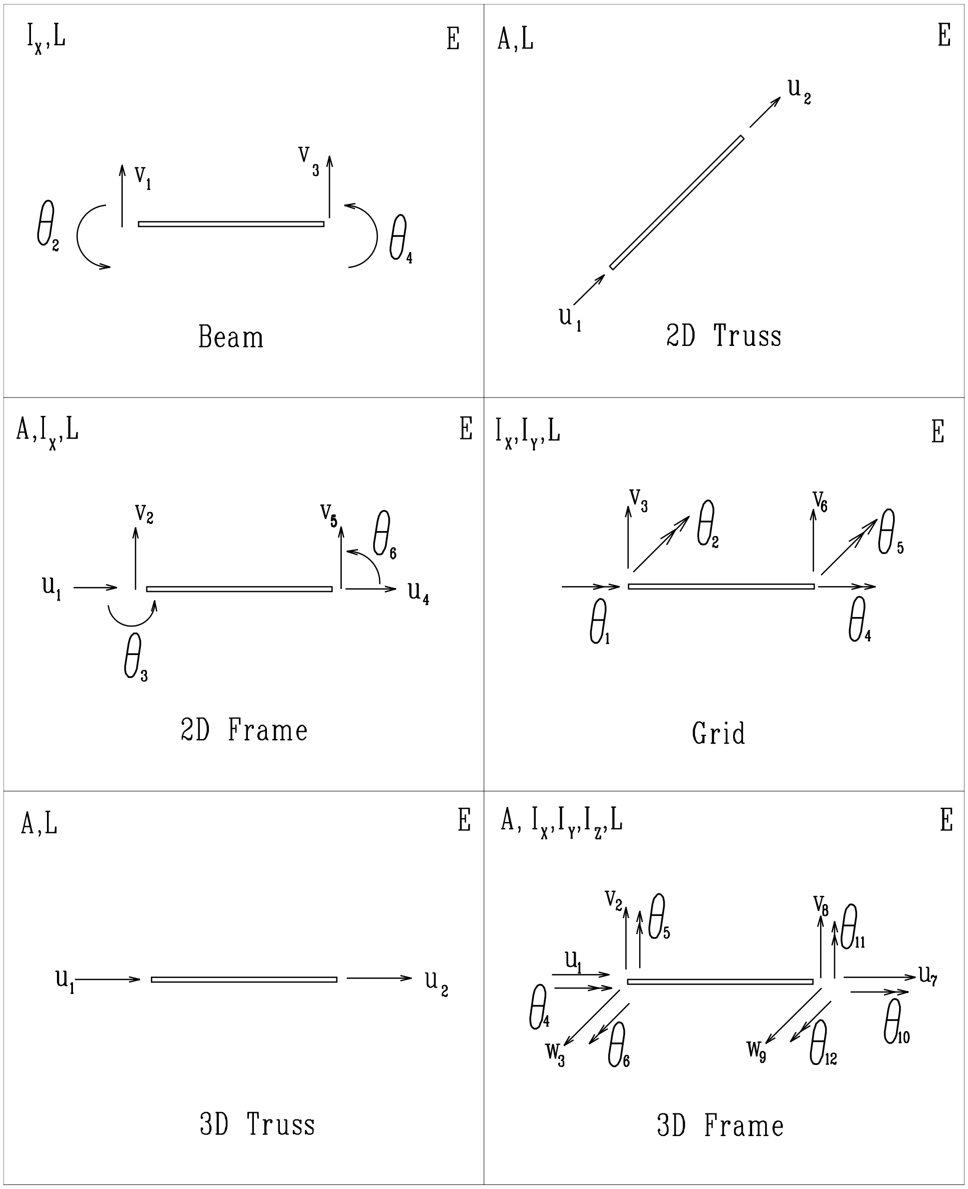


Figure 1.4: Total Degrees of Freedom for various Type of Elements

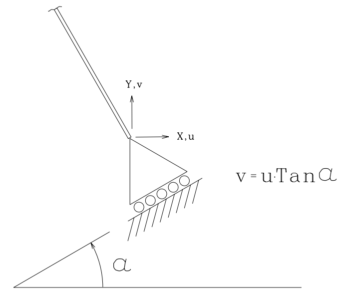


Figure 1.5: Independent Displacements

34 This table shows the degree of freedoms and the corresponding generalized forces.

35 We should distinguish between local and global d.o.f.'s. The numbering scheme follows the following simple rules:

Local: d.o.f. for a given element: Start with the first node, number the local d.o.f. in the same order as the subscripts of the relevant local coordinate system, and repeat for the second node.

Global: d.o.f. for the entire structure: Starting with the 1st node, number all the unrestrained global d.o.f.'s, and then move to the next one until all global d.o.f have been numbered, Fig. 1.6.

1.5 Course Organization

Type		Node 1	Node 2	$[\mathbf{k}]$ (Local)	$[\mathbf{K}]$ (Global)
1 Dimensional					
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\boldsymbol{\delta}\}$	u_1	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathbf{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}	2×2	6×6
	$\{\boldsymbol{\delta}\}$	u_1,	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}	12×12	12×12
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}		

Table 1.4: Degrees of Freedom of Different Structure Types Systems

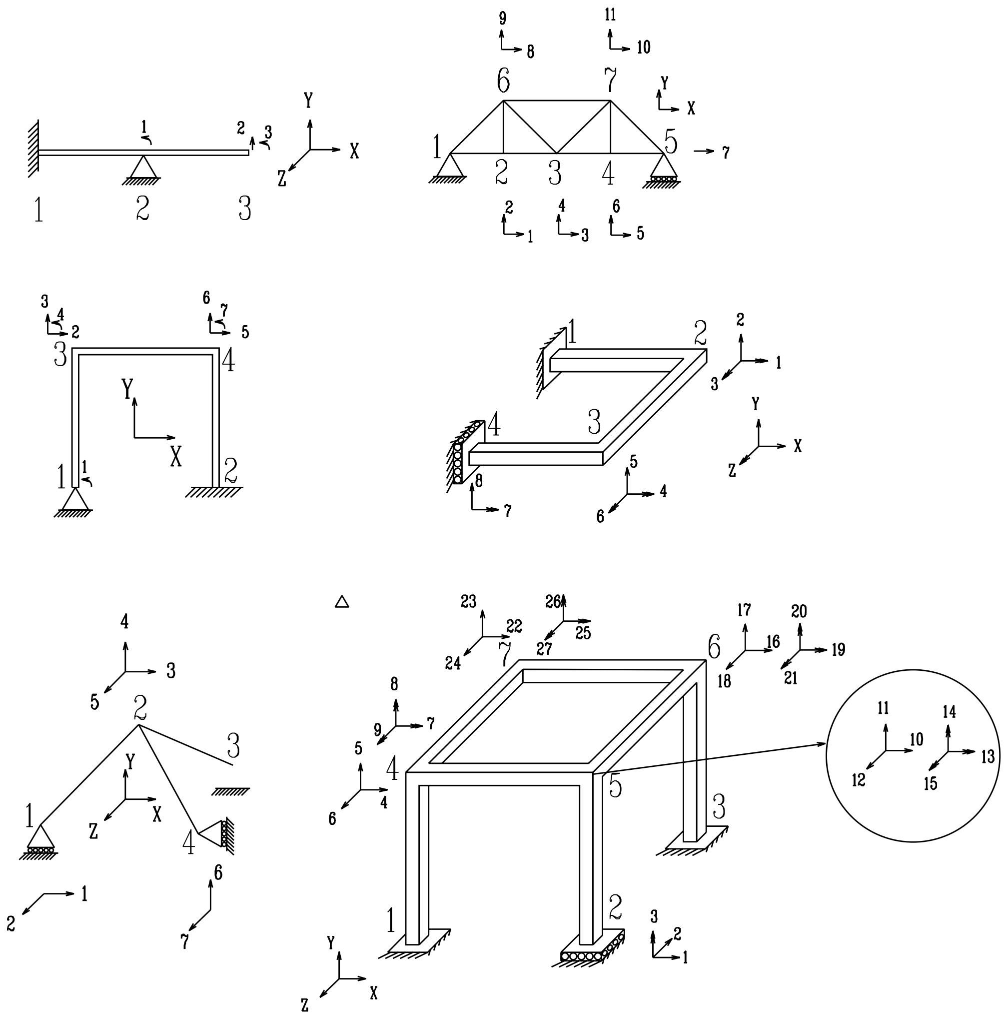


Figure 1.6: Examples of Global Degrees of Freedom

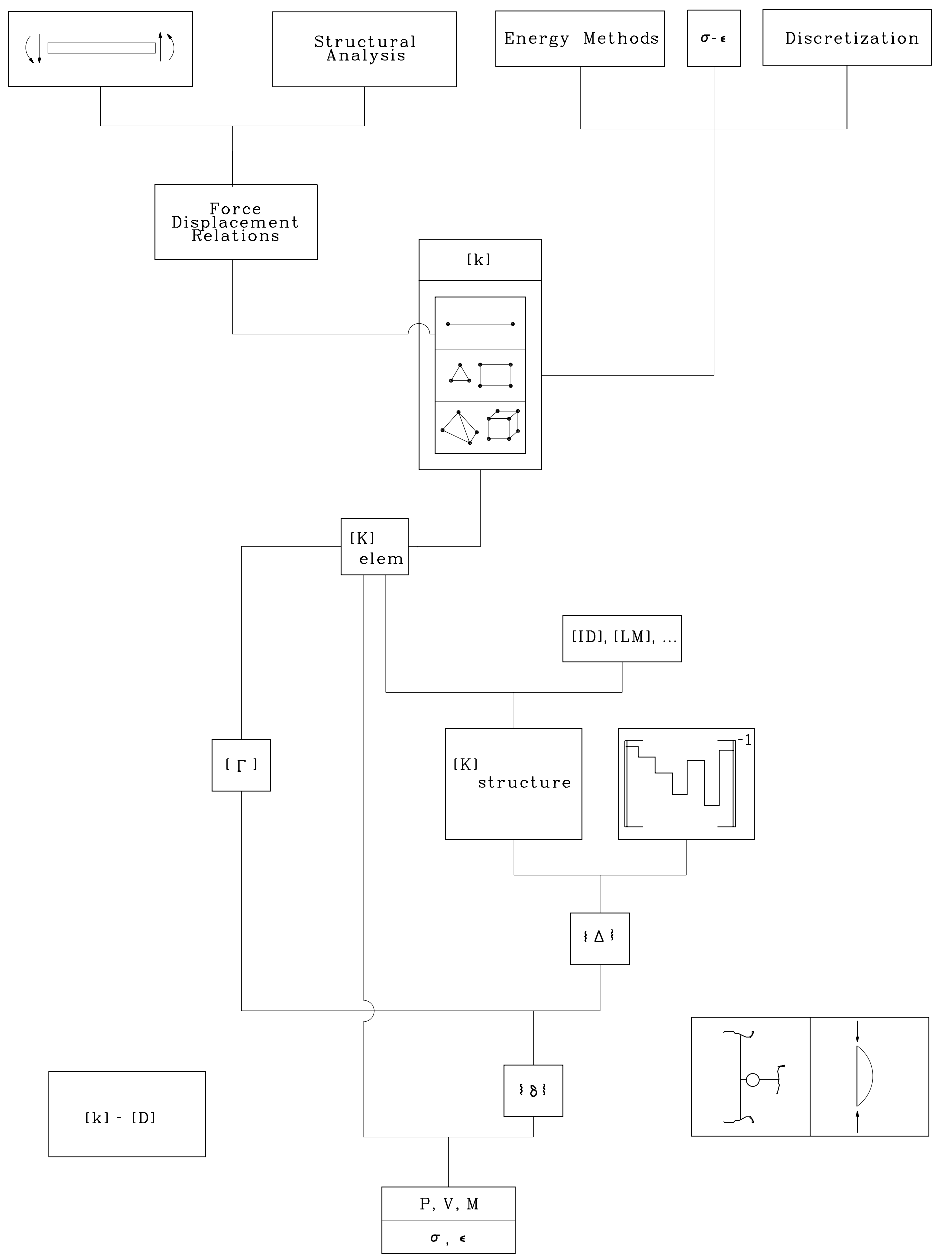


Figure 1.7: Organization of the Course

Part I

Matrix Structural Analysis of Framed Structures

Draft

Chapter 2

ELEMENT STIFFNESS MATRIX

2.1 Introduction

1 In this chapter, we shall derive the element stiffness matrix [**k**] of various one dimensional elements. Only after this important step is well understood, we could expand the theory and introduce the structure stiffness matrix [**K**] in its global coordinate system.

2 As will be seen later, there are two fundamentally different approaches to derive the stiffness matrix of one dimensional element. The first one, which will be used in this chapter, is based on classical methods of structural analysis (such as moment area or virtual force method). Thus, in deriving the element stiffness matrix, we will be reviewing concepts earlier seen.

3 The other approach, based on energy consideration through the use of assumed shape functions, will be examined in chapter 11. This second approach, exclusively used in the finite element method, will also be extended to two and three dimensional continuum elements.

2.2 Influence Coefficients

4 In structural analysis an influence coefficient C_{ij} can be defined as the effect on d.o.f. i due to a unit action at d.o.f. j for an individual element or a whole structure. Examples of Influence Coefficients are shown in Table 2.1.

5 It should be recalled that influence lines are associated with the analysis of structures subjected to moving loads (such as bridges), and that the flexibility and stiffness coefficients are components of matrices used in structural analysis.

	Unit Action	Effect on
Influence Line	Load	Shear
Influence Line	Load	Moment
Influence Line	Load	Deflection
Flexibility Coefficient	Load	Displacement
Stiffness Coefficient	Displacement	Load

Table 2.1: Examples of Influence Coefficients

2.3 Flexibility Matrix (Review)

6 Considering the simply supported beam shown in Fig. 2.1, and using the local coordinate system, we have

$$\left\{ \begin{array}{c} \delta_1 \\ \delta_2 \end{array} \right\} = \left[\begin{array}{cc} d_{11} & d_{12} \\ d_{21} & d_{22} \end{array} \right] \left\{ \begin{array}{c} p_1 \\ p_2 \end{array} \right\} \tag{2.1}$$

Using the virtual work, or more specifically, the virtual force method to analyze this problem, (more about energy methods in Chapter 9), we have:

$$\underbrace{\int_0^l \delta\overline{M}\,\frac{M}{EI_z}dx}_{\text{Internal}} = \underbrace{\delta\overline{P}\,\Delta + \delta\overline{M}\,\theta}_{\text{External}} \tag{2.2}$$

where $\delta\overline{M}$, $\frac{M}{EI_z}$, $\delta\overline{P}$ and Δ are the virtual internal force, real internal displacement, virtual external load, and real external displacement respectively. Here, both the external virtual force and moment are usualy taken as unity.

Virtual Force:

$$\left.\begin{array}{l} \left.\begin{array}{rcl} \delta U &=& \int \delta\overline{\sigma}_x \, \varepsilon_x \, d\text{vol} \\ \delta\overline{\sigma}_x &=& \frac{\overline{M}_x y}{I} \\ \varepsilon_x &=& \frac{\sigma_x}{E} = \frac{My}{EI} \\ \int y^2 d\text{A} &=& I \\ d\text{vol} &=& d\text{A}dx \end{array}\right\} \delta U = \int_0^l \delta\overline{M}\frac{M}{EI}dx \\ \delta V = \delta\overline{P}\Delta \\ \delta U = \delta V \end{array}\right\} \int_0^l \delta\overline{M}\frac{M}{EI}dx = \delta\overline{P}\Delta \tag{2.3}$$

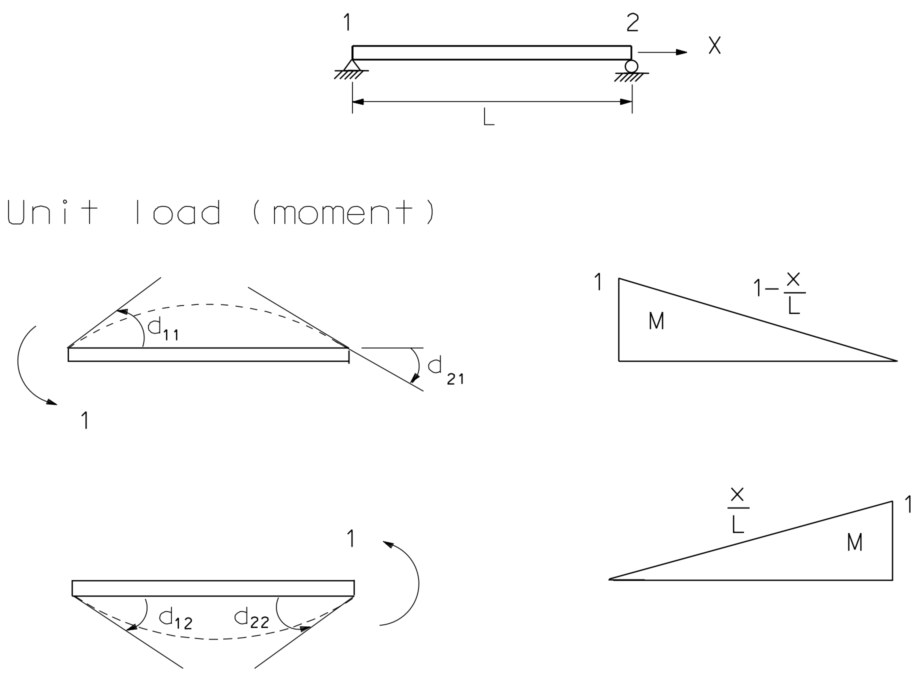


Figure 2.1: Example for Flexibility Method

Hence:

$$EI \underbrace{1}_{\delta\overline{M}} \underbrace{d_{11}}_{\Delta} = \underbrace{\int_0^L \left(1 - \frac{x}{L}\right)^2 dx}_{\delta\overline{M}\cdot M} = \frac{L}{3} \tag{2.4}$$

Similarly, we would obtain:

$$EId_{22} = \int_0^L \left(\frac{x}{L}\right)^2 dx = \frac{L}{3} \tag{2.5}$$

$$EId_{12} = \int_0^L \left(1 - \frac{x}{L}\right) \frac{x}{L} dx = -\frac{L}{6} = EId_{21} \tag{2.6}$$

7 Those results can be summarized in a matrix form as:

$$\boxed{[\mathbf{d}] = \frac{L}{6EI_z} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \tag{2.7}$$

8 The flexibility method will be covered in more detailed, in chapter 7.

2.4 Stiffness Coefficients

9 In the flexibility method, we have applied a unit force at a time and determined all the induced displacements in the statically determinate structure.

10 In the stiffness method, we

1. Constrain all the degrees of freedom
2. Apply a unit displacement at each d.o.f. (while restraining all others to be zero)
3. Determine the reactions associated with all the d.o.f.

$$\boxed{\{\mathbf{p}\} = [\mathbf{k}]\{\boldsymbol{\delta}\}} \tag{2.8}$$

11 Hence k_{ij} will correspond to the reaction at dof i due to a unit deformation (translation or rotation) at dof j, Fig. 2.2.

12 The actual stiffness coefficients are shown in Fig. 2.3 for truss, beam, and grid elements in terms of elastic and geometric properties.

13 In the next sections, we shall derive those stiffness coefficients.

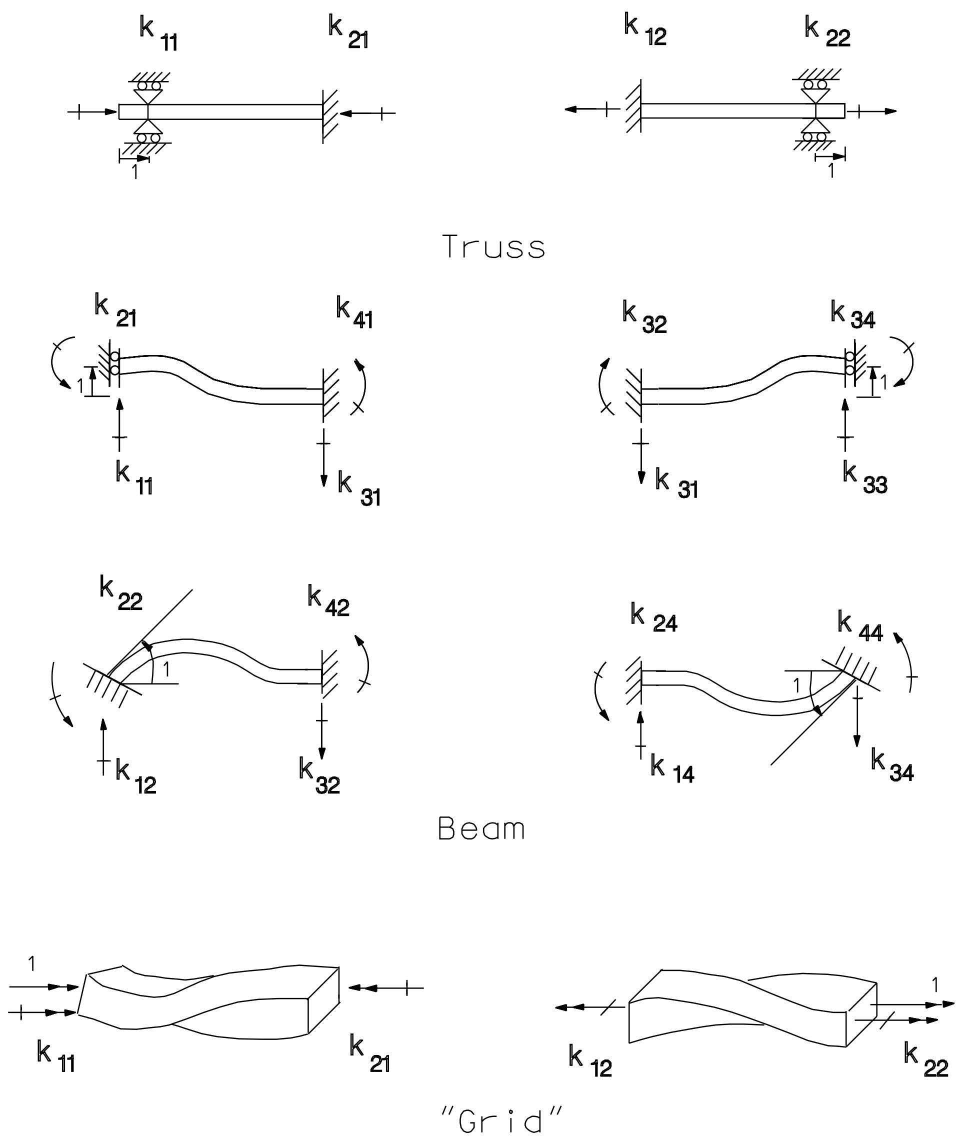


Figure 2.2: Definition of Element Stiffness Coefficients

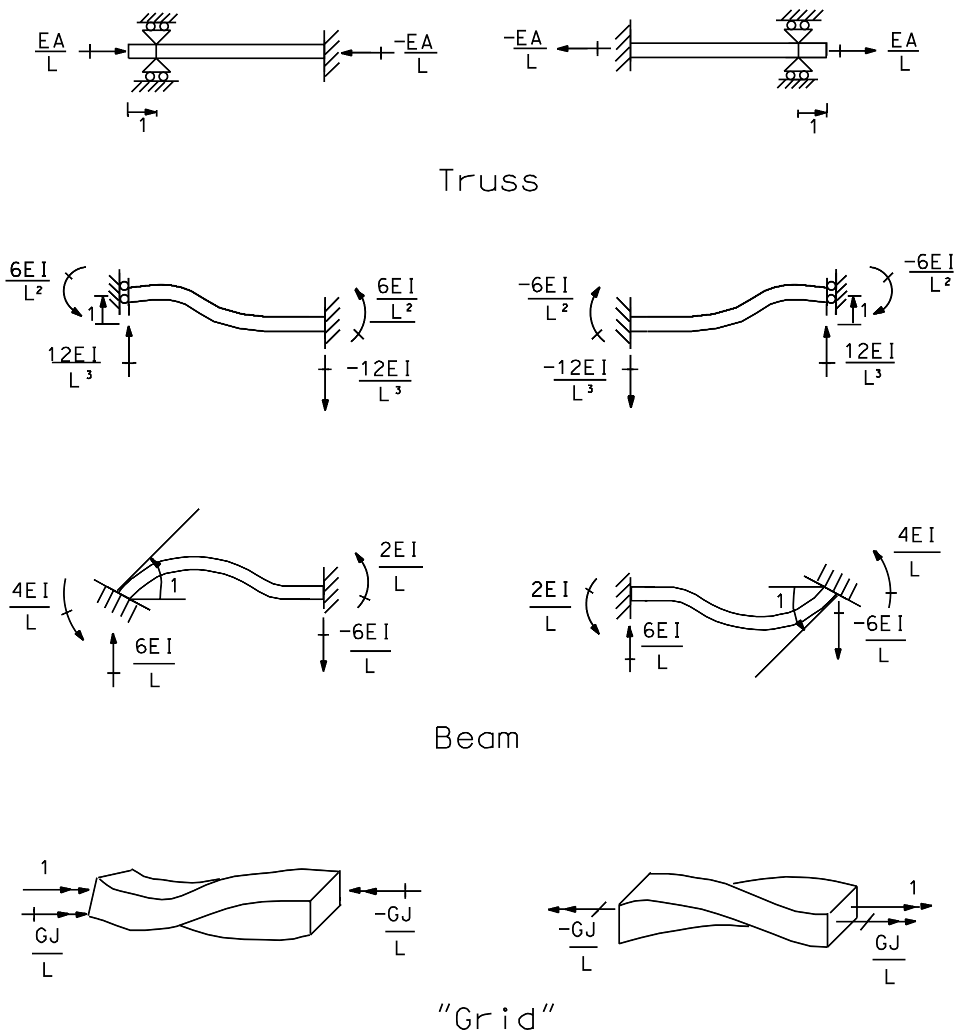


Figure 2.3: Stiffness Coefficients for One Dimensional Elements

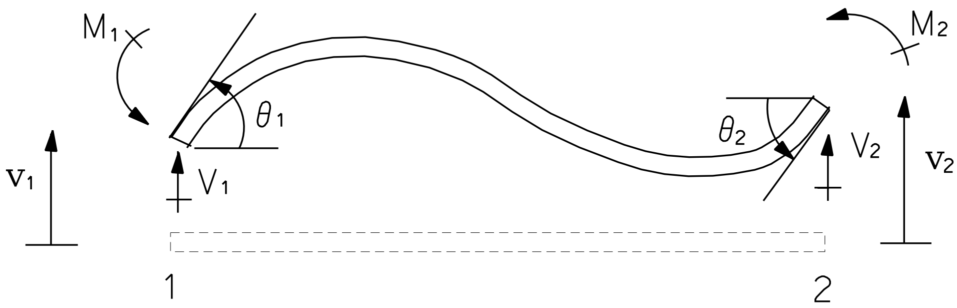


Figure 2.4: Flexural Problem Formulation

2.5 Force-Displacement Relations

2.5.1 Axial Deformations

14 From strength of materials, the force/displacement relation in axial members is

$$
\begin{aligned}
\sigma &= E\epsilon \\
\underbrace{A\sigma}_{P} &= \frac{AE}{L}\underbrace{\Delta}_{1}
\end{aligned}
\tag{2.9}
$$

Hence, for a unit displacement, the applied force should be equal to $\frac{AE}{L}$. From statics, the force at the other end must be equal and opposite.

2.5.2 Flexural Deformation

15 Our objective is to seek a relation for the shear and moments at each end of a beam, in terms of known displacements and rotations at each end.

$$V_1 = V_1(v_1, \theta_1, v_2, \theta_2) \tag{2.10-a}$$
$$M_1 = M_1(v_1, \theta_1, v_2, \theta_2) \tag{2.10-b}$$
$$V_2 = V_2(v_1, \theta_1, v_2, \theta_2) \tag{2.10-c}$$
$$M_2 = M_2(v_1, \theta_1, v_2, \theta_2) \tag{2.10-d}$$

16 We start from the differential equation of a beam, Fig. 2.4 in which we have all positive known displacements, we have from strength of materials

$$M = -EI\frac{d^2v}{dx^2} = M_1 - V_1 x \tag{2.11}$$

17 Integrating twice

$$-EIv' = M_1 x - \frac{1}{2}V_1 x^2 + C_1 \tag{2.12}$$

$$-EIv = \frac{1}{2}M_1 x^2 - \frac{1}{6}V_1 x^3 + C_1 x + C_2 \tag{2.13}$$

18 Applying the boundary conditions at $x = 0$

$$\left.\begin{array}{lcl} v' & = & \theta_1 \\ v & = & v_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{lcl} C_1 & = & -EI\theta_1 \\ C_2 & = & -EIv_1 \end{array}\right. \tag{2.14}$$

19 Applying the boundary conditions at $x = L$ and combining with the expressions for C_1 and C_2

$$\left.\begin{array}{lcl} v' & = & \theta_2 \\ v & = & v_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{lcl} -EI\theta_2 & = & M_1 L - \frac{1}{2}V_1 L^2 - EI\theta_1 \\ -EIv_2 & = & \frac{1}{2}M_1 L^2 - \frac{1}{6}V_1 L^3 - EI\theta_1 L - EIv_1 \end{array}\right. \tag{2.15}$$

20 Since equilibrium of forces and moments must be satisfied, we have:

$$V_1 + V_2 = 0 \qquad M_1 - V_1 L + M_2 = 0 \tag{2.16}$$

or

$$V_1 = \frac{(M_1 + M_2)}{L} \qquad V_2 = -V_1 \tag{2.17}$$

21 Substituting V_1 into the expressions for θ_2 and v_2 in Eq. 2.15 and rearranging

$$\left\{\begin{array}{lcl} M_1 - M_2 & = & \frac{2EI_z}{L}\theta_1 - \frac{2EI_z}{L}\theta_2 \\ 2M_1 - M_2 & = & \frac{6EI_z}{L}\theta_1 + \frac{6EI_z}{L^2}v_1 - \frac{6EI_z}{L^2}v_2 \end{array}\right. \tag{2.18}$$

22 Solving those two equations, we obtain:

$$\boxed{\begin{array}{lcll} M_1 & = & \frac{2EI_z}{L}(2\theta_1 + \theta_2) + \frac{6EI_z}{L^2}(v_1 - v_2) & (2.19) \\ M_2 & = & \frac{2EI_z}{L}(\theta_1 + 2\theta_2) + \frac{6EI_z}{L^2}(v_1 - v_2) & (2.20) \end{array}}$$

23 Finally, we can substitute those expressions in Eq. 2.17

$$\boxed{\begin{array}{lcll} V_1 & = & \frac{6EI_z}{L^2}(\theta_1 + \theta_2) + \frac{12EI_z}{L^3}(v_1 - v_2) & (2.21) \\ V_2 & = & -\frac{6EI_z}{L^2}(\theta_1 + \theta_2) - \frac{12EI_z}{L^3}(v_1 - v_2) & (2.22) \end{array}}$$

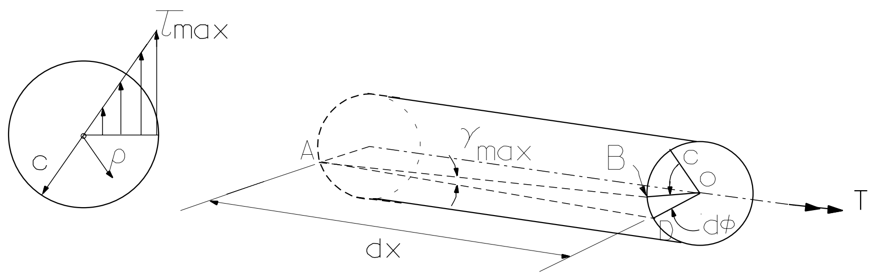


Figure 2.5: Torsion Rotation Relations

2.5.3 Torsional Deformations

24 From Fig. 2.2-d. Since torsional effects are seldom covered in basic structural analysis, and students may have forgotten the derivation of the basic equations from the Strength of Material course, we shall briefly review them.

25 Assuming a linear elastic material, and a linear strain (and thus stress) distribution along the radius of a circular cross section subjected to torsional load, Fig. 2.5 we have:

$$T = \underbrace{\underbrace{\int_A \underbrace{\frac{\rho}{c}\tau_{max}}_{stress} \underbrace{dA}_{area}}_{Force} \underbrace{\rho}_{arm}}_{torque} \tag{2.23}$$

$$= \frac{\tau_{max}}{c} \underbrace{\int_A \rho^2 dA}_{J} \tag{2.24}$$

$$\tau_{max} = \frac{Tc}{J} \tag{2.25}$$

Note the analogy of this last equation with $\sigma = \frac{Mc}{I_z}$.

26 $\int_A \rho^2 dA$ is the polar moment of inertia J. It is also referred to as the St. Venant's torsion constant. For circular cross sections

$$J = \int_A \rho^2 dA = \int_0^c \rho^2 \left(2\pi\rho d\rho\right)$$

$$= \frac{\pi c^4}{2} = \frac{\pi d^4}{32} \tag{2.26}$$

For rectangular sections $b \times d$, and $b < d$, an approximate expression is given by

$$J = kb^3 d \tag{2.27-a}$$

$$k = \frac{0.3}{1+\left(\frac{b}{d}\right)^2} \tag{2.27-b}$$

For other sections, J is often tabulated.

27 Note that J corresponds to I_x where x is the axis along the element.

28 Having developed a relation between torsion and shear stress, we now seek a relation between torsion and torsional rotation. In Fig. 2.5-b, we consider the arc length BD

$$\left.\begin{array}{rcl}\gamma_{max}dx & = & d\Phi c \\ \Rightarrow \frac{d\Phi}{dx} & = & \frac{\gamma_{max}}{c} \\ \gamma_{max} & = & \frac{\tau_{max}}{G}\end{array}\right\} \left.\begin{array}{rcl}\frac{d\Phi}{dx} & = & \frac{\tau_{max}}{Gc} \\ \tau_{max} & = & \frac{TC}{J}\end{array}\right\} \frac{d\Phi}{dx} = \frac{T}{GJ} \tag{2.28}$$

29 Finally, we can rewrite this last equation as $\int T dx = \int G j d\Phi$ and obtain:

$$\boxed{T = \frac{GJ}{L}\Phi} \tag{2.29}$$

Note the similarity between this equation and Equation 2.9.

2.5.4 Shear Deformation

30 In general, shear deformations are quite small. However, for beams with low span to depth ratio, those deformations can not be neglected.

31 Considering an infinitesimal element subjected to shear, Fig. 2.6 and for linear elastic material, the shear strain (assuming small displacement, i.e. $\tan\gamma \approx \gamma$) is given by

$$\tan\gamma \approx \gamma = \underbrace{\frac{dv_s}{dx}}_{Kinematics} = \underbrace{\frac{\tau}{G}}_{Material} \tag{2.30}$$

where $\frac{dv_s}{dx}$ is the slope of the beam neutral axis from the horizontal while the vertical sections remain undeformed, G is the shear modulus, τ the shear stress, and v_s the shear induced displacement.

32 In a beam cross section, the shear stress is not constant. For example for rectangular sections, it varies parabolically, and in I sections, the flange shear components can be neglected.

$$\tau = \frac{VQ}{Ib} \tag{2.31}$$

where V is the shear force, Q is the first moment (or static moment) about the neutral axis of the portion of the cross-sectional area which is outside of the section where the shear stress is

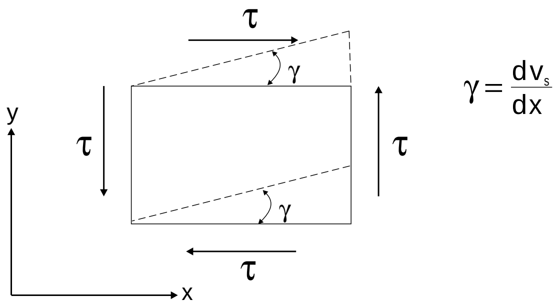


Figure 2.6: Deformation of an Infinitesimal Element Due to Shear

to be determined, I is the moment of inertia of the cross sectional area about the neutral axis, and b is the width of the rectangular beam.

33 The preceding equation can be simplified as

$$\tau = \frac{V}{A_s} \tag{2.32}$$

where A_s is the effective cross section for shear (which is the ratio of the cross sectional area to the area shear factor)

34 Let us derive the expression of A_s for rectangular sections. The exact expression for the shear stress is

$$\tau = \frac{VQ}{Ib} \tag{2.33}$$

where Q is the moment of the area from the external fibers to y with respect to the neutral axis; For a rectangular section, this yields

$$\begin{aligned} \tau &= \frac{VQ}{Ib} && \text{(2.34-a)} \\ &= \frac{V}{Ib}\int_y^{h/2} by'dy' = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right) && \text{(2.34-b)} \\ &= \frac{6V}{bh^3}\left(\frac{h^2}{4} - 4\right) && \text{(2.34-c)} \end{aligned}$$

and we observe that the shear stress is zero for $y = h/2$ and maximum at the neutral axis where it is equal to $1.5\frac{V}{bh}$.

To determine the form factor λ of a rectangular section

$$\left.\begin{array}{rcl}\tau & = & \frac{VQ}{Ib} \\ & = & k\frac{V}{A} \\ Q & = & \displaystyle\int_y^{h/2} by'dy' = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)\end{array}\right\} \left.\begin{array}{rcl} k & = & \frac{A}{2I}\left(\frac{h^2}{4} - y^2\right) \\ \lambda \underbrace{bh}_{A} & = & \displaystyle\int_A k^2 dydz\end{array}\right\} \boxed{\lambda = 1.2} \tag{2.35}$$

Thus, the form factor λ may be taken as 1.2 for rectangular beams of ordinary proportions, and $\boxed{A_s = 1.2A}$

For I beams, k can be also approximated by 1.2, provided A is the area of the web.

35 Combining Eq. 2.31 and 2.32 we obtain

$$\frac{dv_s}{dx} = \frac{V}{GA_s} \tag{2.36}$$

Assuming V to be constant, we integrate

$$v_s = \frac{V}{GA_s}x + C_1 \tag{2.37}$$

36 If the displacement v_s is zero at the opposite end of the beam, then we solve for C_1 and obtain

$$v_s = -\frac{V}{GA_s}(x - L) \tag{2.38}$$

37 We define

$$\boxed{\begin{array}{rcll}\Phi & \stackrel{\text{def}}{=} & \frac{12EI}{GA_sL^2} & (2.39) \\ & = & 24(1+\nu)\frac{A}{A_s}\left(\frac{r}{L}\right)^2 & (2.40)\end{array}}$$

Hence for small slenderness ratio $\frac{r}{L}$ compared to unity, we can neglect Φ.

38 Next, we shall consider the effect of shear deformations on both translations and rotations

Effect on Translation Due to a unit vertical translation, the end shear force is obtained from Eq. 2.21 and setting $v_1 = 1$ and $\theta_1 = \theta_2 = v_2 = 0$, or $V = \frac{12EI_z}{L^3}$. At $x = 0$ we have, Fig. 2.7

$$\left.\begin{array}{rcl} v_s & = & \frac{VL}{GA_s} \\ V & = & \frac{12EI_z}{L^3} \\ \Phi & = & \frac{12EI}{GA_sL^2}\end{array}\right\} \boxed{v_s = \Phi} \tag{2.41}$$

Hence, the shear deformation has increased the total translation from 1 to $1+\Phi$. Similar arguments apply to the translation at the other end.

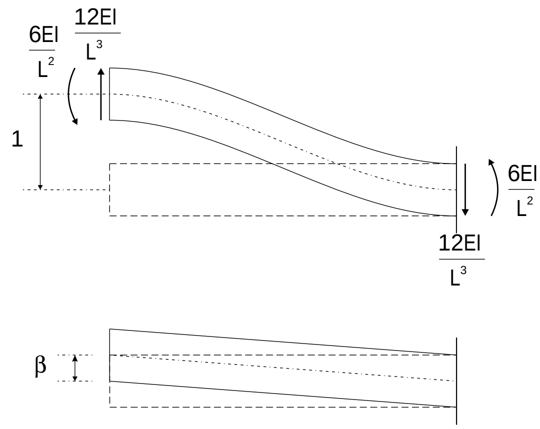


Figure 2.7: Effect of Flexure and Shear Deformation on Translation at One End

Effect on Rotation Considering the beam shown in Fig. 2.8, even when a rotation θ_1 is applied, an internal shear force is induced, and this in turn is going to give rise to shear deformations (translation) which must be accounted for. The shear force is obtained from Eq. 2.21 and setting $\theta_1 = 1$ and $\theta_2 = v_1 = v_2 = 0$, or $V = \frac{6EI_z}{L^2}$. As before

$$\left.\begin{array}{rcl} v_s & = & \frac{VL}{GA_s} \\ V & = & \frac{6EI_z}{L^2} \\ \Phi & = & \frac{12EI}{GA_sL^2} \end{array}\right\} \boxed{v_s = 0.5\Phi L} \tag{2.42}$$

in other words, the shear deformation has moved the end of the beam (which was supposed to have zero translation) by $0.5\Phi L$.

2.6 Putting it All Together, [k]

39 Using basic structural analysis methods we have derived various force displacement relations for axial, flexural, torsional and shear imposed displacements. At this point, and keeping in mind the definition of degrees of freedom, we seek to assemble the individual element stiffness matrices [**k**]. We shall start with the simplest one, the truss element, then consider the beam, 2D frame, grid, and finally the 3D frame element.

40 In each case, a table will cross-reference the force displacement relations, and then the element stiffness matrix will be accordingly defined.

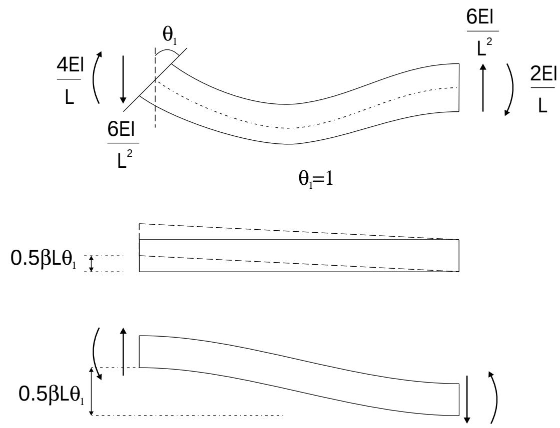


Figure 2.8: Effect of Flexure and Shear Deformation on Rotation at One End

2.6.1 Truss Element

41 The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node. Hence, from Eq. 2.9, we have

$$\boxed{[\mathbf{k}^t] = \frac{AE}{L} \begin{array}{c} \\ p_1 \\ p_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}} \tag{2.43}$$

2.6.2 Beam Element

42 There are two major beam theories:

Euler-Bernoulli which is the classical formulation for beams.

Timoshenko which accounts for transverse shear deformation effects.

2.6.2.1 Euler-Bernoulli

43 Using Equations 2.19, 2.20, 2.21 and 2.22 we can determine the forces associated with each unit displacement.

$$[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \text{Eq. } 2.21(v_1=1) & \text{Eq. } 2.21(\theta_1=1) & \text{Eq. } 2.21(v_2=1) & \text{Eq. } 2.21(\theta_2=1) \\ \text{Eq. } 2.19(v_1=1) & \text{Eq. } 2.19(\theta_1=1) & \text{Eq. } 2.19(v_2=1) & \text{Eq. } 2.19(\theta_2=1) \\ \text{Eq. } 2.22(v_1=1) & \text{Eq. } 2.22(\theta_1=1) & \text{Eq. } 2.22(v_2=1) & \text{Eq. } 2.22(\theta_2=1) \\ \text{Eq. } 2.20(v_1=1) & \text{Eq. } 2.20(\theta_1=1) & \text{Eq. } 2.20(v_2=1) & \text{Eq. } 2.20(\theta_2=1) \end{bmatrix} \end{array} \tag{2.44}$$

44 The stiffness matrix of the beam element (neglecting shear and axial deformation) will thus be

$$\boxed{[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \end{array}} \tag{2.45}$$

2.6.2.2 Timoshenko Beam

45 If shear deformations are present, we need to alter the stiffness matrix given in Eq. 2.45 in the following manner

1. Due to translation, we must divide (or normalize) the coefficients of the first and third columns of the stiffness matrix by $1+\Phi$ so that the net translation at both ends is unity.
2. Due to rotation and the effect of shear deformation
   (a) The forces induced at the ends due to a unit rotation at end 1 (second column) neglecting shear deformations are

$$V_1 = -V_2 = \frac{6EI}{L^2} \tag{2.46-a}$$

$$M_1 = \frac{4EI}{L} \tag{2.46-b}$$

$$M_2 = \frac{2EI}{L} \tag{2.46-c}$$

   (b) There is a net positive translation of $0.5\Phi L$ at end 1 when we applied a unit rotation (this "parasitic" translation is caused by the shear deformation) but no additional forces are induced.

(c) When we apply a unit rotation, all other displacements should be zero. Hence, we should counteract this parasitic shear deformation by an equal and opposite one. Hence, we apply an additional vertical displacement of $-0.5\Phi L$ and the forces induced at the ends (first column) are given by

$$V_1 = -V_2 = \underbrace{\frac{12EI}{L^3}\frac{1}{1+\Phi}}_{k^{bt}_{11}} \underbrace{(-0.5\Phi L)}_{v_s} \tag{2.47-a}$$

$$M_1 = M_2 = \underbrace{\frac{6EI}{L^2}\frac{1}{1+\Phi}}_{k^{bt}_{21}} \underbrace{(-0.5\Phi L)}_{v_s} \tag{2.47-b}$$

Note that the denominators have already been divided by $1 + \Phi$ in k^{bt}.

(d) Summing up all the forces, we have the forces induced as a result of a unit rotation only when the effects of both bending and shear deformations are included.

$$V_1 = -V_2 = \underbrace{\frac{6EI}{L^2}}_{\text{Due to Unit Rotation}} + \underbrace{\underbrace{\frac{12EI}{L^3}\frac{1}{1+\Phi}}_{k^{bt}_{11}} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to Parasitic Shear}} \tag{2.48-a}$$

$$= -\frac{6EI}{L^2}\frac{1}{1+\Phi} \tag{2.48-b}$$

$$M_1 = \underbrace{\frac{4EI}{L}}_{\text{Due to Unit Rotation}} + \underbrace{\underbrace{\frac{6EI}{L^2}\frac{1}{1+\Phi}}_{k^{bt}_{21}} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to Parasitic Shear}} \tag{2.48-c}$$

$$= \frac{4+\Phi}{1+\Phi}\frac{EI}{L} \tag{2.48-d}$$

$$M_2 = \underbrace{\frac{2EI}{L}}_{\text{Due to Unit Rotation}} + \underbrace{\underbrace{\frac{6EI}{L^2}\frac{1}{1+\Phi}}_{k^{bt}_{21}} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to Parasitic Shear}} \tag{2.48-e}$$

$$= \frac{2-\Phi}{1+\Phi}\frac{EI}{L} \tag{2.48-f}$$

46 Thus, the element stiffness matrix given in Eq. 2.45 becomes

$$
[\mathbf{k}^{bV}] = \begin{array}{c c}
 & \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\
\begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} &
\left[\begin{array}{cccc}
\frac{12EI_z}{L^3(1+\Phi_y)} & \frac{6EI_z}{L^2(1+\Phi_y)} & -\frac{12EI_z}{L^3(1+\Phi_y)} & \frac{6EI_z}{L^2(1+\Phi_y)} \\
\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(4+\Phi_y)EI_z}{(1+\Phi_y)L} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
-\frac{12EI_z}{L^3(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{12EI_z}{L^3(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} \\
\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{array}\right]
\end{array} \tag{2.49}
$$

2.6.3 2D Frame Element

47 The stiffness matrix of the two dimensional frame element is composed of terms from the truss and beam elements where $\mathbf{k}^b$ and $\mathbf{k}^t$ refer to the beam and truss element stiffness matrices respectively.

$$
[\mathbf{k}^{2dfr}] = \begin{array}{c c}
 & \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\
\begin{array}{c} P_1 \\ V_1 \\ M_1 \\ P_2 \\ V_2 \\ M_2 \end{array} &
\left[\begin{array}{cccccc}
k_{11}^t & 0 & 0 & k_{12}^t & 0 & 0 \\
0 & k_{11}^b & k_{12}^b & 0 & k_{13}^b & k_{14}^b \\
0 & k_{21}^b & k_{22}^b & 0 & k_{23}^b & k_{24}^b \\
k_{21}^t & 0 & 0 & k_{22}^t & 0 & 0 \\
0 & k_{31}^b & k_{32}^b & 0 & k_{33}^b & k_{34}^b \\
0 & k_{41}^b & k_{42}^b & 0 & k_{43}^b & k_{44}^b
\end{array}\right]
\end{array} \tag{2.50}
$$

Thus, we have:

$$
[\mathbf{k}^{2dfr}] = \begin{array}{c c}
 & \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\
\begin{array}{c} P_1 \\ V_1 \\ M_1 \\ P_2 \\ V_2 \\ M_2 \end{array} &
\left[\begin{array}{cccccc}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right]
\end{array} \tag{2.51}
$$

48 Note that if shear deformations must be accounted for, the entries corresponding to shear and flexure must be modified in accordance with Eq. 2.49

2.6.4 Grid Element

49 The stiffness matrix of the grid element is very analogous to the one of the 2D frame element, except that the axial component is replaced by the torsional one. Hence, the stiffness matrix is

$$
[\mathbf{k}^g] = \begin{array}{c} \\ T_1 \\ M_1 \\ V_1 \\ T_2 \\ M_2 \\ V_2 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \alpha_1 & \beta_1 & u_1 & \alpha_2 & \beta_2 & u_2 \end{array} \\
\left[\begin{array}{cccccc}
\text{Eq. 2.29} & 0 & 0 & \text{Eq. 2.29} & 0 & 0 \\
0 & k^b_{22} & -k^b_{21} & 0 & k^b_{24} & -k^b_{23} \\
0 & -k^b_{12} & k^b_{11} & 0 & -k^b_{14} & k^b_{13} \\
\text{Eq. 2.29} & 0 & 0 & \text{Eq. 2.29} & 0 & 0 \\
0 & k^b_{42} & -k^b_{41} & 0 & k^b_{44} & -k^b_{43} \\
0 & -k^b_{32} & k^b_{31} & 0 & -k^b_{34} & k^b_{32}
\end{array}\right]
\end{array}
\tag{2.52}
$$

Upon substitution, the grid element stiffness matrix is given by

$$
\boxed{[\mathbf{k}^g] = \begin{array}{c} \\ T_1 \\ M_1 \\ V_1 \\ T_2 \\ M_2 \\ V_2 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \alpha_1 & \beta_1 & v_1 & \alpha_2 & \beta_2 & v_2 \end{array} \\
\left[\begin{array}{cccccc}
\frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\
0 & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & 0 & \frac{2EI_z}{L} & \frac{6EI_z}{L^2} \\
0 & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & 0 & -\frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} \\
-\frac{Gi_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & 0 & \frac{4EI_z}{L} & \frac{6EI_z}{L^2} \\
0 & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & 0 & \frac{6EI_z}{L^2} & \frac{12EI_z}{L^3}
\end{array}\right]
\end{array}}
\tag{2.53}
$$

50 Note that if shear deformations must be accounted for, the entries corresponding to shear and flexure must be modified in accordance with Eq. 2.49

2.6.5 3D Frame Element

$$
[\mathbf{k}^{3dfr}] = \begin{array}{c} \\ P_{x1} \\ V_{y1} \\ Vz1 \\ T_{x1} \\ M_{y1} \\ M_{z1} \\ P_{x2} \\ V_{y2} \\ Vz2 \\ T_{x2} \\ M_{y2} \\ M_{z2} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{array} \\
\left[\begin{array}{cccccccccccc}
k^t_{11} & 0 & 0 & 0 & 0 & 0 & k^t_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & k^b_{11} & 0 & 0 & 0 & k^b_{12} & 0 & k^b_{13} & 0 & 0 & 0 & k^b 14 \\
0 & 0 & k^b_{11} & 0 & -k^b_{12} & 0 & 0 & 0 & k^b_{13} & 0 & -k^b_{14} & 0 \\
0 & 0 & 0 & k^g_{11} & 0 & 0 & 0 & 0 & 0 & k^g_{12} & 0 & 0 \\
0 & 0 & k^b_{32} & 0 & k^b_{22} & 0 & 0 & 0 & k^b_{12} & 0 & k^b_{24} & 0 \\
0 & k^b_{21} & 0 & 0 & 0 & k^b_{22} & 0 & -k^b_{12} & 0 & 0 & 0 & k^b_{24} \\
k^t_{21} & 0 & 0 & 0 & 0 & 0 & k^t_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & k^b_{13} & 0 & 0 & 0 & k^b_{14} & 0 & k^b_{33} & 0 & 0 & 0 & k^b_{34} \\
0 & 0 & k^b_{13} & 0 & -k^b_{14} & 0 & 0 & 0 & k^b_{33} & 0 & -k^b_{34} & 0 \\
0 & 0 & 0 & k^g_{12} & 0 & 0 & 0 & 0 & 0 & k^g_{22} & 0 & 0 \\
0 & 0 & k^b_{12} & 0 & k^b_{24} & 0 & 0 & 0 & -k^b_{43} & 0 & k^b_{44} & 0 \\
0 & -k^b_{12} & 0 & 0 & 0 & k^b_{24} & 0 & k^b_{43} & 0 & 0 & 0 & k^b_{44}
\end{array}\right]
\end{array}
\tag{2.54}
$$

For $[\mathbf{k}_{11}^{3D}]$ and with we obtain:

$$
[\mathbf{k}^{3dfr}] = \begin{array}{cc}
 & \begin{array}{cccccccccccc} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{array} \\
\begin{array}{c} P_{x1} \\ V_{y1} \\ V_{z1} \\ T_{x1} \\ M_{y1} \\ M_{z1} \\ P_{x2} \\ V_{y2} \\ V_{z2} \\ T_{x2} \\ M_{y2} \\ M_{z2} \end{array} &
\left[\begin{array}{cccccccccccc}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -6\frac{EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & 6\frac{EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]
\end{array}
\tag{2.55}
$$

51 Note that if shear deformations must be accounted for, the entries corresponding to shear and flexure must be modified in accordance with Eq. 2.49

2.7 Remarks on Element Stiffness Matrices

Singularity: All the derived stiffness matrices are singular, that is there is at least one row and one column which is a linear combination of others. For example in the beam element, row 4 = −row 1; and L times row 2 is equal to the sum of row 3 and 6. This singularity (not present in the flexibility matrix) is caused by the linear relations introduced by the equilibrium equations which are embedded in the formulation.

Symmetry: All matrices are symmetric due to Maxwell-Betti's reciprocal theorem, and the stiffness flexibility relation.

52 More about the stiffness matrix properties later.

Chapter 3

STIFFNESS METHOD; Part I: ORTHOGONAL STRUCTURES

3.1 Introduction

1 In the previous chapter we have first derived displacement force relations for different types of rod elements, and then used those relations to define element stiffness matrices in local coordinates.

2 In this chapter, we seek to perform similar operations, but for an orthogonal structure in global coordinates.

3 In the previous chapter our starting point was basic displacement-force relations resulting in element stiffness matrices $[\mathbf{k}]$.

4 In this chapter, our starting point are those same element stiffness matrices $[\mathbf{k}]$, and our objective is to determine the structure stiffness matrix $[\mathbf{K}]$, which when inverted, would yield the nodal displacements.

5 The element stiffness matrices were derived for fully restrained elements.

6 This chapter will be restricted to orthogonal structures, and generalization will be discussed later.

The stiffness matrices will be restricted to the unrestrained degrees of freedom.

7 From these examples, the interrelationships between structure stiffness matrix, nodal displacements, and fixed end actions will become apparent. Then the method will be generalized in chapter 5 to describe an algorithm which can automate the assembly of the structure global stiffness matrix in terms of the one of its individual elements.

3.2 The Stiffness Method

8 As a "vehicle" for the introduction to the stiffness method let us consider the problem in Fig 3.1-a, and recognize that there are only two unknown displacements, or more precisely, two global d.o.f: θ_1 and θ_2.

9 If we were to analyse this problem by the force (or flexibility) method, then

1. We make the structure statically determinate by removing *arbitrarily* two reactions (as long as the structure remains stable), and the beam is now statically determinate.

2. Assuming that we remove the two roller supports, then we determine the corresponding deflections due to the actual laod (Δ_B and Δ_C).

3. Apply a unit load at point B, and then C, and compute the deflections f_{ij} at note i due to a unit force at node j.

4. Write the compatibility of displacement equation

$$\begin{bmatrix} f_{BB} & f_{BC} \\ f_{CB} & f_{CC} \end{bmatrix} \begin{Bmatrix} R_B \\ R_C \end{Bmatrix} - \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.1}$$

5. Invert the matrix, and solve for the reactions

10 We will analyze this simple problem by the stiffness method.

1. The first step consists in making it kinematically determinate (as opposed to statically determinate in the flexibility method). Kinematically determinate in this case simply means restraining all the d.o.f. and thus prevent joint rotation, Fig 3.1-b.

2. We then determine the fixed end actions caused by the element load, and sum them for each d.o.f., Fig 3.1-c: ΣFEM_1 and ΣFEM_2.

3. In the third step, we will apply a unit displacement (rotation in this case) at each degree of freedom at a time, and in each case we shall determine the reaction forces, K_{11}, K_{21}, and K_{12}, K_{22} respectively. Note that we use $[\mathbf{K}]$, rather than $\mathbf{k}$ since those are forces in the global coordinate system, Fig 3.1-d. Again note that we are focusing only on the reaction forces corresponding to a global degree of freedom. Hence, we are not attempting to determine the reaction at node A.

4. Finally, we write the equation of equilibrium at each node:

$$\boxed{\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \Sigma\text{FEM}_1 \\ \Sigma\text{FEM}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}} \tag{3.2}$$

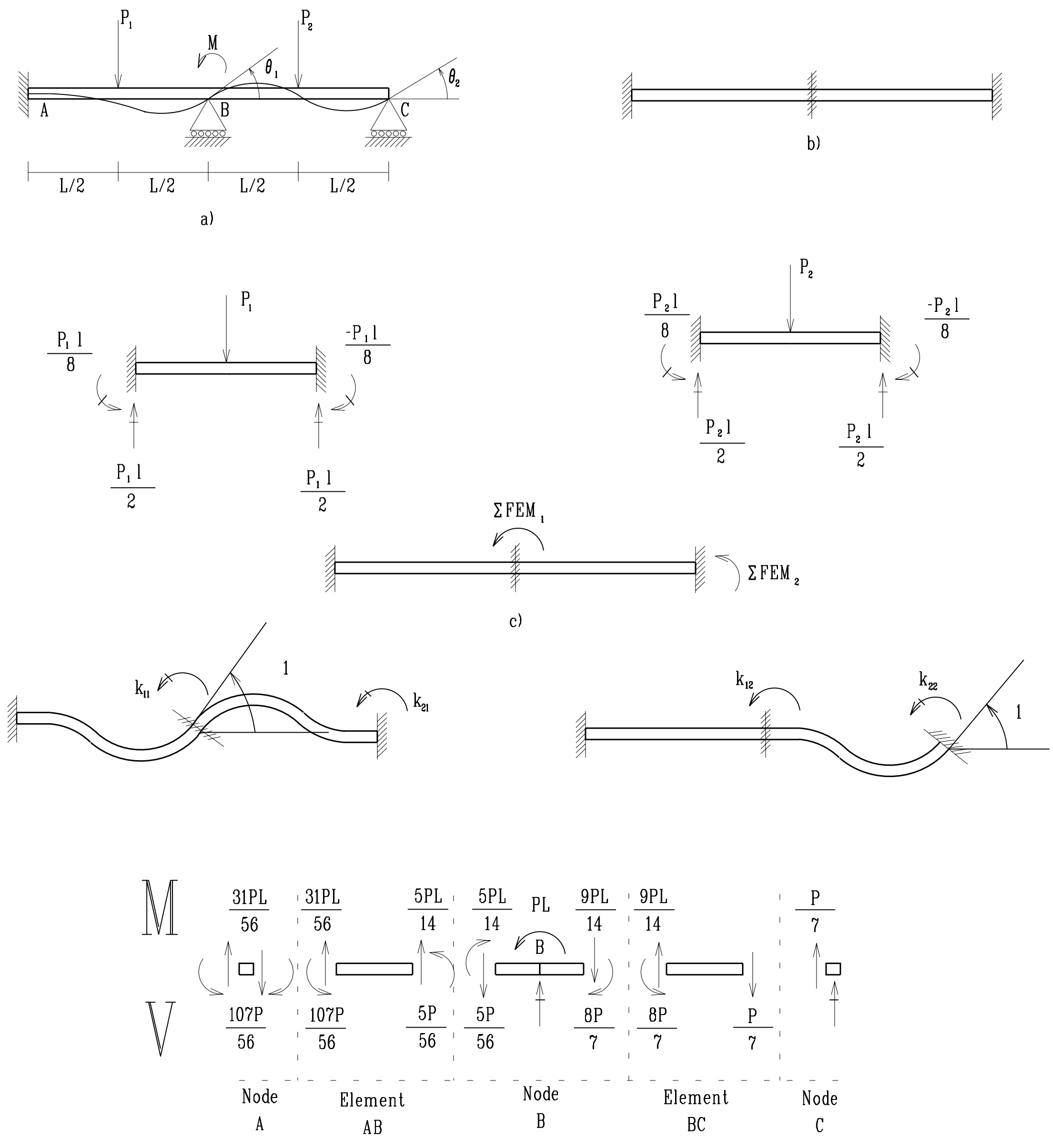


Figure 3.1: Problem with 2 Global d.o.f. θ_1 and θ_2

11 Note that the FEM being on the right hand side, they are detemined as the reactions to the applied load. Strictly speaking, it is a load which should appear on the left hand side of the equation, and are the nodal equivalent loads to the element load (more about this later).

12 As with the element stiffness matrix, each entry in the global stiffness matrix K_{ij}, corresponds to the internal force along d.o.f. i due to a unit displacement (generalized) along d.o.f. j (both in global coordinate systems).

3.3 Examples

■ **Example 3-1: Beam**

Considering the previous problem, Fig. 3.1-a, let $P_1 = 2P$, $M = PL$, $P_2 = P$, and $P_3 = P$, Solve for the displacements.

Solution:

1. Using the previously defined sign convention:

$$\Sigma\text{FEM}_1 = \underbrace{-\frac{P_1 L}{8}}_{BA} + \underbrace{\frac{P_2 L}{8}}_{BC} = -\frac{2PL}{8} + \frac{PL}{8} = -\frac{PL}{8} \tag{3.3}$$

$$\Sigma\text{FEM}_2 = \underbrace{-\frac{PL}{8}}_{CB} \tag{3.4}$$

2. If it takes $\frac{4EI}{L}$ (k_{44}^{AB}) to rotate AB (Eq. 2.45) and $\frac{4EI}{L}$ (k_{22}^{BC}) to rotate BC, it will take a total force of $\frac{8EI}{L}$ to simultaneously rotate AB and BC, (Note that a rigid joint is assumed).
3. Hence, K_{11} which is the sum of the rotational stiffnesses at global d.o.f. 1. will be equal to $K_{11} = \frac{8EI}{L}$; similarly, $K_{21} = \frac{2EI}{L}$ (k_{42}^{BC}) .
4. If we now rotate dof 2 by a unit angle, then we will have $K_{22} = \frac{4EI}{L}$ (k_{22}^{BC}) and $K_{12} = \frac{2EI}{L}$ (k_{42}^{BC}) .
5. The equilibrium relation can thus be written as:

$$\underbrace{\left\{ \begin{array}{c} PL \\ 0 \end{array} \right\}}_{\mathbf{M}} = \underbrace{\left\{ \begin{array}{c} -\frac{PL}{8} \\ -\frac{PL}{8} \end{array} \right\}}_{FEM} + \underbrace{\left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]}_{\mathbf{K}} \underbrace{\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\}}_{\mathbf{\Delta}} \tag{3.5}$$

or

$$\left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} \tag{3.6}$$

6. The two by two matrix is next inverted

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]^{-1} \left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left\{ \begin{array}{c} \frac{17}{112}\frac{PL^2}{EI} \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array} \right\} \tag{3.7}$$

7. Next we need to determine both the reactions and the internal forces.
8. Recall that for each element $\{\mathbf{p}\} = [\mathbf{k}]\{\boldsymbol{\delta}\}$, and in this case $\{\mathbf{p}\} = \{\mathbf{P}\}$ and $\{\boldsymbol{\delta}\} = \{\boldsymbol{\Delta}\}$ for element AB. The element stiffness matrix has been previously derived, Eq. 2.45, and in this case the global and local d.o.f. are the same.
9. Hence, the equilibrium equation for element AB, at the element level, can be written as:

$$\underbrace{\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right\}}_{\{\mathbf{p}\}} = \underbrace{\left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array} \right]}_{[\mathbf{k}]} \underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{17}{112}\frac{PL^2}{EI} \end{array} \right\}}_{\{\boldsymbol{\delta}\}} + \underbrace{\left\{ \begin{array}{c} \frac{2P}{2} \\ \frac{2PL}{8} \\ \frac{2P}{2} \\ -\frac{2PL}{8} \end{array} \right\}}_{\text{FEM}} \tag{3.8}$$

solving

$$\lfloor\ p_1 \quad p_2 \quad p_3 \quad p_4\ \rfloor = \lfloor\ \frac{107}{56}P \quad \frac{31}{56}PL \quad \frac{5}{56}P \quad \frac{5}{14}PL\ \rfloor \tag{3.9}$$

10. Similarly, for element BC:

$$\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right\} = \left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} 0 \\ \frac{17}{112}\frac{PL^2}{EI} \\ 0 \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array} \right\} + \left\{ \begin{array}{c} \frac{P}{2} \\ \frac{PL}{8} \\ \frac{P}{2} \\ -\frac{PL}{8} \end{array} \right\} \tag{3.10}$$

or

$$\lfloor\ p_1 \quad p_2 \quad p_3 \quad p_4\ \rfloor = \lfloor\ \frac{7}{8}P \quad \frac{9}{14}PL \quad -\frac{P}{7} \quad 0\ \rfloor \tag{3.11}$$

11. This simple example calls for the following observations:

1. Node A has contributions from element AB only, while node B has contributions from both AB and BC.

   12. We observe that $p_3^{AB} \neq p_1^{BC}$ eventhough they both correspond to a shear force at node B, the difference betweeen them is equal to the reaction at B. Similarly, $p_4^{AB} \neq p_2^{BC}$ due to the externally applied moment at node B.

2. From this analysis, we can draw the complete free body diagram, Fig. 3.1-e and then the shear and moment diagrams which is what the Engineer is most interested in for design purposes.

■

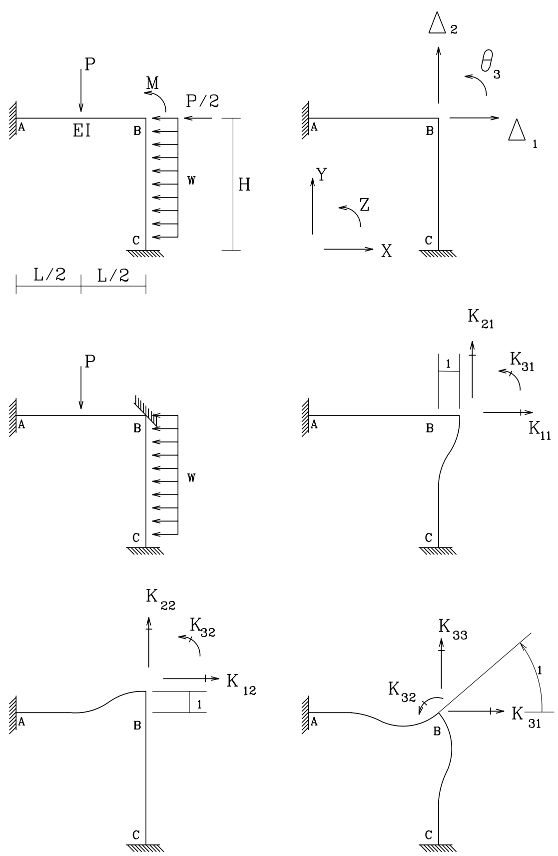


Figure 3.2: *Frame Example (correct K_{23})

■ **Example 3-2: Frame**

Whereas in the first example all local coordinate systems were identical to the global one, in this example we consider the orthogonal frame shown in Fig. 3.2,
Solution:

1. Assuming axial deformations, we do have three global degrees of freedom, Δ_1, Δ_2, and θ_3.
2. Constrain all the degrees of freedom, and thus make the structure kinematically determinate.
3. Determine the fixed end actions for each element in its own local coordinate system:

$$\underbrace{\lfloor\, P_1 \quad V_1 \quad M_1 \,\big|\, P_2 \quad V_2 \quad M_2 \,\rfloor}_{AB} = \lfloor\, 0 \quad \frac{P}{2} \quad \frac{PL}{8} \,\big|\, 0 \quad \frac{P}{2} \quad -\frac{PL}{8} \,\rfloor \tag{3.12}$$

$$\underbrace{\lfloor\ P_1 \quad V_1 \quad M_1 \,|\, P_2 \quad V_2 \quad M_2\ \rfloor}_{BC} \ = \ \lfloor\ 0 \quad \frac{wH}{2} \quad \frac{wH^2}{12} \,\Big|\, 0 \quad \frac{wH}{2} \quad -\frac{wH^2}{12}\ \rfloor \tag{3.13}$$

$$\tag{3.14}$$

4. Summing the fixed end actions at node B in global coordinates we have

$$\lfloor\ P_1 \quad P_2 \quad P_3\ \rfloor = \lfloor\ \frac{wH}{2} \quad \frac{P}{2} \quad -\frac{PL}{8}+\frac{wH^2}{12}\ \rfloor \tag{3.15}$$

5. Next, we apply a unit displacement in each of the 3 global degrees of freedom, and we seek to determine the structure global stiffness matrix. Each entry K_{ij} of the global stiffness matrix will correspond to the internal force in degree of freedom i, due to a unit displacement in degree of freedom j.

6. Recalling the force displacement relations derived earlier, we can assemble the global stiffness matrix in terms of contributions from both AB and BC:

		K_{i1} Δ_1	K_{i2} Δ_2	K_{i3} θ_3
K_{1j}	AB	$\frac{EA}{L}$	0	0
	BC	$\frac{12EI^c}{H^3}$	0	$\frac{6EI^c}{H^2}$
K_{2j}	AB	0	$\frac{12EI^b}{L^3}$	$-\frac{6EI^b}{L^2}$
	BC	0	$\frac{EA}{H}$	0
K_{3j}	AB	0	$-\frac{6EI^b}{L^2}$	$\frac{4EI^b}{L}$
	BC	$\frac{6EI^c}{H^2}$	0	$\frac{4EI^c}{H}$

7. Summing up, the structure global stiffness matrix $[\mathbf{K}]$ is:

$$[\mathbf{K}] = \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \quad\Delta_1 \quad & \quad\Delta_2\quad & \quad\theta_3\quad \end{array} \\ \left[\begin{array}{ccc} \frac{EA}{L}+\frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\ 0 & \frac{12EI^b}{L^3}+\frac{EA}{H} & -\frac{6EI^b}{L^2} \\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L}+\frac{4EI^c}{H} \end{array}\right] \end{array} \tag{3.16-a}$$

$$= \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \quad\Delta_1 \quad & \quad\Delta_2\quad & \quad\theta_3\quad \end{array} \\ \left[\begin{array}{ccc} k_{44}^{AB}+k_{22}^{BC} & k_{45}^{AB}+k_{21}^{BC} & k_{46}^{AB}+k_{23}^{BC} \\ k^{AB}+k^{BC} & k_{55}^{AB}+k_{11}^{BC} & k_{56}^{AB}+k_{13}^{BC} \\ k_{64}^{AB}+k_{32}^{BC} & k_{65}^{AB}+k_{31}^{BC} & k_{66}^{AB}+k_{33}^{BC} \end{array}\right] \end{array} \tag{3.16-b}$$

8. The global equation of equilibrium can now be written

$$\left\{\begin{array}{c} -\frac{P}{2} \\ 0 \\ M \end{array}\right\} = \underbrace{\left\{\begin{array}{c} \frac{wH}{2} \\ \frac{P}{2} \\ -\frac{PL}{8}+\frac{wH^2}{12} \end{array}\right\}}_{FEA} + \underbrace{\left[\begin{array}{ccc} \frac{EA}{L}+\frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\ 0 & \frac{12EI^b}{L^3}+\frac{EA}{H} & -\frac{6EI^b}{L^2} \\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L}+\frac{4EI^c}{H} \end{array}\right]}_{[\mathbf{K}]} \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\} \tag{3.17}$$

9. Solve for the displacemnts

$$
\begin{aligned}
\left\{\begin{array}{c}\Delta_1\\ \Delta_2\\ \theta_3\end{array}\right\} &= \left[\begin{array}{ccc}\frac{EA}{L}+\frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2}\\ 0 & \frac{12EI^b}{L^3}+\frac{EA}{H} & -\frac{6EI^b}{L^2}\\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L}+\frac{4EI^c}{H}\end{array}\right]^{-1}\left\{\begin{array}{c}-\frac{P}{2}-\frac{wH}{2}\\ -\frac{P}{2}\\ M+\frac{PL}{8}-\frac{wH^2}{12}\end{array}\right\}\\
&= \left\{\begin{array}{c}\frac{-\left(L^3\left(84Ib+19AL^2\right)P\right)}{32E(3Ib+AL^2)(12Ib+AL^2)}\\ \frac{-\left(L^3\left(12Ib+13AL^2\right)P\right)}{32E(3Ib+AL^2)(12Ib+AL^2)}\\ \frac{L^2\left(12Ib+AL^2\right)P}{64EIb(3Ib+AL^2)}\end{array}\right\}
\end{aligned}
\tag{3.18}
$$

10. To obtain the element internal forces, we will multiply each element stiffness matrix by the local displacements. For element AB, the local and global coordinates match, thus

$$
\begin{aligned}
\left\{\begin{array}{c}p_1\\ p_2\\ p_3\\ \hline p_4\\ p_5\\ p_6\end{array}\right\} &= \left[\begin{array}{ccc|ccc}\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0\\ 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\\ 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L}\\ \hline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0\\ 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2}\\ 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L}\end{array}\right]\left\{\begin{array}{c}0\\ 0\\ 0\\ \hline \frac{-\left(L^3\left(84Ib+19AL^2\right)P\right)}{32E(3Ib+AL^2)(12Ib+AL^2)}\\ \frac{-\left(L^3\left(12Ib+13AL^2\right)P\right)}{32E(3Ib+AL^2)(12Ib+AL^2)}\\ \frac{L^2\left(12Ib+AL^2\right)P}{64EIb(3Ib+AL^2)}\end{array}\right\}\\
&+\left\{\begin{array}{c}0\\ \frac{P}{2}\\ \frac{PL}{8}\\ 0\\ \frac{P}{2}\\ -\frac{PL}{8}\end{array}\right\}
\end{aligned}
\tag{3.19}
$$

11. For element BC, the local and global coordinates do not match, hence we will need to transform the displacements from their global to their local coordinate components. But since, vector (displacement and load), and matrix transformation have not yet been covered, we note by inspection that the relationship between global and local coordinates for element BC is

Local	δ_1	δ_2	θ_3	δ_4	δ_5	θ_6
Global	0	0	0	Δ_2	$-\Delta_1$	θ_3

and we observe that there are no local or global displacements associated with dof 1-3; Hence

the internal forces for element BC are given by:

$$
\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} =
\left[\begin{array}{ccc|ccc}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\
0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L}
\end{array}\right]
\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \frac{-\left(L^3\left(12I^b+13AL^2\right)P\right)}{32E\left(3I^b+AL^2\right)\left(12I^b+AL^2\right)} \\ \frac{\left(L^3\left(84I^b+19AL^2\right)P\right)}{32E\left(3I^b+AL^2\right)\left(12I^b+AL^2\right)} \\ \frac{L^2\left(12I^b+AL^2\right)P}{64EI^b\left(3I^b+AL^2\right)} \end{array}\right\}
+ \left\{\begin{array}{c} 0 \\ \frac{wH}{2} \\ \frac{wH^2}{12} \\ 0 \\ \frac{wH}{2} \\ -\frac{wH^2}{12} \end{array}\right\} \tag{3.20}
$$

Mathematica:

```
Ic=Ib
M= 0
w= 0
H= L
alpha=
K={
{E A /L + 12 E Ic /H^3, 0,                    6 E Ic/H^2},
{0,                     12 E Ib/L^3 + E A/H, -6 E Ib/L^2},
{6 E Ic/H^2,            -6 E Ib/L^2,          4 E Ib/L + 4 E Ic/H}
}
d=Inverse[K]
load={-P/2 - w H/2, -P/2, M+P L/8 -w H^2/12}
displacement=Simplify[d . load]
```

■

■ Example 3-3: Grid

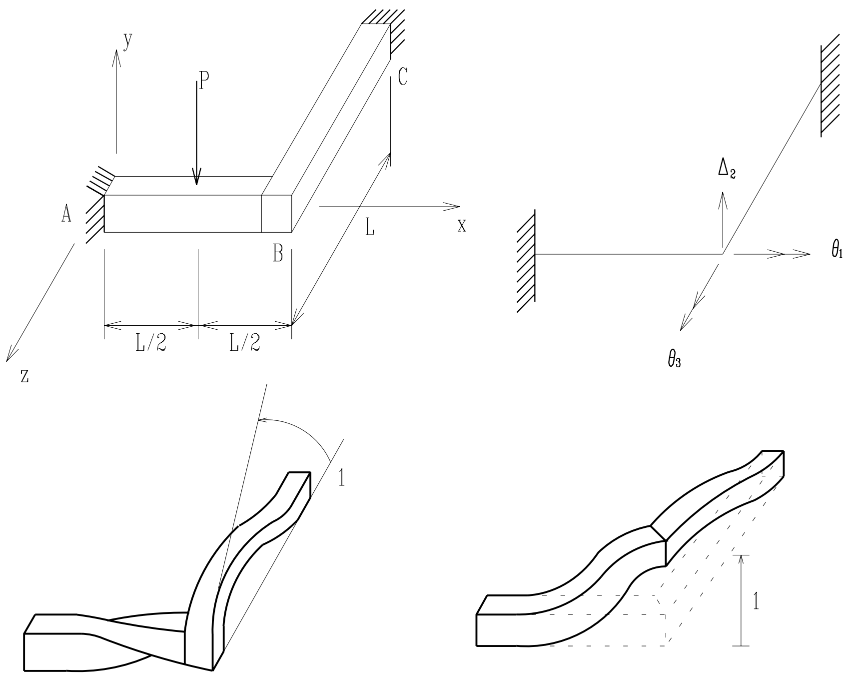


Figure 3.3: Grid Example

Analyse the orhtogonal grid shown in Fig. 3.3. The two elements have identical flexural and torsional rigidity, EI and GJ.
Solution:

1. We first identify the three degrees of freedom, θ_1, Δ_2, and θ_3.
2. Restrain all the degrees of freedom, and determine the fixed end actions:

$$\left\{ \begin{array}{c} T_1 \\ V_2 \\ M_3 \end{array} \right\} = \underbrace{\left\{ \begin{array}{c} 0 \\ \frac{P}{2} \\ \frac{PL}{8} \end{array} \right\}}_{\text{@node A}} = \underbrace{\left\{ \begin{array}{c} 0 \\ \frac{P}{2} \\ -\frac{Pl}{8} \end{array} \right\}}_{\text{@node B}} \tag{3.21}$$

3. Apply a unit displacement along each of the three degrees of freedom, and determine the internal forces:

1. Apply unit rotation along global d.o.f. 1.
   (a) AB (Torsion) $K_{11}^{AB} = \frac{GJ}{L}$, $K_{21}^{AB} = 0$, $K_{31}^{AB} = 0$
   (b) BC (Flexure) $K_{11}^{BC} = \frac{4EI}{L}$, $K_{21}^{BC} = \frac{6EI}{L^2}$, $K_{31}^{BC} = 0$
4. Apply a unit displacement along global d.o.f. 2.
   (a) AB (Flexure): $K_{12}^{AB} = 0$, $K_{22}^{AB} = \frac{12EI}{L^3}$, $K_{32}^{AB} = -\frac{6EI}{L^2}$
   (b) BC (Flexure): $K_{12}^{BC} = \frac{6EI}{L^2}$, $K_{22}^{BC} = \frac{12EI}{L^2}$, $K_{32}^{BC} = 0$
5. Apply unit displacement along global d.o.f. 3.
   (a) AB (Flexure): $K_{13}^{AB} = 0$, $K_{23}^{AB} = -\frac{6EI}{L^2}$, $K_{33}^{AB} = \frac{4EI}{L}$
   (b) BC (Torsion): $K_{13}^{BC} = 0$, $K_{23}^{BC} = 0$, $K_{33}^{BC} = \frac{GJ}{L}$

4. The structures stiffness matrix will now be *assembled*:

$$\begin{aligned}
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} &= \underbrace{\begin{bmatrix} \frac{GJ}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[\mathbf{K}_{AB}]} + \underbrace{\begin{bmatrix} \frac{4EI}{L} & \frac{6EI}{L^2} & 0 \\ \frac{6EI}{L^2} & \frac{12EI}{L^2} & 0 \\ 0 & 0 & \frac{GJ}{L} \end{bmatrix}}_{[\mathbf{K}_{BC}]} \\
&= \frac{EI}{L^3} \begin{bmatrix} \alpha EIL^2 & 0 & 0 \\ 0 & 12 & -6L \\ 0 & -6L & 4L^2 \end{bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 4L^2 & 6L & 0 \\ 6L & 12 & 0 \\ 0 & 0 & \alpha EIL^2 \end{bmatrix} \\
&= \underbrace{\frac{EI}{L^3} \begin{bmatrix} (4+\alpha EI)L^2 & 6L & 0 \\ 6L & 24 & -6L \\ 0 & -6L & (4+\alpha EI)L^2 \end{bmatrix}}_{[\mathbf{K}_{Structure}]}
\end{aligned} \tag{3.22}$$

$$= \begin{bmatrix} k_{44}^{AB}+k_{55}^{BC} & k_{46}^{AB}+k_{56}^{BC} & k_{45}^{AB}+k_{54}^{BC} \\ k_{64}^{AB}+k_{65}^{BC} & k_{66}^{AB}+k_{66}^{BC} & k_{65}^{AB}+k_{64}^{BC} \\ k_{54}^{AB}+k_{45}^{BC} & k_{56}^{AB}+k_{46}^{BC} & k_{55}^{AB}+k_{44}^{BC} \end{bmatrix} \tag{3.23}$$

where $\alpha = \frac{GJ}{EI}$, and in the last equation it is assumed thatfor element BC, node 1 corresponds to C and 2 to B.

5. The structure equilibrium equation in matrix form:

$$\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}}_{\{\mathbf{P}\}} = \underbrace{\begin{Bmatrix} 0 \\ \frac{P}{2} \\ -\frac{PL}{8} \end{Bmatrix}}_{\text{FEA @B}} + \underbrace{\frac{EI}{L^3}\begin{bmatrix} (4+\alpha)L^2 & 6L & 0 \\ 6L & 24 & -6L \\ 0 & -6L & (4+\alpha)L^2 \end{bmatrix}}_{[\mathbf{K}]} \underbrace{\begin{Bmatrix} \theta_1 \\ \Delta_2 \\ \theta_3 \end{Bmatrix}}_{\{\boldsymbol{\Delta}\}} \tag{3.24}$$

or

$$\begin{Bmatrix} \theta_1 \\ \Delta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \frac{PL^2}{16EI}\frac{5+2\alpha}{(1+\alpha)(4+\alpha)} \\ -\frac{PL^3}{96EI}\frac{5+2\alpha}{1+\alpha} \\ -\frac{3PL^2}{16EI}\frac{1}{(1+\alpha)(4+\alpha)} \end{Bmatrix} \tag{3.25}$$

6. Determine the element internal forces. This will be accomplished by multiplying each element stiffness matrix [**k**] with the vector of nodal displacement $\{\boldsymbol{\delta}\}$. Note these operations should be accomplished in local coordinate system, and great care should be exercized in writing the nodal displacements in the same local coordinate system as the one used for the derivation of the element stiffness matrix, Eq. 2.53.

7. For element AB and BC, the vector of nodal displacements are

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \hline \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline \theta_1 \\ -\theta_3 \\ \Delta_2 \end{Bmatrix}}_{AB} = \underbrace{\begin{Bmatrix} -\theta_3 \\ -\theta_1 \\ \Delta_2 \\ \hline 0 \\ 0 \\ 0 \end{Bmatrix}}_{BC} \tag{3.26}$$

8. Hence, for element AB we have

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} \\ \hline -\frac{Gi_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline \theta_1 \\ -\theta_3 \\ \Delta_2 \end{Bmatrix} = \tag{3.27}$$

9. For element BC:

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \left[\begin{array}{ccc|ccc} \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} \\ \hline -\frac{Gi_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} \end{array}\right] \left\{\begin{array}{c} -\theta_3 \\ -\theta_1 \\ \Delta_2 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\} \tag{3.28}$$

■

3.4 Observations

13 On the basis of these two illustrative examples we note that the global structure equilibrium equation can be written as

$$\boxed{\{\mathbf{P}\} = \{\mathbf{FEA}\} + [\mathbf{K}]\{\boldsymbol{\Delta}\}} \tag{3.29}$$

where $[\mathbf{K}]$ is the global structure stiffness matrix (in terms of the unrestrained d.o.f.) $\{\mathbf{P}\}$ the vector containing both the nodal load and the nodal equivalent load caused by element loading, $\{\boldsymbol{\Delta}\}$ is the vector of generalized nodal displacements.

14 Whereas the preceding two examples were quite simple to analyze, we seek to generalize the method to handle any arbitrary structure. As such, some of the questions which arise are:

1. How do we determine the element stiffness matrix in global coordinate systems, $[\mathbf{K}^e]$, from the element stiffness matrix in local coordinate system $[\mathbf{k}^e]$?
2. How to assemble the structure $[\mathbf{K}^S]$ from each element $[\mathbf{K}^E]$?
3. How to determine the $\{\mathbf{FEA}\}$ or the nodal equivalent load for an element load?
4. How to determine the local nodal displacements from the global ones?
5. How do we compute reactions in the restrained d.o.f?
6. How can we determine the internal element forces (P, V, M, and T)?
7. How do we account for temperature, initial displacements or prestrain?

Those questions, and others, will be addressed in the next chapters which will outline the general algorithm for the direct stiffness method.

Chapter 4

TRANSFORMATION MATRICES

4.1 Derivations

4.1.1 $[\mathbf{k}^e]$ $[\mathbf{K}^e]$ Relation

1 In the previous chapter, in which we focused on orthogonal structures, the assembly of the structure's stiffness matrix $[\mathbf{K}^e]$ in terms of the element stiffness matrices was relatively straightforward.

2 The determination of the element stiffness matrix in global coordinates, from the element stiffness matrix in local coordinates requires the introduction of a transformation.

3 This chapter will examine the 2D and 3D transformations required to obtain an element stiffness matrix in global coordinate system prior to assembly (as discussed in the next chapter).

4 Recalling that

$$\{\mathbf{p}\} = [\mathbf{k}^e]\{\boldsymbol{\delta}\} \tag{4.1}$$

$$\{\mathbf{P}\} = [\mathbf{K}^e]\{\boldsymbol{\Delta}\} \tag{4.2}$$

5 Let us define a transformation matrix $[\boldsymbol{\Gamma}]$ such that:

$$\{\boldsymbol{\delta}\} = [\boldsymbol{\Gamma}]\{\boldsymbol{\Delta}\} \tag{4.3}$$

$$\{\mathbf{p}\} = [\boldsymbol{\Gamma}]\{\mathbf{P}\} \tag{4.4}$$

Note that we use the same matrix $\boldsymbol{\Gamma}$ since both $\{\boldsymbol{\delta}\}$ and $\{\mathbf{p}\}$ are vector quantities (or tensors of order one).

6 Substituting Eqn. 4.3 and Eqn. 4.4 into Eqn. 4.1 we obtain

$$[\boldsymbol{\Gamma}]\{\mathbf{P}\} = [\mathbf{k}^e][\boldsymbol{\Gamma}]\{\boldsymbol{\Delta}\} \tag{4.5}$$

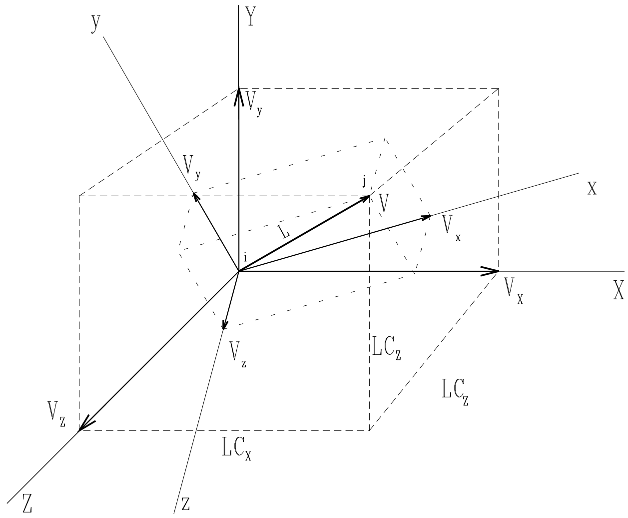


Figure 4.1: Arbitrary 3D Vector Transformation

premultiplying by $[\mathbf{\Gamma}]^{-1}$

$$\{\mathbf{P}\} = [\mathbf{\Gamma}]^{-1}[\mathbf{k}^e][\mathbf{\Gamma}]\{\mathbf{\Delta}\} \tag{4.6}$$

7 But since the rotation matrix is orthogonal, we have $[\mathbf{\Gamma}]^{-1} = [\mathbf{\Gamma}]^T$ and

$$\{\mathbf{P}\} = \underbrace{[\mathbf{\Gamma}]^T[\mathbf{k}^e][\mathbf{\Gamma}]}_{[\mathbf{K}^e]}\{\mathbf{\Delta}\} \tag{4.7}$$

$$\boxed{[\mathbf{K}^e] = [\mathbf{\Gamma}]^T[\mathbf{k}^e][\mathbf{\Gamma}]} \tag{4.8}$$

which is the general relationship between element stiffness matrix in local and global coordinates.

4.1.2 Direction Cosines

8 The problem confronting us is the general transfoormation of a vector $\mathbf{V}$ from $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ coordinate system to $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, Fig. 4.1: where:

$$\left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} = \underbrace{\left[\begin{array}{ccc} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{array} \right]}_{[\gamma]} \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} \tag{4.9}$$

where l_{ij} is the direction cosine of axis i with respect to axis j, and thus the rows of the matrix correspond to the rotated vectors with respect to the original ones corresponding to the columns.

9 Transformation can be accomplished through simple rotation matrices of direction cosines.

10 We define the rotated coordinate system as x, y, z relative to original system X, Y, Z, in terms of direction cosines l_{ij} where:

- l_{ij} direction cosines of rotated axis i with respect to original axis j.
- $\mathbf{l_{x\mathit{j}}} = (l_{\mathbf{xX}}, l_{\mathbf{xY}}, l_{\mathbf{xZ}})$direction cosines of $\mathbf{x}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $\mathbf{l_{y\mathit{j}}} = (l_{\mathbf{yX}}, l_{\mathbf{yY}}, l_{\mathbf{yZ}})$direction cosines of $\mathbf{y}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $\mathbf{l_{z\mathit{j}}} = (l_{\mathbf{zX}}, l_{\mathbf{zY}}, l_{\mathbf{zZ}})$direction cosines of $\mathbf{z}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$

and thus

$$\left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} = \underbrace{\left[\begin{array}{ccc} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{array} \right]}_{[\gamma]} \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} \tag{4.10}$$

and the rows of the matrix correspond to the rotated vectors with respect to the original ones corresponding to the columns.

11 Note that with respect to Fig. 4.2, $l_{xX} = \cos\alpha; l_{xY} = \cos\beta$, and $l_{xZ} = \cos\gamma$ or

$$V_x = V_X \cos\alpha + V_Y \cos\beta + V_Z \cos\gamma \tag{4.11}$$

12 Recalling that the dot product of two vectors

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\mathbf{A}|.|\mathbf{B}| \cos\alpha \tag{4.12}$$

where $||\mathbf{A}||$ is the norm (length) of $\vec{\mathbf{A}}$, and α is the angle between the two vectors. If we write

$$\vec{\mathbf{V}} = V_x\vec{\mathbf{i}} + V_y\vec{\mathbf{j}} + V_z\vec{\mathbf{k}} \tag{4.13}$$

The vector can be normalized

$$\vec{\mathbf{V}}_n = \frac{V_x}{|V|}\vec{\mathbf{i}} + \frac{V_y}{|V|}\vec{\mathbf{j}} + \frac{V_z}{|V|}\vec{\mathbf{k}} \tag{4.14}$$

and hence the to get the three direction cosines of vector $\vec{\mathbf{V}}$ we simply take the dot product of its normalized form with the thrre unit vector forming the orthogonal coordinate system

$$\vec{\mathbf{V}}_n \cdot \vec{\mathbf{i}} = \frac{x_2 - x_1}{L} = l_{vx} \tag{4.15-a}$$

$$\vec{\mathbf{V}}_n \cdot \vec{\mathbf{j}} = \frac{y_2 - y_1}{L} = l_{vy} \tag{4.15-b}$$

$$\vec{\mathbf{V}}_n \cdot \vec{\mathbf{k}} = \frac{z_2 - z_1}{L} = l_{vz} \tag{4.15-c}$$

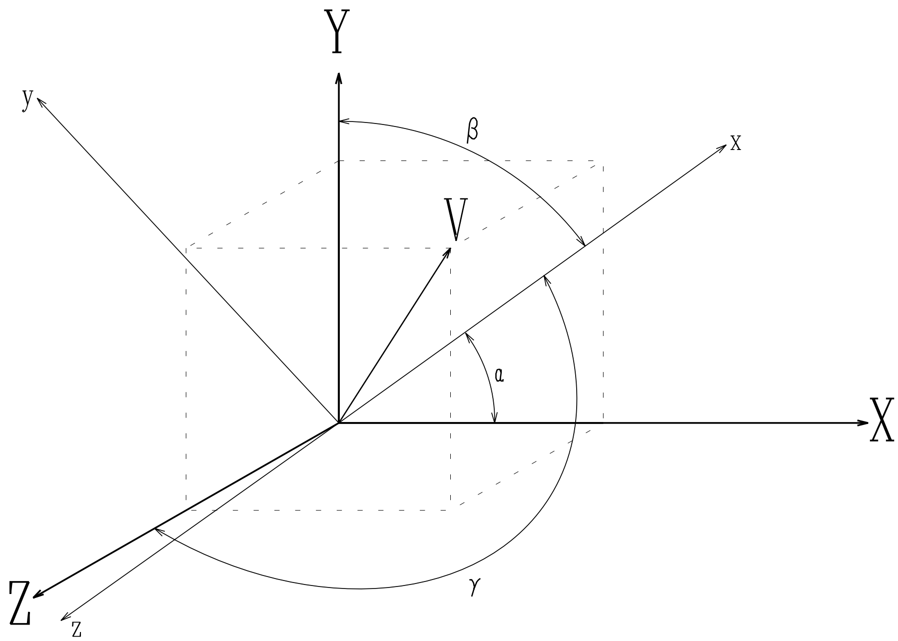


Figure 4.2: 3D Vector Transformation

13 If we use indecis instead of cartesian system, then direction cosines can be expressed as

$$V_x = V_X l_{11} + V_Y l_{12} + V_Z l_{13} \tag{4.16}$$

or by extension:

$$\left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} = \underbrace{\left[\begin{array}{ccc} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{array} \right]}_{[\boldsymbol{\gamma}]} \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} \tag{4.17}$$

Alternatively, $[\boldsymbol{\gamma}]$ is the matrix whose columns are the direction cosines of x, y, z with respect to X, Y, Z:

$$[\gamma]^T = \left[\begin{array}{ccc} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{array} \right] \tag{4.18}$$

The transformation of $\mathbf{V}$ can be written as:

$$\{\mathbf{v}\} = [\boldsymbol{\gamma}] \{\mathbf{V}\} \tag{4.19}$$

where: $\{v\}$ is the rotated coordinate system and $\{\mathbf{V}\}$ is in the original one.

14 Direction cosines are unit orthogonal vectors

$$\sum_{j=1}^{3} l_{ij} l_{ij} = 1 \qquad i = 1, 2, 3 \tag{4.20}$$

i.e:

$$\begin{aligned} l_{11}^2 + l_{12}^2 + l_{13}^2 &= 1 && (4.21) \\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= 1 = \delta_{11} && (4.22) \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{3} l_{ij} l_{kj} &= 0 \left\{ \begin{array}{l} i = 1, 2, 3 \\ k = 1, 2, 3 \\ i \neq k \end{array} \right. && (4.23) \\ l_{11} l_{21} + l_{12} l_{22} + l_{13} l_{23} &= 0 = \delta_{12} && (4.24) \end{aligned}$$

15 By direct multiplication of $[\boldsymbol{\gamma}]^T$ and $[\boldsymbol{\gamma}]$ it can be shown that: $[\boldsymbol{\gamma}]^T [\boldsymbol{\gamma}] = [\mathbf{I}] \Rightarrow [\boldsymbol{\gamma}]^T = [\boldsymbol{\gamma}]^{-1} \Rightarrow [\boldsymbol{\gamma}]$ is an orthogonal matrix.

16 The reverse transformation (from local to global) would be

$$\{\mathbf{V}\} = [\boldsymbol{\gamma}]^T \{\mathbf{v}\} \tag{4.25}$$

17 Finally, recalling that the transofrmation matrix $[\boldsymbol{\gamma}]$ is orthogonal, we have:

$$\left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} = \underbrace{\left[\begin{array}{ccc} l_{xX} & l_{yX} & l_{zX} \\ l_{xY} & l_{yY} & l_{xY} \\ l_{xZ} & l_{yZ} & l_{zZ} \end{array} \right]}_{[\boldsymbol{\gamma}]^{-1} = [\boldsymbol{\gamma}]^T} \left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} \tag{4.26}$$

4.2 Transformation Matrices For Framework Elements

18 The rotation matrix, $[\boldsymbol{\Gamma}]$, will obviously vary with the element type. In the most general case (3D element, 6 d.o.f. per node), we would have to define:

$$\left\{\begin{array}{c} F_{x1}\\ F_{y1}\\ F_{z1}\\ \hline M_{x1}\\ M_{y1}\\ M_{z1}\\ \hline F_{x2}\\ F_{y2}\\ F_{z2}\\ \hline M_{x2}\\ M_{y2}\\ M_{z2} \end{array}\right\} = \underbrace{\left[\begin{array}{c|c|c|c} [\gamma] & & & \\ \hline & [\gamma] & & \\ \hline & & [\gamma] & \\ \hline & & & [\gamma] \end{array}\right]}_{[\boldsymbol{\Gamma}]} \left\{\begin{array}{c} F_{X1}\\ F_{Y1}\\ F_{Z1}\\ \hline M_{X1}\\ M_{Y1}\\ M_{Z1}\\ \hline F_{X2}\\ F_{Y2}\\ F_{Z2}\\ \hline M_{X2}\\ M_{Y2}\\ M_{Z2} \end{array}\right\} \tag{4.27}$$

and should distinguish between the vector transformation $[\gamma]$ and the element transformation matrix $[\boldsymbol{\Gamma}]$.

19 In the next sections, we will examine the transformation matrix of each type of element.

4.2.1 2 D cases

4.2.1.1 2D Frame, and Grid Element

20 The vector rotation matrix $[\gamma]$ is identical for both 2D frame and grid elements, Fig. 4.3, and 4.4 respectively.

21 From Eq. 4.10 the vector rotation matrix is defined in terms of 9 direction cosines of 9 different angles. However for the 2D case, we will note that four angles are interrelated $(l_{xX}, l_{xY}, l_{yX}, l_{yY})$ and can all be expressed in terms of a single one α, where α is the direction of the local x axis (along the member from the first to the second node) with respect to the global X axis. The remaining 5 terms are related to another angle, β, which is between the **Z** axis and the x-y plane. This angle is zero because we select an orthogonal right handed coordinate system. Thus, the rotation matrix can be written as:

$$[\gamma] = \begin{bmatrix} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos(\frac{\pi}{2}-\alpha) & 0 \\ \cos(\frac{\pi}{2}+\alpha) & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.28}$$

and we observe that the angles are defined from the second subscript to the first, and that counterclockwise angles are positive.

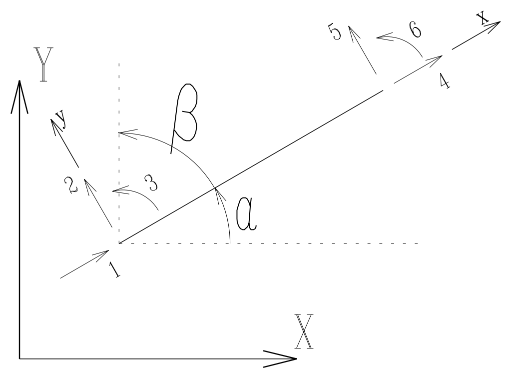


Figure 4.3: 2D Frame Element Rotation

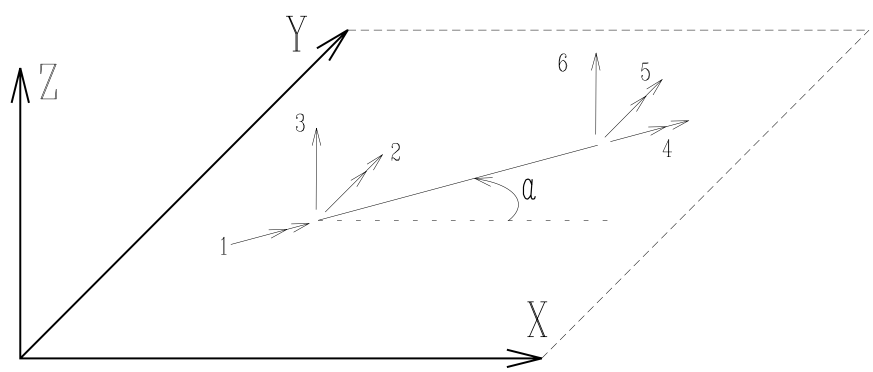


Figure 4.4: Grid Element Rotation

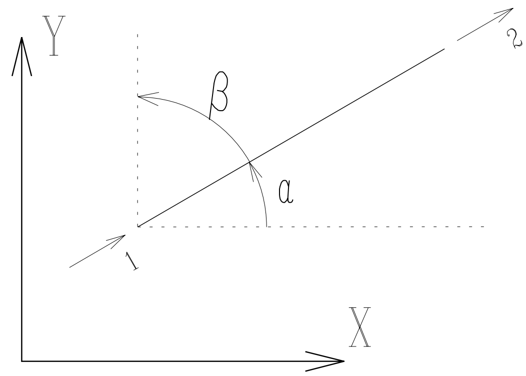


Figure 4.5: 2D Truss Rotation

22 The element rotation matrix $[\mathbf{\Gamma}]$ will then be given by

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \underbrace{\left[\begin{array}{rrr|rrr} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathbf{\Gamma}]} \left\{\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \\ P_6 \end{array}\right\} \tag{4.29}$$

4.2.1.2 2D Truss

23 For the 2D truss element, the global coordinate system is two dimensional, whereas the local one is only one dimensional, hence the vector transformation matrix is, Fig. 4.5.

$$[\boldsymbol{\gamma}] = \left[\begin{array}{cc} l_{xX} & l_{xY} \end{array}\right] = \left[\begin{array}{cc} \cos\alpha & \cos(\frac{\pi}{2}-\alpha) \end{array}\right] = \left[\begin{array}{cc} \cos\alpha & \sin\alpha \end{array}\right] \tag{4.30}$$

24 The element rotation matrix $[\mathbf{\Gamma}]$ will then be assembled from the vector rotation matrix $[\boldsymbol{\gamma}]$.

$$\left\{\begin{array}{c} p_1 \\ \hline p_2 \end{array}\right\} = \left[\begin{array}{cc} [\boldsymbol{\gamma}] & \mathbf{0} \\ \mathbf{0} & [\boldsymbol{\gamma}] \end{array}\right] = \underbrace{\left[\begin{array}{cc|cc} \cos\alpha & \sin\alpha & 0 & 0 \\ \hline 0 & 0 & \cos\alpha & \sin\alpha \end{array}\right]}_{[\mathbf{\Gamma}]} \left\{\begin{array}{c} P_1 \\ P_2 \\ \hline P_3 \\ P_4 \end{array}\right\} \tag{4.31}$$

4.2.2 3D Frame

25 Given that rod elements, are defined in such a way to have their local x axis aligned with their major axis, and that the element is defined by the two end nodes (of known coordinates),

then recalling the definition of the direction cosines it should be apparent that the evaluation of the first row, only, is quite simple. However evaluation of the other two is more complex.

26 This generalized transformation from $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ to $\mathbf{x}, \mathbf{y}, \mathbf{z}$ was accomplished in one step in the two dimensional case, but intermediary ones will have to be defined in the 3D case.

27 Starting with a reference (X_1, Y_1, Z_1) coordinate system which corresponds to the global coordinate system, we can define another one, X_2, Y_2, Z_2, such that X_2 is aligned along the element, Fig. **??**.

28 In the 2D case this was accomplished through one single rotation α, and all other angles where defined in terms of it.

29 In the 3D case, it will take a minimum of two rotations β and γ, and possibly a third one α (different than the one in 2D) to achieve this transformation.

30 We can start with the first row of the transformation matrix which corresponds to the direction cosines of the reference axis $(\mathbf{X_1}, \mathbf{Y_1}, \mathbf{Z_1})$ with respect to $\mathbf{X_2}$. This will define the first row of the vector rotation matrix $[\boldsymbol{\gamma}]$:

$$[\boldsymbol{\gamma}] = \begin{bmatrix} C_X & C_Y & C_Z \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \tag{4.32}$$

where $C_X = \frac{x_j - x_i}{L}$, $C_Y = \frac{y_j - y_i}{L}$, $C_Z = \frac{z_j - z_i}{L}$, $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$.

31 Note that this does not uniquely define the new coordinate system. This will be done in two ways: a simple and a general one.

4.2.2.1 Simple 3D Case

32 We start by looking at a simplified case, Fig. 4.6, one in which Z_2 is assumed to be horizontal in the $X_1 - Z_1$ plane, this will also define Y_2. We note that there will be no ambiguity unless the member is vertical.

33 This transformation can be used if:

1. The principal axis of the cross section lie in the horizontal and vertical plane (i.e the web of an I Beam in the vertical plane).

2. If the member has 2 axis of symmetry in the cross section and same moment of inertia about each one of them (i.e circular or square cross section).

34 The last two rows of Eq. 4.32 can be determined through two successive rotations (assuming that $(X_1, Y_1, Z_1$ and X_2, Y_2, Z_2 are originally coincident):

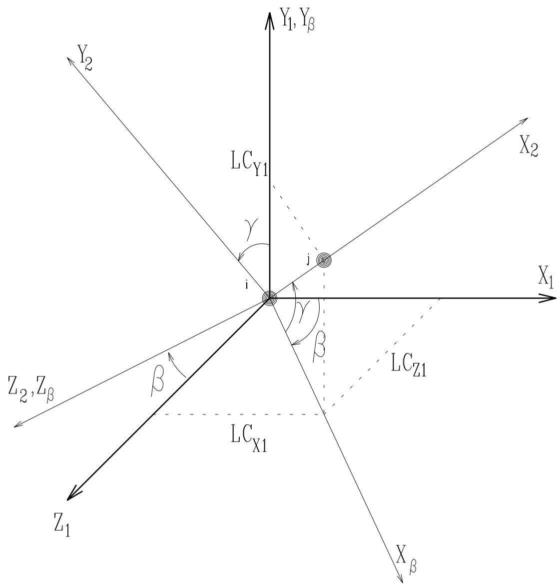


Figure 4.6: Simple 3D Rotation

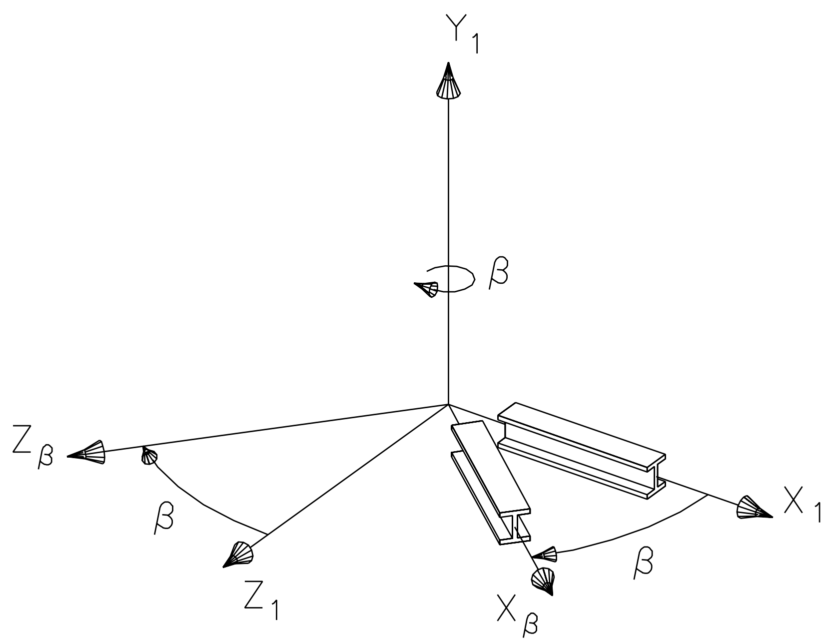


Figure 4.7: Arbitrary 3D Rotation; Rotation with respect to β

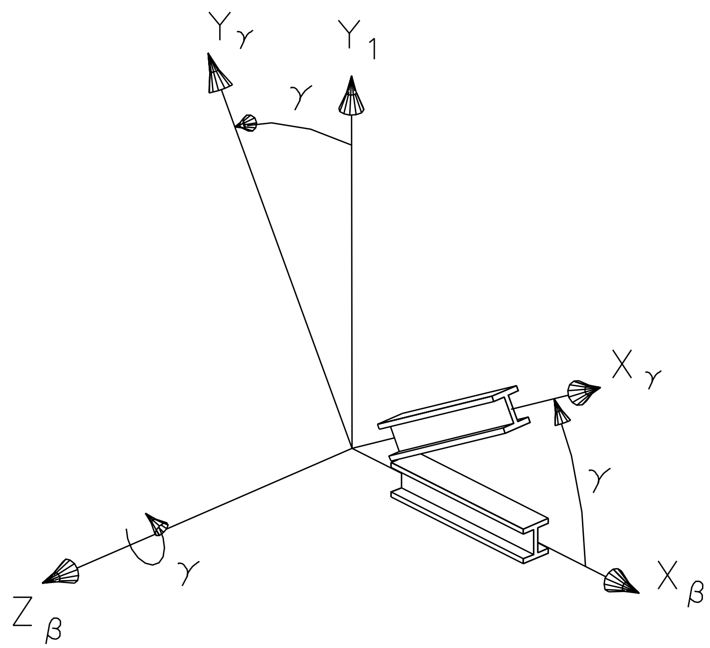


Figure 4.8: Arbitrary 3D Rotation; Rotation with respect to γ

1. Rotation by β about the Y_1 axis, 4.7 this will place the X_1 axis along X_β. This rotation $[\mathbf{R}_\beta]$ is made of the direction cosines of the β axis $(X_\beta, Y_\beta, Z_\beta)$ with respect to (X_1, Y_1, Z_1):

$$[\mathbf{R}_\beta] = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{4.33}$$

   we note that: $\cos\beta = \frac{C_X}{C_{XZ}}$, $\sin\beta = \frac{C_Z}{C_{XZ}}$, and $C_{XZ} = \sqrt{C_X^2 + C_Z^2}$.

2. Rotation by γ about the Z_2 axis 4.8

$$[\mathbf{R}_\gamma] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.34}$$

   where $\cos\gamma = C_{XZ}$, and $\sin\gamma = C_Y$.

35 Combining Eq. 4.33 and 4.34 yields:

$$[\boldsymbol{\gamma}] = [\mathbf{R}_\gamma][\mathbf{R}_\beta] = \begin{bmatrix} C_X & C_Y & C_Z \\ \frac{-C_X C_Y}{C_{XZ}} & C_{XZ} & \frac{-C_Y C_Z}{C_{XZ}} \\ \frac{-C_Z}{C_{XZ}} & 0 & \frac{C_X}{C_{XZ}} \end{bmatrix} \tag{4.35}$$

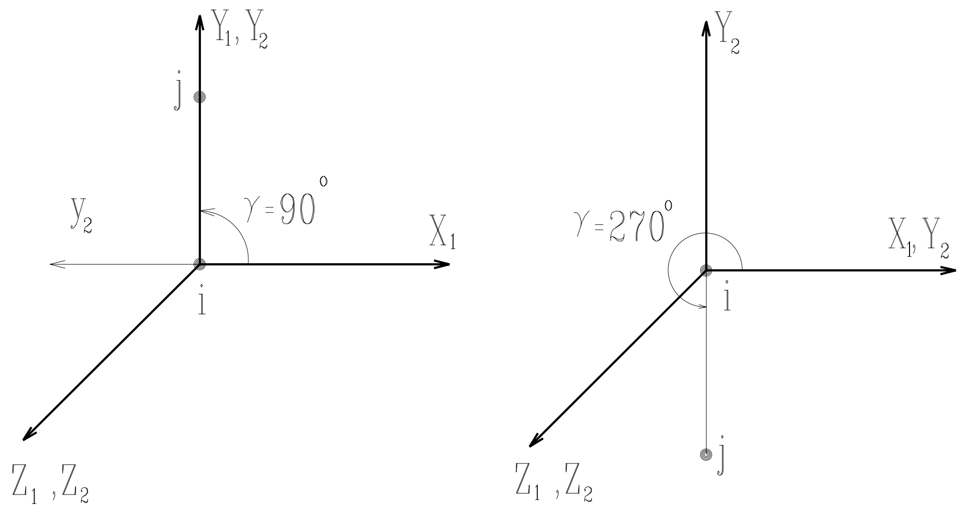


Figure 4.9: Special Case of 3D Transformation for Vertical Members

36 For vertical member the preceding matrix is no longer valid as C_{XZ} is undefined. However we can obtain the matrix by simple inspection, Fig. 4.9 as we note that:

1. X_2 axis aligned with Y_1
2. Y_2 axis aligned with -X_1
3. Z_2 axis aligned with Z_1

hence the rotation matrix with respect to the y axis, is similar to the one previously derived for rotation with respect to the z axis, except for the reordering of terms:

$$[\boldsymbol{\gamma}] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.36}$$

which is valid for both cases ($C_Y = 1$ for $\gamma = 90\,\text{deg}$, and $C_Y = -1$ for $\gamma = 270\,\text{deg}$).

4.2.2.2 General Case

37 In the most general case, we need to define an additional rotation to the preceding transformation of an angle α about the X_γ axis, Fig. 4.10. This rotation is defined such that:

1. X_α is aligned with X_2 and normal to both Y_2 and Z_2
2. Y_α makes an angle 0, α and $\beta = \frac{\pi}{2} - \alpha$, with respect to X_2, Y_2 and Z_2 respectively

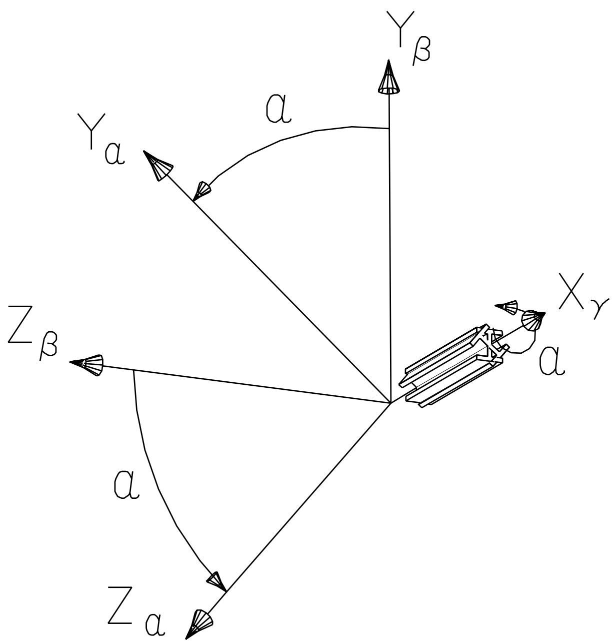


Figure 4.10: Arbitrary 3D Rotation; Rotation with respect to α

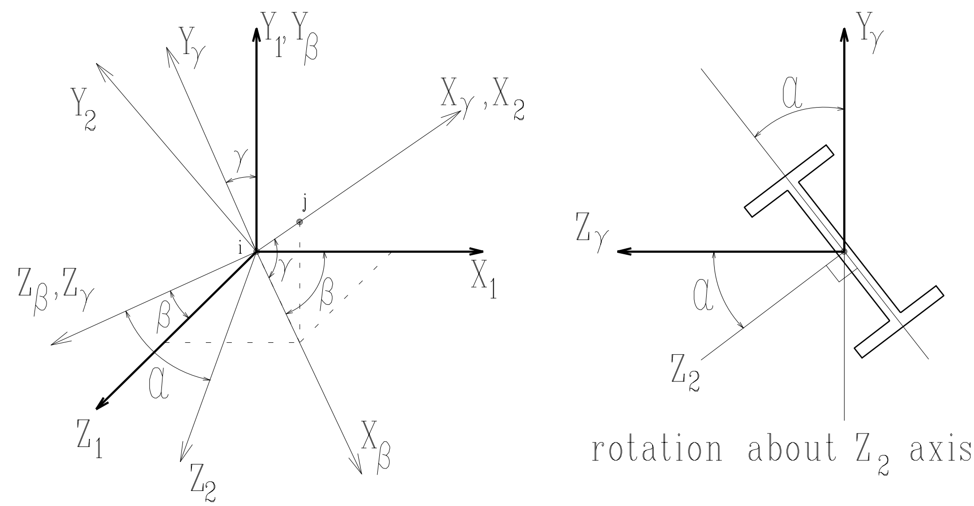


Figure 4.11: Rotation of Cross-Section by α

3. Z_α makes an angle 0, $\frac{\pi}{2}+\alpha$ and α, with respect to X_2, Y_2 and Z_2 respectively

38 Noting that $\cos(\frac{\pi}{2}+\alpha) = -\sin\alpha$ and $\cos\beta = \sin\alpha$, the direction cosines of this transformation are given by:

$$[\mathbf{R}_\alpha] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{4.37}$$

causing the $Y_2 - Z_2$ axis to coincide with the principal axes of the cross section, Fig. 4.12. This will yield:

$$\boxed{[\boldsymbol{\gamma}] = [\mathbf{R}_\alpha][\mathbf{R}_\gamma][\mathbf{R}_\beta]} \tag{4.38}$$

$$[\boldsymbol{\gamma}] = \begin{bmatrix} C_X & C_Y & C_Z \\ \frac{-C_XC_Y\cos\alpha - C_Z\sin\alpha}{C_{XZ}} & C_{XZ}\cos\alpha & \frac{-C_YC_Z\cos\alpha + C_X\sin\alpha}{C_{XZ}} \\ \frac{C_XC_Y\sin\alpha - C_Z\cos\alpha}{C_{XZ}} & -C_{XZ}\sin\alpha & \frac{C_YC_Z\sin\alpha + C_X\cos\alpha}{C_{XZ}} \end{bmatrix} \tag{4.39}$$

39 As for the simpler case, the preceding equation is undefined for vertical members, and a counterpart to Eq. 4.36 must be derived. This will be achieved in two steps:

1. Rotate the member so that:
   (a) X_2 axis aligned with Y_1
   (b) Y_2 axis aligned with -X_1
   (c) Z_2 axis aligned with Z_1

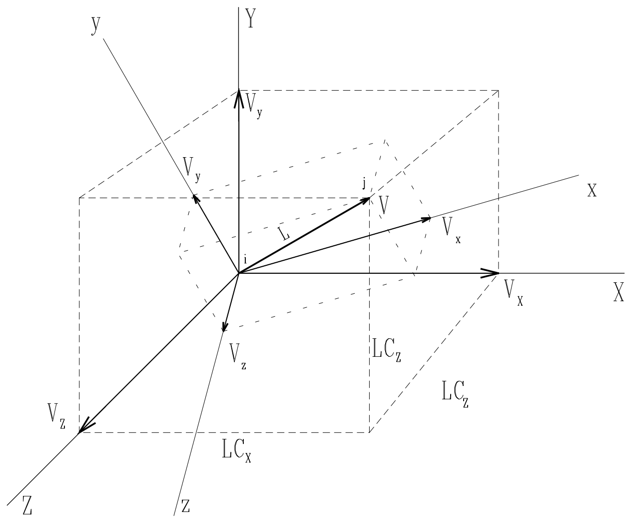


Figure 4.12: Arbitrary 3D Element Transformation

this was previously done and resulted in Eq. 4.36

$$[\mathbf{R}_\gamma] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4.40}$$

2. The second step consists in performing a rotation of angle α with respect to the new X_2 as defined in Eq. 4.37.

3. Finally, we multiply the two transformation matrices $[\mathbf{R}_\gamma][\mathbf{R}_\alpha]$ given by Eq. 4.40 and 4.37 to obtain:

$$[\mathbf{\Gamma}] = [\mathbf{R}_\gamma][\mathbf{R}_\alpha] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y\cos\alpha & 0 & \sin\alpha \\ C_Y\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{4.41}$$

Note with $\alpha = 0$, we recover Eq. 4.36.

4.2.3 3D Truss

40 With reference to the first part of the derivation of the transformation of 3D frame element, the transformation matrix of 3D truss elements is

$$[\mathbf{\Gamma}^{3D\ T}] = \begin{bmatrix} C_X & C_Y & C_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_X & C_Y & C_Z \end{bmatrix} \tag{4.42}$$

Chapter 5

STIFFNESS METHOD; Part II

5.1 Introduction

1 The direct stiffness method, covered in *Advanced Structural Analysis* is briefly reviewed in this lecture.

2 A slightly different algorithm will be used for the assembly of the global stiffness matrix.

Preliminaries: First we shall

1. Identify type of structure (Plane stress/strain/Axisymmetric/Plate/Shell/3D) and determine the
   (a) Number of spatial coordinates (1D, 2D, or 3D)
   (b) Number of degree of freedom per node
   (c) Number of material properties
2. Determine the global unrestrained degree of freedom equation numbers for each node, to be stored in the [`ID`] matrix.

Analysis :

1. For each element, determine
   (a) Vector `LM` relating local to global degree of freedoms.
   (b) Element stiffness matrix $[\mathbf{K}^e]$. This may require a numerical integration
2. Assemble the structure stiffness matrix $[\mathbf{K}^S]$ of unconstrained degree of freedom's.
3. Decompose $[\mathbf{K}^S]$ into $[\mathbf{K}^S] = [\mathbf{L}][\mathbf{L}]^T$ where $[\mathbf{L}]$ is a lower triangle matrix[1].
4. For traction, body forces, determine the nodal equivalent load.
5. Assemble load vector $\{\mathbf{P}\}$

[1] More about these operations in chapter C.

6. Backsubstitute and obtain nodal displacements
7. For each element, determine strain and stresses.
8. For each restrained degree of freedom compute its reaction from $\{\mathbf{R}\} = \Sigma_{i=1}^{\#\text{ of elem.}}[\mathbf{K}_i]\{\mathbf{\Delta}\}$

3 Some of the prescribed steps are further discussed in the next sections.

5.2 [ID] Matrix

4 Because of the boundary condition restraints, the total structure number of *active* degrees of freedom (i.e unconstrained) will be less than the number of nodes times the number of degrees of freedom per node.

5 To obtain the global degree of freedom for a given node, we need to define an [ID] matrix such that:

ID has dimensions $l \times k$ where l is the number of degree of freedom per node, and k is the number of nodes).

ID matrix is initialized to zero.

1. At input stage read `ID(idof,inod)` of each degree of freedom for every node such that:

$$\mathtt{ID}(\mathtt{idof},\mathtt{inod}) = \begin{cases} 0 & \text{if unrestrained d.o.f.} \\ 1 & \text{if restrained d.o.f.} \end{cases} \tag{5.1}$$

2. After all the node boundary conditions have been read, assign incrementally equation numbers

   (a) First to all the active dof

   (b) Then to the other (restrained) dof.

   (c) Multiply by -1 all the passive dof.

   Note that the total number of dof will be equal to the number of nodes times the number of dof/node NEQA.

3. The largest *positive* global degree of freedom number will be equal to NEQ (Number Of Equations), which is the size of the square matrix which will have to be decomposed.

6 For example, for the frame shown in Fig. 5.1:

1. The input data file may contain:

Node No.	$[\mathbf{ID}]^T$
1	0 0 0
2	1 1 0
3	0 0 0
4	1 0 0

2. At this stage, the [ID] matrix is equal to:

$$\mathbf{ID} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.2}$$

3. After we determined the equation numbers, we would have:

$$\mathbf{ID} = \begin{bmatrix} 1 & -10 & 5 & -12 \\ 2 & -11 & 6 & 8 \\ 3 & 4 & 7 & 9 \end{bmatrix} \tag{5.3}$$

5.3 LM Vector

7 The LM vector of a given element gives the global degree of freedom of each one of the element degree of freedom's. For the structure shown in Fig. 5.1, we would have:

$$\begin{array}{lcll} \lfloor \mathtt{LM} \rfloor & = & \lfloor -10 \quad -11 \quad 4 \quad 5 \quad 6 \quad 7 \rfloor & \text{element 1 } (2 \rightarrow 3) \\ \lfloor \mathtt{LM} \rfloor & = & \lfloor 5 \quad 6 \quad 7 \quad 1 \quad 2 \quad 3 \rfloor & \text{element 2 } (3 \rightarrow 1) \\ \lfloor \mathtt{LM} \rfloor & = & \lfloor 1 \quad 2 \quad 3 \quad -12 \quad 8 \quad 9 \rfloor & \text{element 3 } (1 \rightarrow 4) \end{array}$$

5.4 Assembly of Global Stiffness Matrix

8 As for the element stiffness matrix, the global stiffness matrix [**K**] is such that K_{ij} is the force in degree of freedom i caused by a unit displacement at degree of freedom j.

9 Whereas this relationship was derived from basic analysis at the element level, at the structure level, this term can be obtained from the contribution of the element stiffness matrices $[\mathbf{K}^e]$ (written in global coordinate system).

10 For each K_{ij} term, we shall add the contribution of all the elements which can connect degree of freedom i to degree of freedom j, assuming that those forces are readily available from the individual element stiffness matrices written in global coordinate system.

11 K_{ij} is non-zero if and only if degree of freedom i and degree of freedom j are connected by an element or share a node.

12 There are usually more than one element connected to a dof. Hence, individual element stiffness matrices terms must be added up.

13 Because each term of all the element stiffness matrices must find its position inside the global stiffness matrix $[\mathbf{K}]$, it is found computationally most effective to initialize the global stiffness matrix $[\mathbf{K}^S]_{NEQA\times NEQA}$ to zero, and then loop through all the elements, and then through each entry of the respective element stiffness matrix K_{ij}^e.

14 The assignment of the element stiffness matrix term K_{ij}^e (note that e, i, and j are all known since we are looping on e from 1 to the number of elements, and then looping on the rows and columns of the element stiffness matrix i, j) into the global stiffness matrix K_{kl}^S is made through the `LM` vector (note that it is k and l which must be determined).

15 Since the global stiffness matrix is also symmetric, we would need to only assemble one side of it, usually the upper one.

16 Contrarily to *Matrix Structural Analysis*, we will assemble the full *augmented* stiffness matrix.

17 The algorithm for this assembly is illustrated in Fig. 5.2.

■ Example 5-1: Global Stiffness Matrix Assembly

Assemble the global stiffness matrix in terms of element 2 and 3 of the example problem shown in Fig. 5.1.

Solution:

Given the two elements 2 and 3, their respective stiffness matrices in global coordinate systems may be symbolically represented by:

$$
[\mathbf{K}_2^e] = \begin{array}{cc|cccccc}
 & & 5 & 6 & 7 & 1 & 2 & 3 \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
5 & 1 & a & b & c & d & e & f \\
6 & 2 & & g & h & i & j & k \\
7 & 3 & & & l & m & n & o \\
1 & 4 & & & & p & q & r \\
2 & 5 & & & & & s & t \\
3 & 6 & & & & & & u
\end{array}
\begin{array}{l}
\rightarrow \text{structure d.o.f. LM} \\
\rightarrow \text{element d.o.f.} \\
\\ \\ \\ \\ \\ \\
\end{array}
$$

and

$$
[\mathbf{K}_3^e] = \begin{array}{cc|cccccc}
 & & 1 & 2 & 3 & -12 & 8 & 9 \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline
1 & 1 & A & B & C & D & E & F \\
2 & 2 & & G & H & I & J & K \\
3 & 3 & & & L & M & N & O \\
-12 & 4 & & & & P & Q & R \\
8 & 5 & & & & & S & T \\
9 & 6 & & & & & & U
\end{array}
\begin{array}{l}
\rightarrow \text{structure d.o.f. LM} \\
\rightarrow \text{element d.o.f.} \\
\\ \\ \\ \\ \\ \\
\end{array}
$$

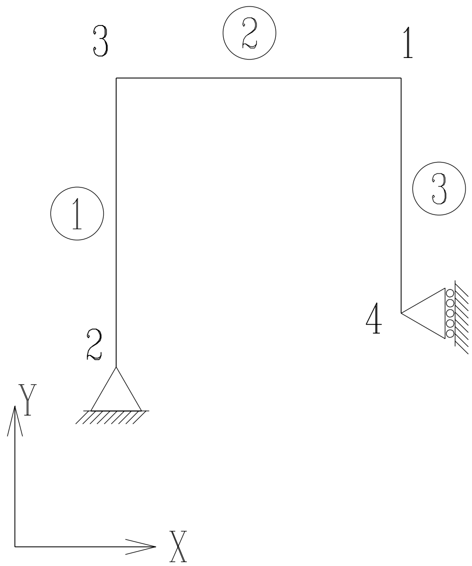


Figure 5.1: Example for [ID] Matrix Determination

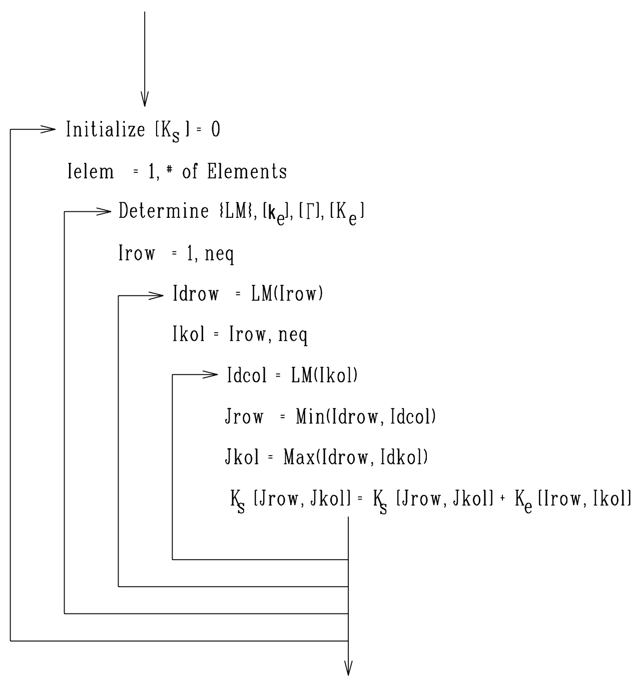


Figure 5.2: Flowchart for Assembling Global Stiffness Matrix

The partially assembled structure global stiffness matrix will then be given by (**check**:

$$
\mathbf{K}^S = \begin{array}{c|cccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
1 & A+p & B+q & C+r & 0 & d & i & m & E & F & & D & \\
2 & & G+s & H+t & 0 & e & j & n & J & K & & I & \\
3 & & & L+u & 0 & f & k & o & N & O & & M & \\
4 & & & & & & & & & & & & \\
6 & & & & & & g & h & & & & & \\
7 & & & & & & & l & & & & & \\
8 & & & & & & & & S & T & & & \\
9 & & & & & & & & & U & & & \\
10 & & & & & & & & & & & & \\
11 & & & & & & & & & & & & \\
12 & & & & & & & & & & & P & \\
\hline
\end{array}
$$

■

18 From this example problem, we note that:

1. Many entries in the global stiffness matrix are left as zero, because they correspond to unconnected degrees of freedom (such as $K_{4,6}$).

2. All entries in the element stiffness matrix do find a storage space in the global stiffness matrix.

5.5 Skyline Storage of Global Stiffness Matrix, `MAXA` Vector

19 The stiffness matrix of a structure will be a square matrix of dimension `NEQxNEQ`.

20 We first observe that the matrix is symmetric, thus only the upper half needs to be stored. Furthermore, we observe that this matrix has a certain "bandwidth", BW, defined as $| K_{ij} - K_{ii} |_{\max}$, when $K_{ij} \neq 0$, Fig. 5.3.

21 Thus, we could as a first space saving solution store the global stiffness matrix inside a rectangular matrix of length `NEQ` and width `BW`, which can be obtained from the `LM` vector (largest difference of terms of `LM` for all the elements).

22 It is evident that numbering of nodes is extremely important as it controls the size of the bandwidth, and hence the storage requirement, Fig. 5.4. In this context, we observe that the stiffness matrix really has a variable bandwidth, or variable "skyline". Hence if we want to store only those entries below the "skyline" inside a vector rather than a matrix for maximum storage efficiency, then we shall define a vector `MAXA` which provides the address of the diagonal terms.

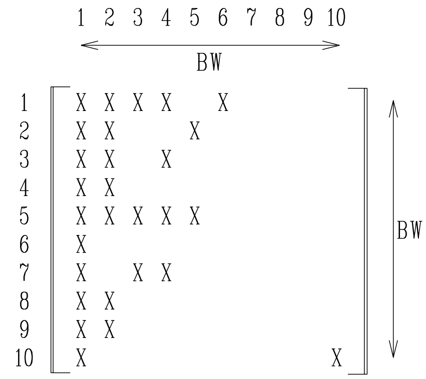


Figure 5.3: Example of Bandwidth

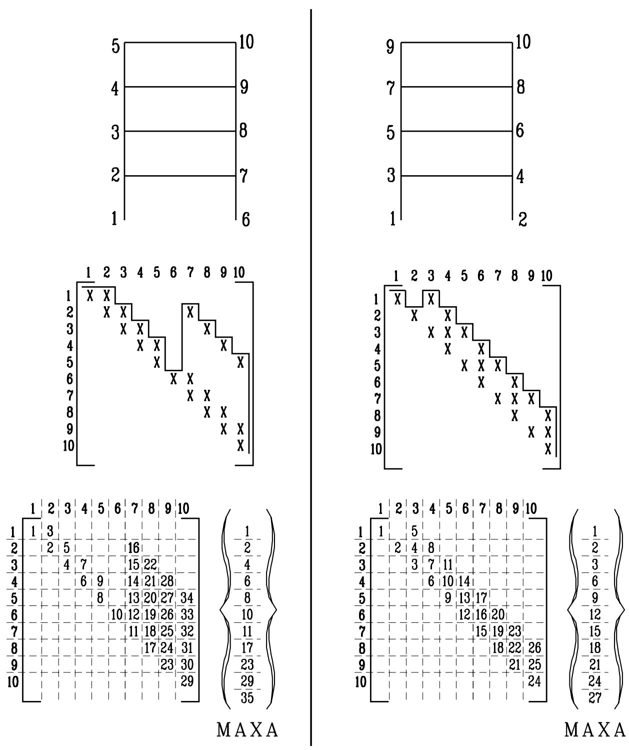


Figure 5.4: Numbering Schemes for Simple Structure

23 In the following global stiffness matrix, the individual entries which must be stored in the global stiffness matrix are replaced by their address in the vector representation of this same matrix. Also shown is the corresponding `MAXA` vector.

$$
\mathbf{K} \;=\; \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left[\begin{array}{cccccccccccc}
x & x &   &   & x & x & x & x & x &   &   & x \\
  & x & x &   & x & x & x & x & x &   &   & x \\
  &   & x &   & x & x & x & x & x &   &   & x \\
  &   &   & x & x & x & x &   &   & x & x &   \\
  &   &   &   & x & x & x &   &   & x & x &   \\
  &   &   &   &   & x & x &   &   & x & x &   \\
  &   &   &   &   &   & x &   &   & x & x &   \\
  &   &   &   &   &   &   & x & x &   &   & x \\
  &   &   &   &   &   &   &   & x &   &   & x \\
  &   &   &   &   &   &   &   &   & x & x &   \\
  &   &   &   &   &   &   &   &   &   & x &   \\
  &   &   &   &   &   &   &   &   &   &   & x
\end{array}\right]
\end{array}
$$

$$
\mathbf{K} \;=\; \left[\begin{array}{ccccccccc|ccc}
1 & 3 &   &   & 11 & 17 & 24 & 32 & 41 &   &   & 68 \\
  & 2 & 5 &   & 10 & 16 & 23 & 31 & 40 &   &   & 67 \\
  &   & 4 &   & 9 & 15 & 22 & 30 & 39 &   &   & 66 \\
  &   &   & 6 & 8 & 14 & 21 & 29 & 38 & 48 & 56 & 65 \\
  &   &   &   & 7 & 13 & 20 & 28 & 37 & 47 & 55 & 64 \\
  &   &   &   &   & 12 & 19 & 27 & 36 & 46 & 54 & 63 \\
  &   &   &   &   &   & 18 & 26 & 35 & 45 & 53 & 62 \\
  &   &   &   &   &   &   & 25 & 34 & 44 & 52 & 61 \\
  &   &   &   &   &   &   &   & 33 & 43 & 51 & 60 \\
\hline
  &   &   &   &   &   &   &   &   & 42 & 50 & 59 \\
  &   &   &   &   &   &   &   &   &   & 49 & 58 \\
  &   &   &   &   &   &   &   &   &   &   & 57
\end{array}\right]
\qquad
\texttt{MAXA} = \left\{\begin{array}{c} 1\\2\\4\\6\\7\\12\\18\\25\\33\\42\\49\\57 \end{array}\right\}
$$

Thus, to locate an element within the stiffness matrix, we use the following formula:

$$\boxed{K_{ij} = \text{MAXA}(j) + (j - i)} \tag{5.4}$$

if $i \leq j$ (since we are storing only the upper half).

24 Using this formula, we will have:

$$K_{58} = \text{MAXA}(8) + (8 - 5) = 18 + 3 = 21 \tag{5.5}$$

$$K_{42} = \text{MAXA}(4) + (4 - 2) = 6 + 2 = 8 \tag{5.6}$$

25 We should note that the total number of non-zero entries inside the global stiffness matrix is always the same, irrespective of our numbering scheme. However by properly numbering the nodes, we can minimize the number of zero terms[2] which would fall below the skyline and which storage would be ineffective.

[2]As we shall see later, all the terms below the skyline (including the zeros) must be stored. Following matrix decomposition, all zero terms outside the skyline terms remain zero, and all others are altered.

5.6 Augmented Stiffness Matrix

26 Previous exposure to the Direct Stiffness Method is assumed.

27 We can conceptually partition the global stiffness matrix into two groups with respective subscript 'u' over Γ_u where the displacements are known (zero or otherwise), and t where the tractions are known.

$$\left\{\begin{array}{c}\mathbf{P}_t\surd\\ \hline \mathbf{R}_u?\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{K}_{tt} & \mathbf{K}_{tu}\\ \hline \mathbf{K}_{ut} & \mathbf{K}_{uu}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{\Delta}_t?\\ \hline \boldsymbol{\Delta}_u\surd\end{array}\right\}$$

28 The first equation enables the calculation of the unknown displacements on Γ_t

$$\boxed{\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\left(\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u\right)} \tag{5.7}$$

29 The second equation enables the calculation of the reactions on Γ_u

$$\boxed{\mathbf{R}_t = \mathbf{K}_{ut}\boldsymbol{\Delta}_t + \mathbf{K}_{uu}\boldsymbol{\Delta}_u} \tag{5.8}$$

30 For internal book-keeping purpose, since we are assembling the *augmented* stiffness matrix, we proceed in two stages:

1. First number all the unrestrained degrees of freedom, i.e. those on Γ_t.
2. Then number all the degrees of freedom with known displacements, on Γ_u, and multiply by -1.

31 Considering a simple beam, Fig. 5.5 the full stiffness matrix is equal to

$$[\mathbf{K}] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad \theta_1 \quad & \quad v_2 \quad & \quad \theta_2 \quad \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array} \tag{5.9}$$

This matrix is singular, it has a rank 2 and order 4 (as it embodies also 2 rigid body motions).

32 We shall consider 3 different cases, Fig. 5.6

Cantilivered Beam/Point Load

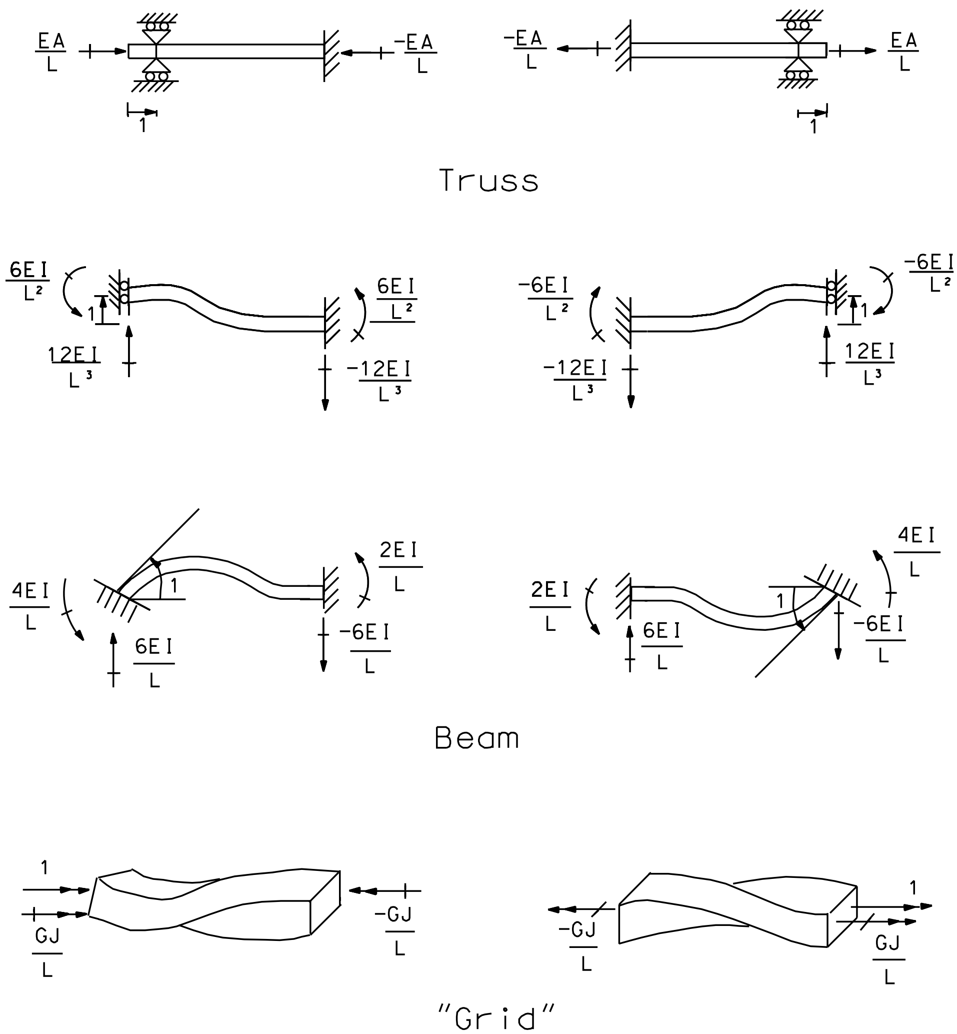


Figure 5.5: Beam Element

1. The *element* stiffness matrix is

$$
\mathbf{k}^e = \begin{array}{c} \\ -3 \\ -4 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{cccc} \;\;-3\;\; & \;\;\;-4\;\;\; & \;\;\;1\;\;\; & \;\;\;2\;\;\; \end{array} \\
\left[\begin{array}{cccc}
12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\
6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\
-12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\
6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L
\end{array}\right]
\end{array}
$$

2. The *structure* stiffness matrix is assembled

$$
\mathbf{K}^S = \begin{array}{c} \\ 1 \\ 2 \\ -3 \\ -4 \end{array}
\begin{array}{c}
\begin{array}{cccc} \;\;\;1\;\;\; & \;\;\;2\;\;\; & \;\;-3\;\; & \;\;-4\;\; \end{array} \\
\left[\begin{array}{cccc}
12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\
-6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\
-12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\
-6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L
\end{array}\right]
\end{array}
$$

3. The global matrix can be rewritten as

$$
\left\{\begin{array}{c} -P\surd \\ 0\surd \\ \hline R_3? \\ R_4? \end{array}\right\}
=
\left[\begin{array}{cc|cc}
12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\
-6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\
\hline
-12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\
-6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L
\end{array}\right]
\left\{\begin{array}{c} \Delta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \theta_4\surd \end{array}\right\}
$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$
\left[\begin{array}{cc|cc}
\boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\
\boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\
\hline
-12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\
-6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L
\end{array}\right]
$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$
\begin{aligned}
\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u &= \left\{\begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array}\right\}
-
\left[\begin{array}{cc|cc}
L^3/3EI & L^2/2EI & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} \\
L^2/2EI & L/EI & \boxed{6EI/L^2} & \boxed{2EI/L} \\
\hline
-12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\
-6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L
\end{array}\right]
\left\{\begin{array}{c} -P \\ 0 \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\} \\
&= \left\{\begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array}\right\}
\end{aligned}
$$

6. Now we solve for the displacement $\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\boldsymbol{\Delta}_t$

$$
\begin{aligned}
\left\{\begin{array}{c} \boxed{\Delta_1} \\ \boxed{\theta_2} \\ \hline 0 \\ 0 \end{array}\right\}
&=
\left[\begin{array}{cc|cc}
\boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\
\boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\
\hline
-12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\
-6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L
\end{array}\right]
\left\{\begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array}\right\} \\
&= \left\{\begin{array}{c} \boxed{-PL^3/3EI} \\ \boxed{-PL^2/2EI} \\ \hline 0 \\ 0 \end{array}\right\}
\end{aligned}
$$

7. Finally, we solve for the reactions, $\mathbf{R}_u = \mathbf{K}_{ut}\boldsymbol{\Delta}_{tt} + \mathbf{K}_{uu}\boldsymbol{\Delta}_u$, and overwrite $\boldsymbol{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} -PL^3/3EI \\ -PL^2/2EI \\ \hline \boxed{R_3} \\ \boxed{R_4} \end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & L^2/2EI & -12EI/L^3 & -6EI/L^2 \\ L^2/2EI & L/EI & 6EI/L^2 & 2EI/L \\ \hline \boxed{-12EI/L^3} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} \\ \boxed{-6EI/L^2} & \boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} \end{array}\right] \left\{\begin{array}{c} \boxed{-PL^3/3EI} \\ \boxed{-PL^2/2EI} \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -PL^3/3EI \\ -PL^2/2EI \\ \hline \boxed{P} \\ \boxed{PL} \end{array}\right\}$$

Simply Supported Beam/End Moment

1. The *element* stiffness matrix is

$$\mathbf{k}^e = \begin{array}{c} \\ -3 \\ 1 \\ -4 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} -3 & 1 & -4 & 2 \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K}^S = \begin{array}{c} \\ 1 \\ 2 \\ -3 \\ -4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & -3 & -4 \end{array} \\ \left[\begin{array}{cccc} 4EI/L & 2EI/L & 6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & 6EI/L^2 & -6EI/L^2 \\ 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \end{array}$$

3. The global stiffness matrix can be rewritten as

$$\left\{\begin{array}{c} 0\surd \\ M\surd \\ \hline R_3? \\ R_4? \end{array}\right\} = \left[\begin{array}{cc|cc} 4EI/L & 2EI/L & 6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \Delta_4\surd \end{array}\right\}$$

4. $\mathbf{K}_{tt}$ is inverted

$$\left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2 \\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right]$$

5. We compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u = \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\} - \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ -L/6EI & L/3EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} 0 \\ M \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\}$$

6. Solve for the displacements, $\mathbf{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P'}_t$, and overwrite $\mathbf{P}_t$ by $\mathbf{\Delta}_t$

$$\left\{\begin{array}{c}\boxed{\theta_1}\\ \boxed{\theta_2}\\ \hline 0\\ 0\end{array}\right\} = \left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2\\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3\\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\boxed{0}\\ \boxed{M}\\ \hline 0\\ 0\end{array}\right\}$$

$$= \left\{\begin{array}{c}\boxed{-ML/6EI}\\ \boxed{ML/3EI}\\ \hline 0\\ 0\end{array}\right\}$$

7. Solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\mathbf{\Delta}_{tt} + \mathbf{K}_{uu}\mathbf{\Delta}_u$, and overwrite $\mathbf{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c}-ML/6EI\\ ML/3EI\\ \hline \boxed{R_1}\\ \boxed{R_2}\end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & 6EI/L^2 & -6EI/L^2\\ -L/6EI & L/3EI & 6EI/L^2 & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{-12EI/L^3}\\ \boxed{-6EI/L^2} & \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{12EI/L^3}\end{array}\right]\left\{\begin{array}{c}\boxed{-ML/6EI}\\ \boxed{ML/3EI}\\ \hline \boxed{0}\\ \boxed{0}\end{array}\right\}$$

$$= \left\{\begin{array}{c}-ML/6EI\\ ML/3EI\\ \hline \boxed{M/L}\\ \boxed{-M/L}\end{array}\right\}$$

Cantilivered Beam/Initial Displacement and Concentrated Moment

1. The *element* stiffness matrix is

$$\mathbf{k}^e = \begin{array}{c} \\ -2\\ -3\\ -4\\ 1\end{array}\begin{array}{c} \begin{array}{cccc} \quad -2 \quad & \quad -3 \quad & \quad -4 \quad & \quad 1 \quad \end{array}\\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2\\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L\\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2\\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L\end{array}\right]\end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K}^S = \begin{array}{c} \\ 1\\ -2\\ -3\\ -4\end{array}\begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad -2 \quad & \quad -3 \quad & \quad -4 \quad \end{array}\\ \left[\begin{array}{cccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2\\ 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]\end{array}$$

3. The global matrix can be rewritten as

$$\left\{\begin{array}{c}M\surd\\ \hline R_2?\\ R_3?\\ R_4?\end{array}\right\} = \left[\begin{array}{c|ccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2\\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\theta_1?\\ \hline \Delta_2\surd\\ \theta_3\surd\\ \Delta_4\surd\end{array}\right\}$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$\left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2\\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u = \left\{\begin{array}{c} \boxed{M} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\} - \left[\begin{array}{c|ccc} L/4EI & \boxed{6EI/L^2} & \boxed{2EI/L} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1 \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta^0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{M + 6EI\Delta^0/L^2} \\ 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

6. Now we solve for the displacements, $\mathbf{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\mathbf{\Delta}_t$

$$\left\{\begin{array}{c} \boxed{\theta_1} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\} = \left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \boxed{M + 6EI\Delta^0/L^2} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{ML/4EI + 3\Delta^0/2L} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

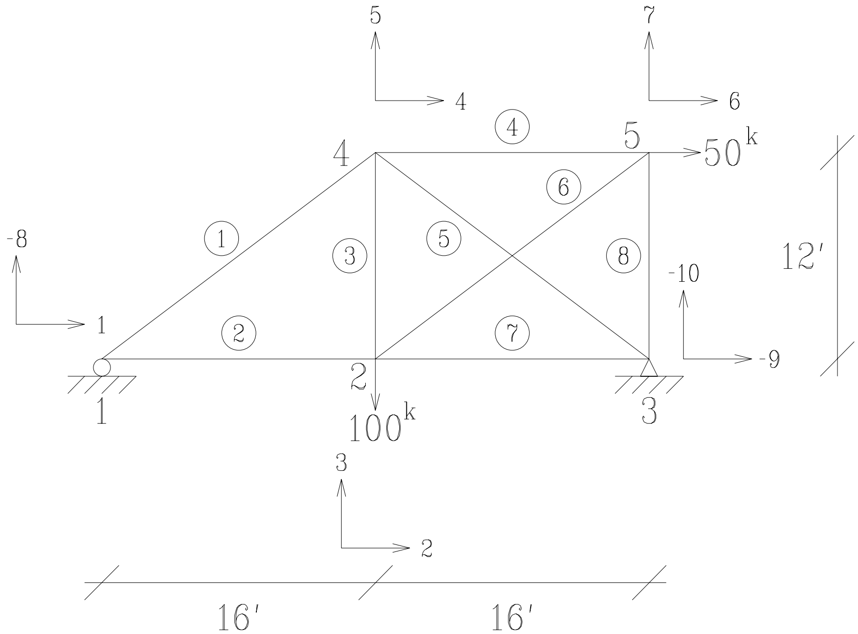
7. Finally, we solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\mathbf{\Delta}_{tt} + \mathbf{K}_{uu}\mathbf{\Delta}_u$, and overwrite $\mathbf{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} ML/4EI + 3\Delta^0/2L \\ \hline \boxed{R_2} \\ \boxed{R_3} \\ \boxed{R_4} \end{array}\right\} = \left[\begin{array}{cc|cc} L/4EI & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} & \boxed{-12EI/L^3} \\ \boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} & \boxed{-6EI/L^2} \\ \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} & \boxed{12EI/L^3} \end{array}\right] \left\{\begin{array}{c} \boxed{ML/4EI + 3\Delta^0/2L} \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta^0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} ML/4EI + 3\Delta^0/2L \\ \hline \boxed{3M/2L - 3EI\Delta^0/L^3} \\ \boxed{M/2 - 3EI\Delta^0/L^2} \\ \boxed{-3M/2L + 3EI\Delta^0/L^3} \end{array}\right\}$$

■ **Example 5-2: Direct Stiffness Analysis of a Truss**

Using the direct stiffness method, analyse the following truss.



Solution:

1. Determine the structure ID matrix and the LM vector for each element Initial ID matrix

$$ID = \begin{array}{c} Node \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \end{bmatrix} \end{array}$$

Final ID matrix

$$ID = \begin{array}{c} Node \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 2 & -9 & 4 & 6 \\ -8 & 3 & -10 & 5 & 7 \end{bmatrix} \end{array}$$

LM vectors for each element

$$\begin{array}{r} \text{Element 1} \\ \text{Element 2} \\ \text{Element 3} \\ \text{Element 4} \\ \text{Element 5} \\ \text{Element 6} \\ \text{Element 7} \\ \text{Element 8} \end{array} \begin{bmatrix} 1 & -8 & 4 & 5 \\ 1 & -8 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ -9 & -10 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 2 & 3 & -9 & -10 \\ -9 & -10 & 6 & 7 \end{bmatrix}$$

2. Derive the element stiffness matrix in global coordinates

$$[K] = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

where $c = cos\alpha = \frac{x_2 - x_1}{L}$, $s = sin\alpha = \frac{Y_2 - Y_1}{L}$

Element 1 $L = 20'$, $c = \frac{16-0}{20} = 0.8$, $s = \frac{12-0}{20} = 0.6$,
$\frac{EA}{L} = \frac{(30{,}000)(10)}{20} = 15,000$

$$[K_1] = \begin{array}{c} \\ 1 \\ -8 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & -8 & 4 & 5 \end{array} \\ \begin{bmatrix} 9600 & 7200 & -9600 & -7200 \\ 7200 & 5400 & -7200 & -5400 \\ -9600 & -7200 & 9600 & 7200 \\ -7200 & -5400 & 7200 & 5400 \end{bmatrix} \end{array}$$

Element 2 $L = 16', c = 1, s = 0, \frac{EA}{L} = 18,750.$

$$[K_2] = \begin{array}{c} \\ 1 \\ -8 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & -8 & 2 & 3 \end{array} \\ \begin{bmatrix} 18,750 & 0 & -18,750 & 0 \\ 0 & 0 & 0 & 0 \\ -18,750 & 0 & 18,750 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 3 $L = 12', c = 0, s = 1, \frac{EA}{L} = 25,000$

$$[K_3] = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}$$

Element 4 $L = 16', c = 1, s = 0, \frac{EA}{L} = 18,750$

$$[K_4] = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 18,750 & 0 & -18,750 & 0 \\ 0 & 0 & 0 & 0 \\ -18,750 & 0 & 18,750 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 5 $L = 20', c = \frac{-16-0}{20} = -0.8, s = 0.6, \frac{EA}{L} = 15,000$

$$[K_5] = \begin{array}{cc} & \begin{array}{cccc} -9 & -10 & 4 & 5 \end{array} \\ \begin{array}{c} -9 \\ -10 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 9600 & -7200 & -9600 & 7200 \\ -7200 & 5400 & 7200 & -5400 \\ -9600 & 7200 & 9600 & -7200 \\ 7200 & -5400 & -7200 & 5400 \end{bmatrix} \end{array}$$

Element 6 $L = 20', c = 0.8, s = 0.6, \frac{EA}{L} = 15,000$

$$[K_6] = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 6 & 7 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 9600 & 7200 & -9600 & -7200 \\ 7200 & 5400 & -7200 & -5400 \\ -9600 & -7200 & 9600 & 7200 \\ -7200 & -5400 & 7200 & 5400 \end{bmatrix} \end{array}$$

Element 7 $L = 16', c = 1, s = 0, \frac{EA}{L} = 18,750$

$$[K_7] = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & -9 & -10 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ -9 \\ -10 \end{array} & \begin{bmatrix} 18,750 & 0 & -18,750 & 0 \\ 0 & 0 & 0 & 0 \\ -18,750 & 0 & 18,750 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 8 $L = 12', c = 0, s = 1, \frac{EA}{L} = 25,000$

$$[K_8] = \begin{array}{cc} & \begin{array}{cccc} -9 & -10 & 6 & 7 \end{array} \\ \begin{array}{c} -9 \\ -10 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}$$

3. Assemble the augmented global stiffness matrix in kips/ft.

$$\mathbf{k}_{tt} = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 9,600+18,750 & -18,750 & 0 & -9,600 & -7,200 & 0 & 0 \\ & 9,600+(2)18,750 & 7,200 & 0 & 0 & -9,600 & -7,200 \\ & & 5,400+25,000 & 0 & -25,000 & -7,200 & -5,400 \\ & & & 18,750+(2)9,600 & 7,200-7,200 & -18,750 & 0 \\ & \text{SYMMETRIC} & & & 25,000+5,400(2) & 0 & 0 \\ & & & & & 18,750+9600 & 7200 \\ & & & & & & 25,000+5,400 \end{bmatrix} \end{array}$$

$$\mathbf{k}_{tu} = \begin{array}{cc} & \begin{array}{ccc} -8 & -9 & -10 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0+7,200 & & \\ 0 & -18,750-18,750 & 0 \\ 0 & 0 & 0 \\ -7,200 & -9,600 & 7,200 \\ -5,400 & 7,200 & -5,400 \\ & & 0 \\ & & -25,000 \end{bmatrix} \end{array}$$

$$
\mathbf{k}_{uu} = \begin{array}{c} \\ -8 \\ -9 \\ -10 \end{array}
\begin{array}{c}
\begin{array}{ccc} -8 & -9 & -10 \end{array} \\
\left[\begin{array}{ccc}
0+5,400 & & \\
 & 9,600+18,750 & 0-7,200+0 \\
 & -7,200+0 & 0+5,400+25,000
\end{array}\right]
\end{array}
$$

4. convert to kips/in and simplify

$$
\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 50 \\ 0 \end{array}\right\} =
\left[\begin{array}{ccccccc}
2,362.5 & -1,562.5 & 0 & -800 & -600 & 0 & 0 \\
 & 3,925.0 & 600 & 0 & 0 & -800 & -600 \\
 & & 2,533.33 & 0 & -2,083.33 & -600 & -450 \\
 & & & 3,162.5 & 0 & -1,562.5 & 0 \\
\text{Symmetric} & & & & 2,983.33 & 0 & 0 \\
 & & & & & 2,362.5 & 600 \\
 & & & & & & 2,533.33
\end{array}\right]
\left\{\begin{array}{c} u_1 \\ u_2 \\ v_3 \\ u_4 \\ v_5 \\ u_6 \\ v_7 \end{array}\right\}
$$

5. Invert stiffness matrix and solve for displacements (inches)

$$
\left\{\begin{array}{c} u_1 \\ u_2 \\ v_3 \\ u_4 \\ v_5 \\ u_6 \\ v_7 \end{array}\right\} =
\left\{\begin{array}{c} -0.0223 \\ 0.00433 \\ -0.116 \\ -0.0102 \\ -0.0856 \\ -0.00919 \\ -0.0174 \end{array}\right\}
$$

6. Solve for member forces in local coordinate systems

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\} =
\left[\begin{array}{cccc} c & s & -c & -s \\ -c & -s & c & s \end{array}\right]
\left\{\begin{array}{c} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{array}\right\}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^1 = \frac{15,000}{12}
\left[\begin{array}{cccc} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{array}\right]
\left\{\begin{array}{c} -0.0223 \\ 0 \\ -0.0102 \\ -0.0856 \end{array}\right\} =
\left\{\begin{array}{c} 52.1 \\ -52.1 \end{array}\right\} \text{Compression}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^2 = \frac{18,750}{12}
\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]
\left\{\begin{array}{c} -0.0233 \\ 0 \\ 0.00433 \\ -0.116 \end{array}\right\} =
\left\{\begin{array}{c} -43.2 \\ 43.2 \end{array}\right\} \text{Tension}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^3 = \frac{25,000}{12}
\left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{array}\right]
\left[\begin{array}{c} 0.00433 \\ -0.116 \\ -0.0102 \\ -0.0856 \end{array}\right] =
\left\{\begin{array}{c} -63.3 \\ 63.3 \end{array}\right\} \text{Tension}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^4 = \frac{18,750}{12}
\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]
\left\{\begin{array}{c} -0.0102 \\ -0.0856 \\ -0.00919 \\ -0. \end{array}\right\} =
\left\{\begin{array}{c} -1.58 \\ 1.58 \end{array}\right\} \text{Tension}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^5 = \frac{15,000}{12}
\left[\begin{array}{cccc} -0.8 & 0.6 & 0.8 & -0.6 \\ 0.8 & -0.6 & -0.8 & 0.6 \end{array}\right]
\left\{\begin{array}{c} -0.0102 \\ -0.0856 \end{array}\right\} =
\left\{\begin{array}{c} 54.0 \\ -54.0 \end{array}\right\} \text{Compression}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^6 = \frac{15,000}{12}
\left[\begin{array}{cccc} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{array}\right]
\left\{\begin{array}{c} -0.116 \\ -0.00919 \\ -0.0174 \end{array}\right\} =
\left\{\begin{array}{c} -60.43 \\ 60.43 \end{array}\right\} \text{Tension}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^7 = \frac{18,750}{12}
\left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]
\left\{\begin{array}{c} -0.116 \\ 0 \\ 0 \end{array}\right\} =
\left\{\begin{array}{c} 6.72 \\ -6.72 \end{array}\right\} \text{Compression}
$$

$$
\left\{\begin{array}{c} v_1 \\ v_2 \end{array}\right\}^8 = \frac{25,000}{12}
\left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{array}\right]
\left\{\begin{array}{c} 0 \\ 0 \\ -0.00919 \\ -0.0174 \end{array}\right\} =
\left\{\begin{array}{c} 36.3 \\ -36.3 \end{array}\right\} \text{Compression}
$$

7. Determine the structure's `MAXA` vector

$$\begin{bmatrix} 1 & 3 & & 9 & 14 & & \\ & 2 & 5 & 8 & 13 & 19 & 25 \\ & & 4 & 7 & 12 & 18 & 24 \\ & & & 6 & 11 & 17 & 23 \\ & & & & 10 & 16 & 22 \\ & & & & & 15 & 21 \\ & & & & & & 20 \end{bmatrix}$$

$$MAXA = \begin{Bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 10 \\ 15 \\ 20 \end{Bmatrix}$$

25 terms would have to be stored.

■

■ Example 5-3: Assembly of the Global Stiffness Matrix

As an example, let us consider the frame shown in Fig. 5.7.
The ID matrix is initially set to:

$$[ID] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{5.10}$$

We then modify it to generate the global degrees of freedom of each node:

$$[ID] = \begin{bmatrix} -4 & 1 & -7 \\ -5 & 2 & -8 \\ -6 & 3 & -9 \end{bmatrix} \tag{5.11}$$

Finally the LM vectors for the two elements (assuming that Element 1 is defined from node 1 to node 2, and element 2 from node 2 to node 3):

$$[LM] = \left[\begin{array}{ccc|ccc} -4 & -5 & -6 & 1 & 2 & 3 \\ 1 & 2 & 3 & -7 & -8 & -9 \end{array}\right] \tag{5.12}$$

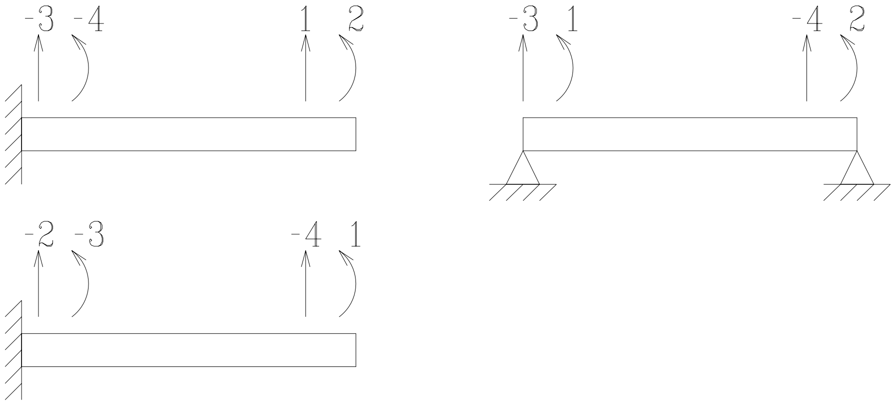


Figure 5.6: ID Values for Simple Beam

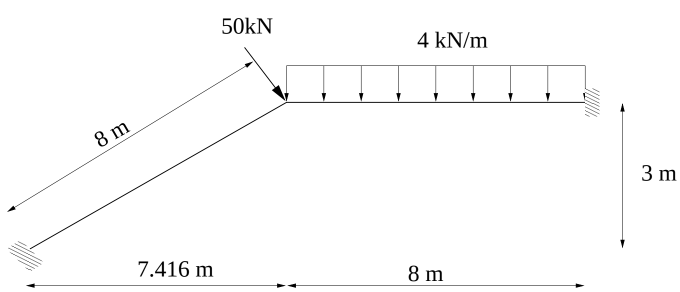


Figure 5.7: Simple Frame Anlysed with the MATLAB Code

Let us simplfy the operation by designating the element stiffness matrices in global coordinates as follows:

$$K^{(1)} = \begin{array}{c c} & \begin{array}{cccccc} -4 & -5 & -6 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} -4 \\ -5 \\ -6 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{array}\right] \end{array} \tag{5.13-a}$$

$$K^{(2)} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & -7 & -8 & -9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ -7 \\ -8 \\ -9 \end{array} & \left[\begin{array}{cccccc} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} \end{array}\right] \end{array} \tag{5.13-b}$$

We note that for each element we have shown the corresponding LM vector.

Now, we assemble the global stiffness matrix

$$K = \left[\begin{array}{ccc|cccccc} A_{44}+B_{11} & A_{45}+B_{12} & A_{46}+B_{13} & A_{41} & A_{42} & A_{43} & B_{14} & B_{15} & B_{16} \\ A_{54}+B_{21} & A_{55}+B_{22} & A_{56}+B_{23} & A_{51} & A_{52} & A_{53} & B_{24} & B_{25} & B_{26} \\ A_{64}+B_{31} & A_{65}+B_{32} & A_{66}+B_{33} & A_{61} & A_{62} & A_{63} & B_{34} & B_{35} & B_{36} \\ \hline A_{14} & A_{15} & A_{16} & A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{24} & A_{25} & A_{26} & A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{34} & A_{35} & A_{36} & A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\ B_{41} & B_{42} & B_{43} & 0 & 0 & 0 & B_{44} & B_{45} & B_{46} \\ B_{51} & B_{52} & B_{53} & 0 & 0 & 0 & B_{54} & B_{55} & B_{56} \\ B_{61} & B_{62} & B_{63} & 0 & 0 & 0 & B_{64} & B_{65} & B_{66} \end{array}\right] \tag{5.14}$$

■

We note that some terms are equal to zero because we do not have a connection between the corresponding degrees of freedom (i.e. node 1 is not connected to node 3).

■ **Example 5-4: Analysis of a Frame with MATLAB**

The simple frame shown in Fig. 5.8 is to be analysed by the direct stiffness method. Assume: $E = 200,000$ MPa, $A = 6,000$ mm^2, and $I = 200 \times 10^6$ mm^4. The complete MATLAB solution is shown below along with the results.

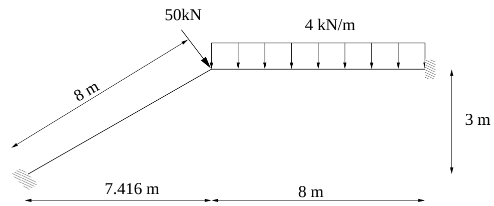


Figure 5.8: Simple Frame Anlysed with the MATLAB Code

```
% zero the matrices
k=zeros(6,6,2);
K=zeros(6,6,2);
Gamma=zeros(6,6,2);
% Structural properties units: mm^2, mm^4, and MPa(10^6 N/m)
A=6000;II=200*10^6;EE=200000;
% Convert units to meter and kN
A=A/10^6;II=II/10^12;EE=EE*1000;
% Element 1
i=[0,0];j=[7.416,3];
[k(:,:,1),K(:,:,1),Gamma(:,:,1)]=stiff(EE,II,A,i,j);
% Element 2
i=j;j=[15.416,3];
[k(:,:,2),K(:,:,2),Gamma(:,:,2)]=stiff(EE,II,A,i,j);
% Define ID matrix
ID=[
   -4 1 -7;
   -5 2 -8;
   -6 3 -9];
% Determine the LM matrix
LM=[
   -4 -5 -6  1  2  3;
    1  2  3 -7 -8 -9];
% Assemble augmented stiffness matrix
Kaug=zeros(9);
for elem=1:2
   for r=1:6
      lr=abs(LM(elem,r));
      for c=1:6
         lc=abs(LM(elem,c));
         Kaug(lr,lc)=Kaug(lr,lc)+K(r,c,elem);
      end
   end
```

```
end
% Extract the structures Stiffness Matrix
Ktt=Kaug(1:3,1:3);
% Determine the fixed end actions in local coordinate system
fea(1:6,1)=0;
fea(1:6,2)=[0,8*4/2,4*8^2/12,0,8*4/2,-4*8^2/12]';
% Determine the fixed end actions in global coordinate system
FEA(1:6,1)=Gamma(:,:,1)*fea(1:6,1);
FEA(1:6,2)=Gamma(:,:,2)*fea(1:6,2);
% FEA_Rest for all the restrained nodes
FEA_Rest=[0,0,0,FEA(4:6,2)'];
% Assemble the load vector for the unrestrained node
P(1)=50*3/8;P(2)=-50*7.416/8-fea(2,2);P(3)=-fea(3,2);
% Solve for the Displacements in meters and radians
Displacements=inv(Ktt)*P'
% Extract Kut
Kut=Kaug(4:9,1:3);
% Compute the Reactions and do not forget to add fixed end actions
Reactions=Kut*Displacements+FEA_Rest'
% Solve for the internal forces and do not forget to include the fixed end actions
dis_global(:,:,1)=[0,0,0,Displacements(1:3)'];
dis_global(:,:,2)=[Displacements(1:3)',0,0,0];
for elem=1:2
   dis_local=Gamma(:,:,elem)*dis_global(:,:,elem)';
   int_forces=k(:,:,elem)*dis_local+fea(1:6,elem)
end

function [k,K,Gamma]=stiff(EE,II,A,i,j)
% Determine the length
L=sqrt((j(2)-i(2))^2+(j(1)-i(1))^2);
% Compute the angle theta (carefull with vertical members!)
if(j(1)-i(1))~=0
   alpha=atan((j(2)-i(2))/(j(1)-i(1)));
else
   alpha=-pi/2;
end
% form rotation matrix Gamma
Gamma=[
cos(alpha)  sin(alpha)  0  0           0           0;
-sin(alpha) cos(alpha)  0  0           0           0;
0           0           1  0           0           0;
0           0           0  cos(alpha)  sin(alpha)  0;
0           0           0 -sin(alpha)  cos(alpha)  0;
```

```
0           0           0  0          0          1];
% form element stiffness matrix in local coordinate system
EI=EE*II;
EA=EE*A;
k=[EA/L,          0,          0, -EA/L,          0,          0;
   0,    12*EI/L^3,  6*EI/L^2,     0, -12*EI/L^3,  6*EI/L^2;
   0,     6*EI/L^2,    4*EI/L,     0,  -6*EI/L^2,    2*EI/L;
 -EA/L,          0,          0,  EA/L,          0,          0;
     0, -12*EI/L^3, -6*EI/L^2,     0,  12*EI/L^3, -6*EI/L^2;
     0,   6*EI/L^2,    2*EI/L,     0,  -6*EI/L^2,    4*EI/L];
% Element stiffness matrix in global coordinate system
K=Gamma'*k*Gamma;
```

This simple proigram will produce the following results:

```
Displacements =

    0.0010
   -0.0050
   -0.0005

Reactions =

  130.4973
   55.6766
   13.3742
 -149.2473
   22.6734
  -45.3557


int_forces =   int_forces =

  141.8530       149.2473
    2.6758         9.3266
   13.3742        -8.0315
 -141.8530      -149.2473
   -2.6758        22.6734
    8.0315       -45.3557
```

We note that the internal forces are consistent with the reactions (specially for the second node of element 2), and amongst themselves, i.e. the moment at node 2 is the same for both elements (8.0315). ■

5.7 Computer Program Flow Charts

33 The main program should, Fig. 5.9:

1. Read
   (a) TITLE CARD
   (b) CONTROL CARD which should include:
      i. Number of nodes
      ii. Number of elements
      iii. Type of structure: beam, grid, truss, or frame; (2D or 3D)
      iv. Number of different element properties
      v. Number of load cases
2. Determine:
   (a) Number of spatial coordinates for the structure
   (b) Number of local and global degrees of freedom per node
3. Set up the pointers of the dynamic memory allocation (if using f77) for:
   (a) Nodal coordinates
   (b) Equation number matrix (ID)
   (c) Element connectivity
   (d) Element properties
   (e) Element stiffness matrices
   (f) Element rotation matrices
4. Loop over all the elements and determine the element stiffness matrices (in local coordinates), and rotation angles.
5. Determine the column heights, and initialize the global stiffness vector to zero.
6. Loop through all the elements, and for each one
   (a) Determine the element stiffness matrices in global coordinates
   (b) Determine the `LM` vector
   (c) Assemble the structure's global stiffness matrix.
7. Decompose the global stiffness matrix using a Cholesky's decomposition).
8. For each load case:

(a) Determine the nodal equivalent loads (fixed end actions), if any.
(b) Assemble the load vector
(c) Backsubstitute and obtain the nodal displacements
(d) Loop through each element and:
   i. Determine the nodal displacements in local coordinates
   ii. Determine the internal forces (include effects of fixed end actions).

34 The tree structure of the program is illustrated in Fig. 5.10

5.7.1 Input

35 The input subroutine should:

1. For each node read:
   (a) Node number
   (b) Boundary conditions of each global degree of freedom [**ID**]
   (c) Spatial coordinates

   Note that all the above are usually written on the same "data card"

2. Determine equation numbers associated with each degree of freedom, and the total number of equations (`NEQ`).

3. For each element, read:
   (a) Element number
   (b) First and second node
   (c) Element Property number

4. For each element property group read the associated elastic and cross sectional characteristics. Note these variables will depend on the structure type.

5.7.2 Element Stiffness Matrices

For each element:

1. Retrieve its properties

2. Determine the length

3. Call the appropriate subroutines which will determine:
   (a) The stiffness matrix in local coordinate systems [$\mathbf{k}^e$].
   (b) The direction cosines.

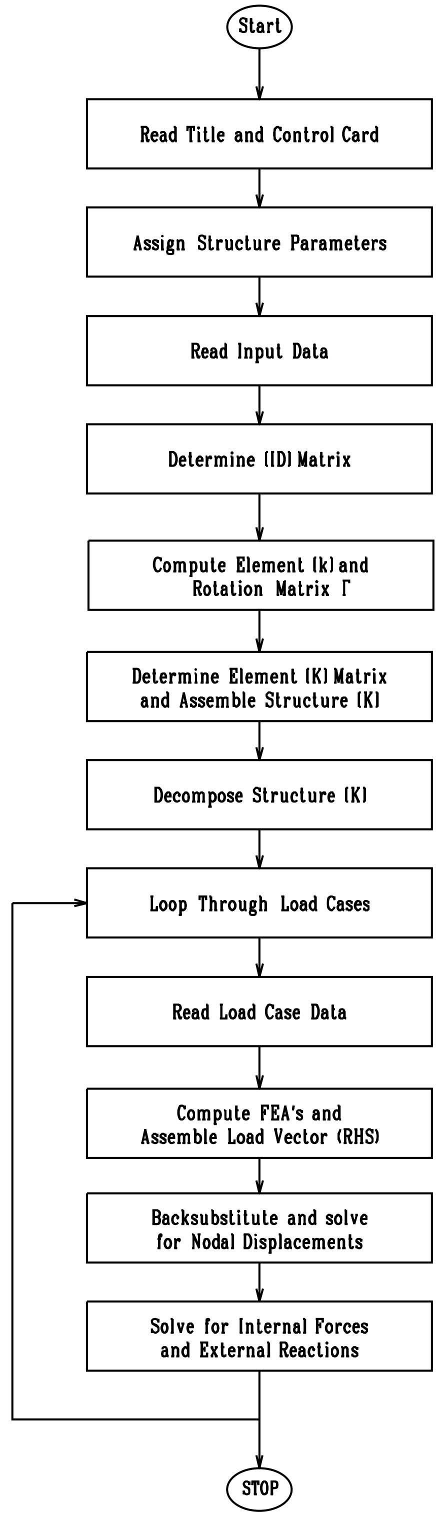


Figure 5.9: Program Flowchart

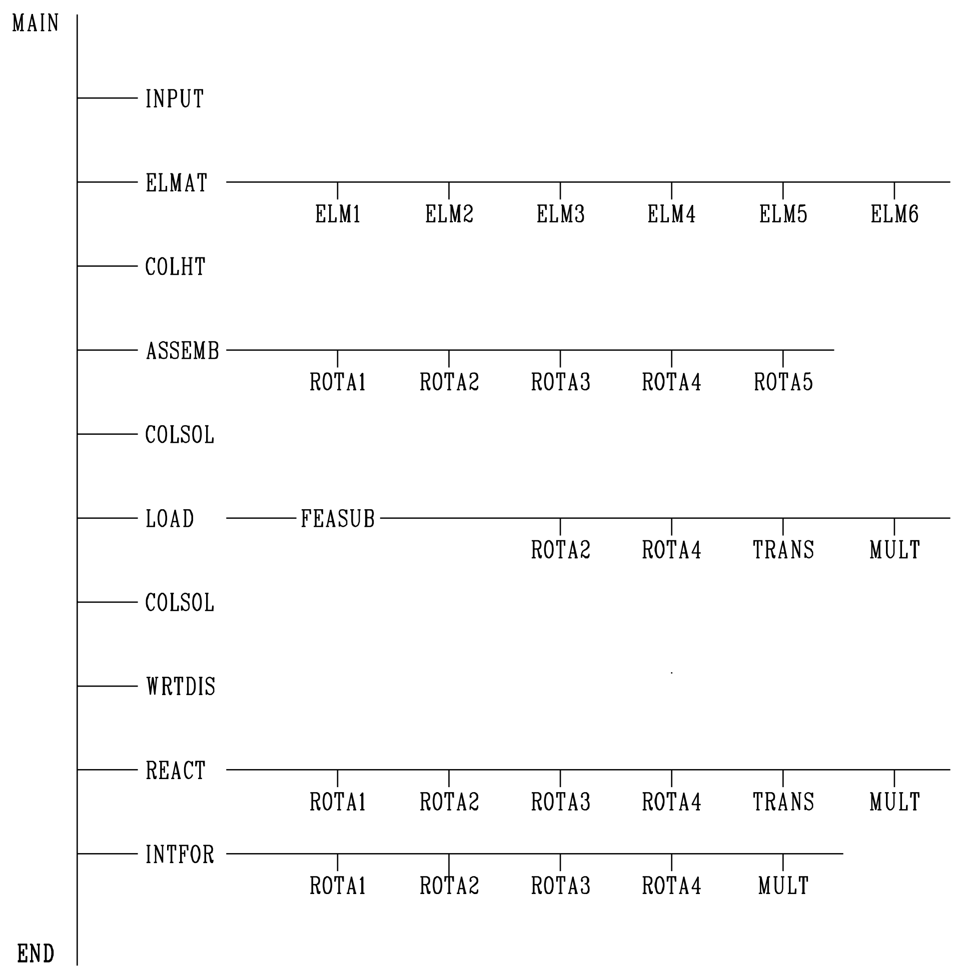


Figure 5.10: Program's Tree Structure

5.7.3 Assembly

Since a skyline solver will be used, we first need to determine the appropriate pointers which will enable us to efficiently store the global stiffness matrix ({**MAXA**}). This is accomplished as follows, Fig. 5.11:

1. Determine the maximum height of the skyline for each column of the global stiffness matrix by first assigning a very large number to each row of {**MAXA**}, and then looping through each element, and for each one:
   (a) Determine the lowest associated global degree of freedom number (from the {**LM**} vectors)
   (b) Compare this "height" with the one currently associated with those degree of freedom stored in the element {**LM**}; if lower overwrite
2. Determine the total height of each skyline (i.e. each column) by determining the difference between **MAXA** (IEQ) (Skyline elevation), and IEQ ("BottomLine"). Overwrite **MAXA** with this height.
3. Determine the total length of the vector storing the compacted structure global stiffness matrix by summing up the height of each skyline
4. Assign to `MAXA(NEQ+1)` this total length +1.
5. Loop backward from the last column to the first, and for each one determine the address of the diagonal term from $\mathbf{MAXA}(IEQ) = \mathbf{MAXA}(IEQ+1) - \mathbf{MAXA}(IEQ)$

36 Once the **MAXA** vector has been determine, then term $K(i,j)$ in the square matrix, would be stored in `KK(MAXA(j)+j-i)` (assuming $j > i$) in the compacted form of {**K**}.

37 The assembly of the global stiffness matrix is next described, Fig. 5.12:

1. Initialize the vector storing the compacted stiffness matrix to zero.
2. Loop through each element, e, and for each element:
   (a) Retrieve its stiffness matrix (in local coordinates) $[\mathbf{k}^e]$, and direction cosines.
   (b) Determine the rotation matrix $[\mathbf{\Gamma}]$ of the element.
   (c) Compute the element stiffness matrix in global coordinates from $[bK^e] = [\mathbf{\Gamma}]^T[\mathbf{k}^e][\mathbf{\Gamma}]$.
   (d) Define the {**LM**} array of the element
   (e) Loop through each row and column of the element stiffness matrix, and for those degree of freedom not equal to zero, add the contributions of the element to the structure's stiffness matrix (note that we assemble only the upper half).

$$\boxed{K^S[LM(i), LM(j)] = K^S[LM(i), LM(j)] + K^e[i,j]} \tag{5.15}$$

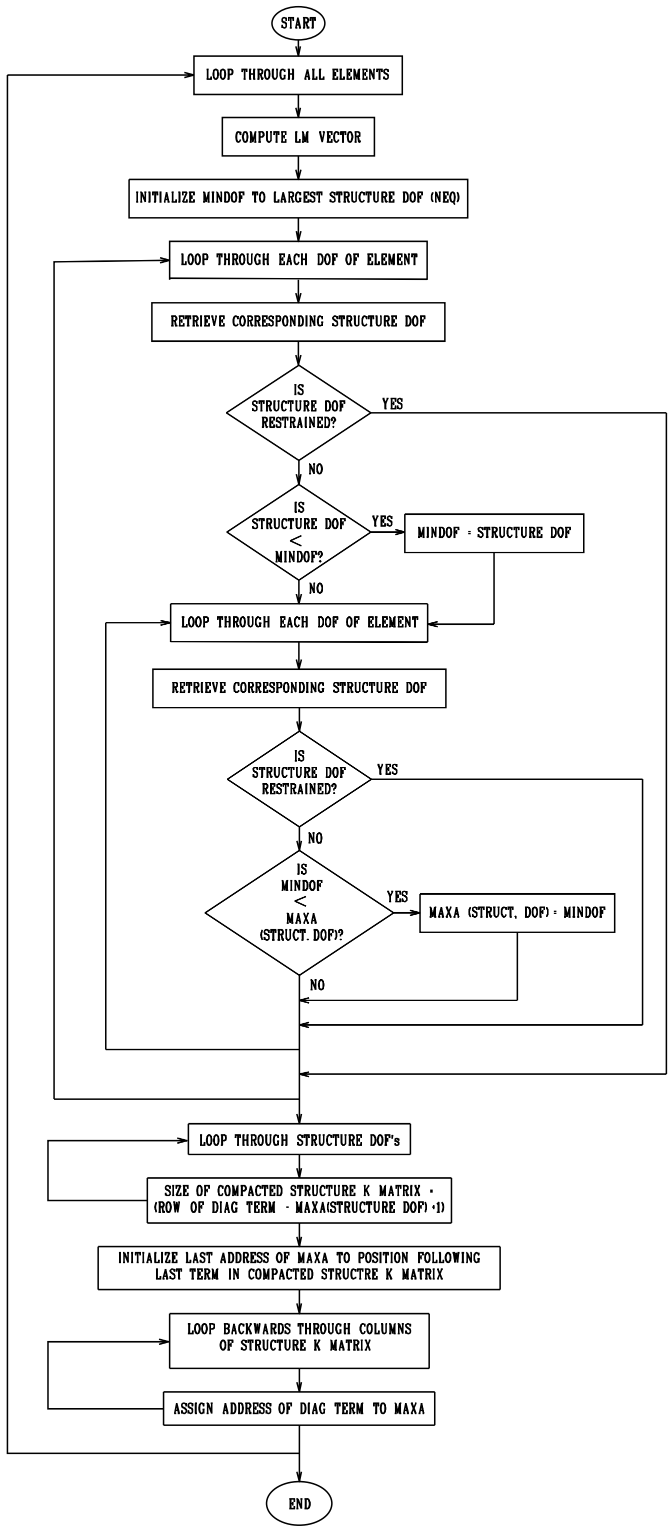


Figure 5.11: Flowchart for the Skyline Height Determination

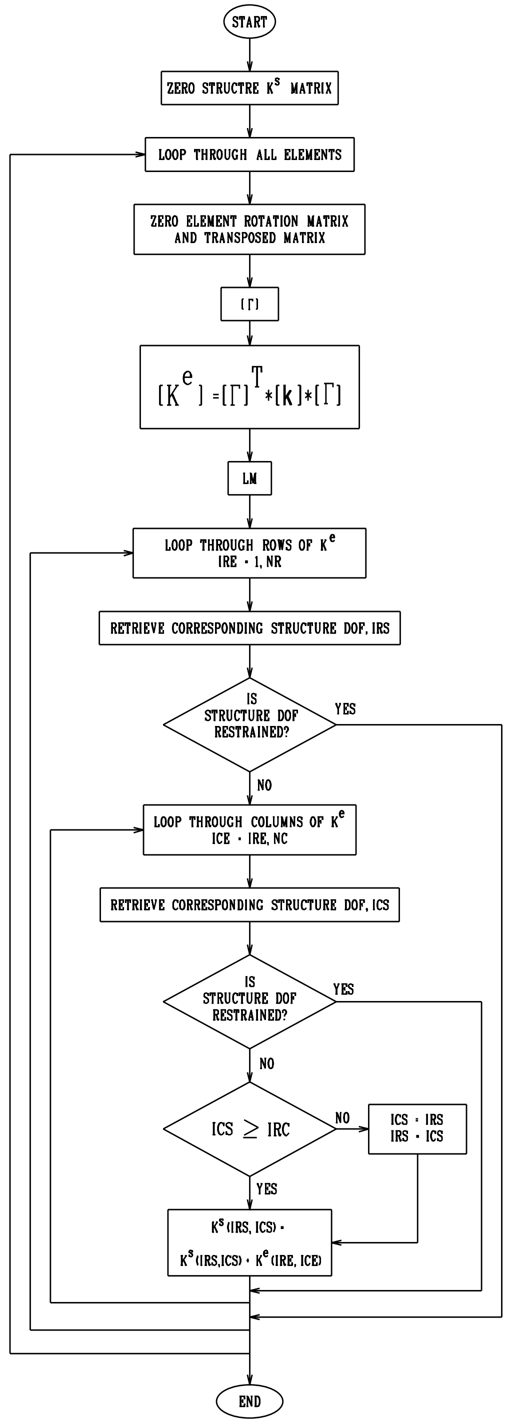


Figure 5.12: Flowchart for the Global Stiffness Matrix Assembly

5.7.4 Decomposition

38 Decompose the global stiffness matrix. Since the matrix is both symmetric and positive definite, the matrix can be decomposed using Cholesky's method into: $[\mathbf{K}] = [\mathbf{L}][\mathbf{L}]^T$. Should a division by zero occur, or an attempt to extract the square root of a negative number happen, then this would be an indication that either the global stiffness matrix is not properly assembled, or that there are not enough restraint to prevent rigid body translation or rotation of the structure.

5.7.5 Load

39 Once the stiffness matrix has been decomposed, than the main program should loop through each load case and, Fig. 5.13

1. Initialize the load vector (of length NEQ) to zero.
2. Read number of loaded nodes. For each loaded node store the non-zero values inside the load vector (using the [**ID**] matrix for determining storage location).
3. Loop on all loaded elements:
   (a) Read element number, and load value
   (b) Compute the fixed end actions and rotate them from local to global coordinates.
   (c) Using the **LM** vector, add the fixed end actions to the nodal load vector (unless the corresponding equation number is zero, ie. restrained degree of freedom).
   (d) Store the fixed end actions for future use.

5.7.6 Backsubstitution

40 Backsubstitution is achieved by multiplying the decomposed stiffness matrix with the load vector. The resulting vector stores the nodal displacements, in global coordinate system, corresponding to the unrestrained degree of freedom.

5.7.7 Internal Forces and Reactions

41 The internal forces for each element, and reactions at each restrained degree of freedom, are determined by, Fig. 5.15

1. Initialize reactions to zero
2. For each element retrieve:
   (a) nodal coordinates

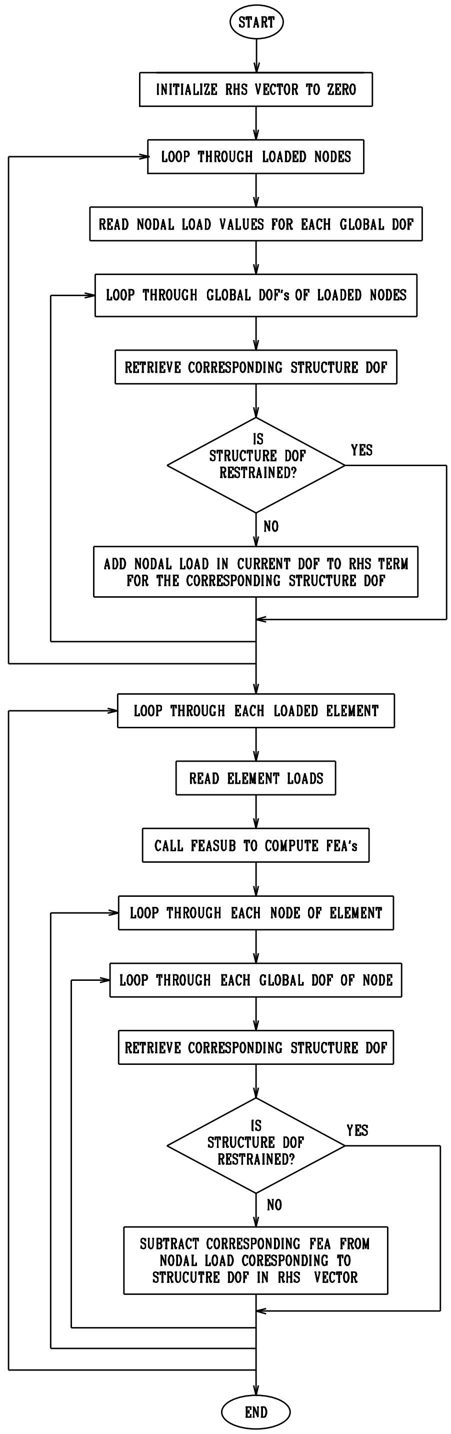


Figure 5.13: Flowchart for the Load Vector Assembly

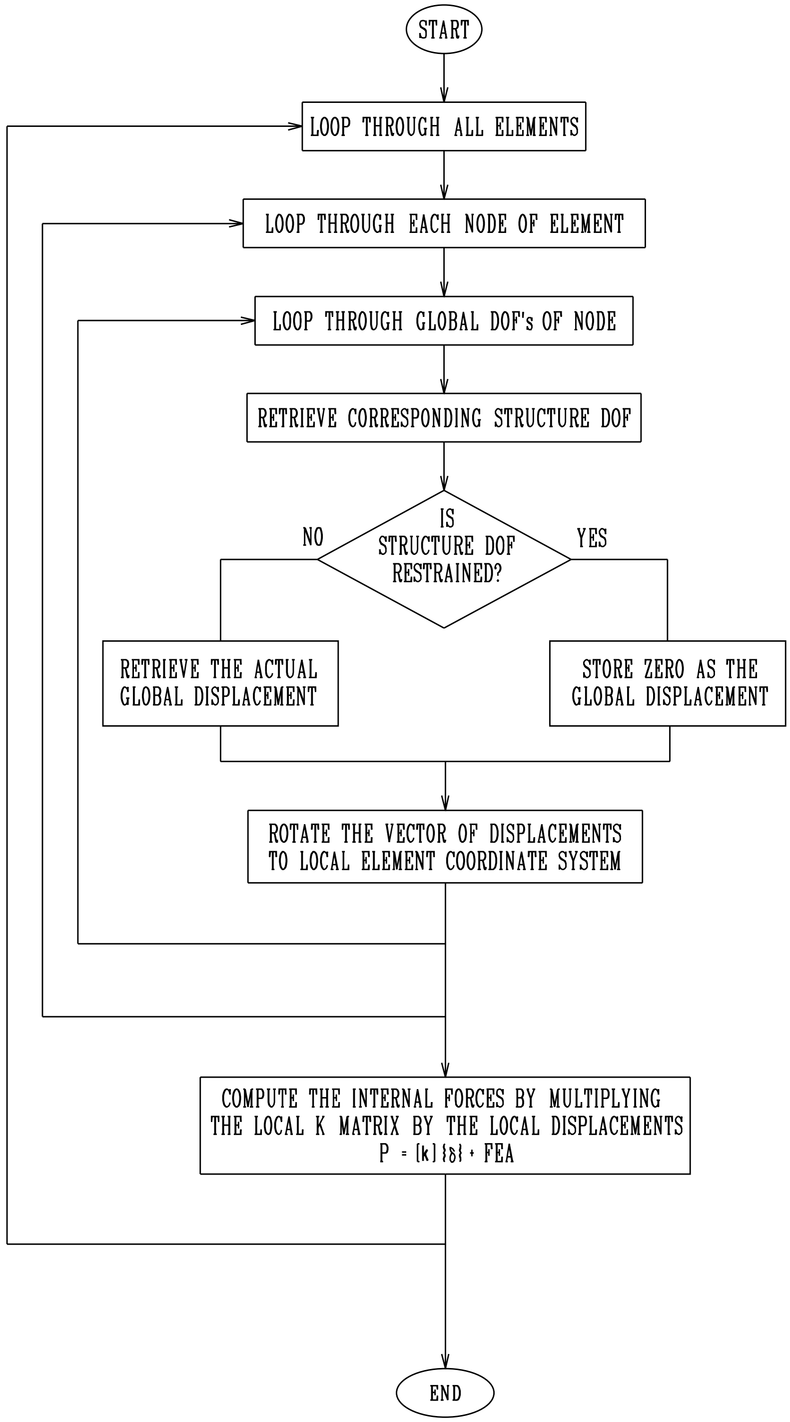


Figure 5.14: Flowchart for the Internal Forces

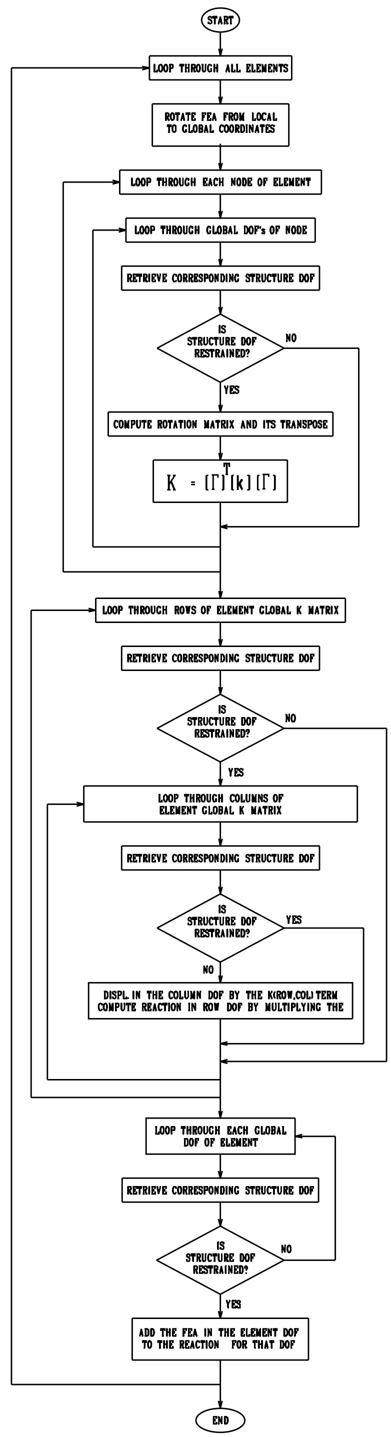


Figure 5.15: Flowchart for the Reactions

   (b) rotation matrix
   (c) element stiffness matrix

3. Compute nodal displacements in local coordinate system from $\boldsymbol{\delta}^e = [\boldsymbol{\Gamma}]\boldsymbol{\Delta}(\mathbf{LM})$
4. Compute element internal forces from $\mathbf{p} = [\mathbf{k}^e]\boldsymbol{\delta}^e$
5. If the element is loaded, add corresponding fixed end actions
6. print the interior forces
7. check if any of its degree of freedom is restrained, if so:
   (a) rotate element forces to global coordinates
   (b) update appropriate reaction

5.8 Computer Implementation with MATLAB

42 You will be required, as part of your term project, to write a simple MATLAB (or whatever other language you choose) program for the analysis of two dimensional frames with nodal load and initial displacement, as well as element load.

43 To facilitate the task, your instructor has taken the liberty of taking a program written by Mr. Dean Frank (as part of his term project with this instructor in the Advanced Structural Analysis course, Fall 1995), modified it with the aid of Mr. Pawel Smolarki, and is making available most, but not all of it to you. Hence, you will be expected to first familiarize yourself with the code made available to you, and then complete it by essentially filling up the missing parts.

5.8.1 Program Input

`From Dean Frank's User's Manual`

44 It is essential that the structure be idealized such that it can be discretized. This discretization should define each node and element uniquely. In order to decrease the required amount of computer storage and computation it is best to number the nodes in a manner that minimizes the numerical separation of the node numbers on each element. For instance, an element connecting nodes 1 and 4, could be better defined by nodes 1 and 2, and so on. As it was noted previously, the user is required to have a decent understanding of structural analysis and structural mechanics. As such, it will be necessary for the user to generate or modify an input file input.m using the following directions. Open the file called input.m and set the existing variables in the file to the appropriate values. The input file has additional helpful directions given as comments for each variable. After setting the variables to the correct values, be sure to save the file. Please note that the program is case-sensitive.

45 In order for the program to be run, the user must supply the required data by setting certain variables in the file called `indat.m` equal to the appropriate values. All the user has to do is open the text file called indat.txt, fill in the required values and save the file as `indat.m` in a directory within MATAB's path. There are helpful hints within this file. It is especially important that the user keep track of units for all of the variables in the input data file. All of the units MUST be consistent. It is suggested that one always use the same units for all problems. For example, always use kips and inches, or kilo- newtons and millimeters.

5.8.1.1 Input Variable Descriptions

46 A brief description of each of the variables to be used in the input file is given below:

`npoin` This variable should be set equal to the number of nodes that comprise the structure. A node is defined as any point where two or more elements are joined.

`nelem` This variable should be set equal to the number of elements in the structure. Elements are the members which span between nodes.

`istrtp` This variable should be set equal to the type of structure. There are six types of structures which this program will analyze: beams, 2-D trusses, 2-D frames, grids, 3-D trusses, and 3-D frames. Set this to 1 for beams, 2 for 2D-trusses, 3 for 2D- frames, 4 for grids, 5 for 3D-trusses, and 6 for 3D-frames. An error will occur if it is not set to a number between 1 and 6. Note only `istrp=3` was kept.

`nload` This variable should be set equal to the number of different load cases to be analyzed. A load case is a specific manner in which the structure is loaded.

`ID` (matrix) The ID matrix contains information concerning the boundary conditions for each node. The number of rows in the matrix correspond with the number of nodes in the structure and the number of columns corresponds with the number of degrees of freedom for each node for that type of structure type. The matrix is composed of ones and zeros. A one indicates that the degree of freedom is restrained and a zero means it is unrestrained.

`nodecoor` (matrix) This matrix contains the coordinates (in the global coordinate system) of the nodes in the structure. The rows correspond with the node number and the columns correspond with the global coordinates x, y, and z, respectively. It is important to always include all three coordinates for each node even if the structure is only two- dimensional. In the case of a two-dimensional structure, the z-coordinate would be equal to zero.

`lnods` (matrix) This matrix contains the nodal connectivity information. The rows correspond with the element number and the columns correspond with the node numbers which the element is connected from and to, respectively.

`E,A,Iy` (arrays) These are the material and cross-sectional properties for the elements. They are arrays with the number of terms equal to the number of elements in the structure. The index number of each term corresponds with the element number. For example, the value of A(3) is the area of element 3, and so on. E is the modulus of elasticity, A is the cross-sectional area, Iy is the moment of inertia about the y axes

`Pnods` This is an array of nodal loads in global degrees of freedom. Only put in the loads in the global degrees of freedom and if there is no load in a particular degree of freedom, then

put a zero in its place. The index number corresponds with the global degree of freedom.

`Pelem` This an array of element loads, or loads which are applied between nodes. Only one load between elements can be analyzed. If there are more than one element loads on the structure, the equivalent nodal load can be added to the nodal loads. The index number corresponds with the element number. If there is not a load on a particular member, put a zero in its place. These should be in local coordinates.

`a` This is an array of distances from the left end of an element to the element load. The index number corresponds to the element number. If there is not a load on a particular member, put a zero in its place. This should be in local coordinates.

`w` This is an array of distributed loads on the structure. The index number corresponds with the element number. If there is not a load on a particular member, put a zero in its place. This should be in local coordinates

`dispflag` Set this variable to 1 if there are initial displacements and 0 if there are none.

`initial_displ` This is an array of initial displacements in all structural degrees of freedom. This means that you must enter in values for all structure degrees of freedom, not just those restrained. For example, if the structure is a 2D truss with 3 members and 3 node, there would be 6 structural degrees of freedom, etc. If there are no initial displacements, then set the values equal to zero.

`angle` This is an array of angles which the x-axis has possibly been rotated. This angle is taken as positive if the element has been rotated towards the z-axis. The index number corresponds to the element number.

`drawflag` Set this variable equal to 1 if you want the program to draw the structure and 0 if you do not.

5.8.1.2 Sample Input Data File

The contents of the input.m file which the user is to fill out is given below:

```
%*********************************************************************************************
% Scriptfile name: indat.m (EXAMPLE 2D-FRAME INPUT DATA)
%
% Main Program: casap.m
%
% This is the main data input file for the computer aided
% structural analysis program CASAP.  The user must supply
% the required numeric values for the variables found in
% this file (see user's manual for instructions).
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************
```

```
% HELPFUL INSTRUCTION COMMENTS IN ALL CAPITALS

% SET NPOIN EQUAL TO THE NUMBER OF NODES IN THE STRUCTURE

npoin=3;

% SET NELEM EQUAL TO THE NUMBER OF ELEMENTS IN THE STRUCTURE

nelem=2;

% SET NLOAD EQUAL TO THE NUMBER OF LOAD CASES

nload=1;

% INPUT THE ID MATRIX CONTAINING THE NODAL BOUNDARY CONDITIONS (ROW # = NODE #)

ID=[1 1 1;
   0 0 0;
   1 1 1];

% INPUT THE NODE COORDINATE (X,Y) MATRIX, NODECOOR (ROW # = NODE #)

nodecoor=[
   0 0;
   7416 3000;
   15416 3000
];

% INPUT THE ELEMENT CONNECTIVITY MATRIX, LNODS (ROW # = ELEMENT #)

lnods=[
   1 2;
   2 3
];

% INPUT THE MATERIAL PROPERTIES ASSOCIATED WITH THIS TYPE OF STRUCTURE
% PUT INTO ARRAYS WHERE THE INDEX NUMBER IS EQUAL TO THE CORRESPONDING ELEMENT NUMBER.
% COMMENT OUT VARIABLES THAT WILL NOT BE USED

E=[200 200];
A=[6000 6000];
Iz=[200000000 200000000];

% INPUT THE LOAD DATA.  NODAL LOADS, PNODS SHOULD BE IN MATRIX FORM.  THE COLUMNS CORRESPOND
% TO THE GLOBAL DEGREE OF FREEDOM IN WHICH THE LOAD IS ACTING AND THE THE ROW NUMBER CORRESPONDS
% WITH THE LOAD CASE NUMBER.  PELEM IS THE ELEMENT LOAD, GIVEN IN A MATRIX, WITH COLUMNS
% CORRESPONDING TO THE ELEMENT NUMBER AND ROW THE LOAD CASE.  ARRAY "A" IS THE DISTANCE FROM
% THE LEFT END OF THE ELEMENT TO THE LOAD, IN ARRAY FORM.  THE DISTRIBUTED LOAD, W SHOULD BE
% IN MATRIX FORM ALSO WITH COLUMNS = ELEMENT NUMBER UPON WHICH W IS ACTING AND ROWS = LOAD CASE.
% ZEROS SHOULD BE USED IN THE MATRICES WHEN THERE IS NO LOAD PRESENT.  NODAL LOADS SHOULD
% BE GIVEN IN GLOBAL COORDINATES, WHEREAS THE ELEMENT LOADS AND DISTRIBUTED LOADS SHOULD BE
% GIVEN IN LOCAL COORDINATES.
```

```
Pnods=[18.75 -46.35 0];
Pelem=[0 0];
a=[0 0];
w=[0 4/1000];

% IF YOU WANT THE PROGRAM TO DRAW THE STUCTURE SET DRAWFLAG=1, IF NOT SET IT EQUAL TO 0.
% THIS IS USEFUL FOR CHECKING THE INPUT DATA.

drawflag=1;

% END OF INPUT DATA FILE
```

5.8.1.3 Program Implementation

In order to "run" the program, open a new MATLAB Notebook. On the first line, type the name of the main program `CASAP` and evaluate that line by typing ctrl-enter. At this point, the main program reads the input file you have just created and calls the appropriate subroutines to analyze your structure. In doing so, your input data is echoed into your MATLAB notebook and the program results are also displayed. As a note, the program can also be executed directly from the MATAB workspace window, without Microsoft Word.

5.8.2 Program Listing

5.8.2.1 Main Program

```
%*******************************************************************************************
%Main Program: casap.m
%
% This is the main program, Computer Aided Structural Analysis Program
% CASAP.  This program primarily contains logic for calling scriptfiles and does not
% perform calculations.
%
% All variables are global, but are defined in the scriptfiles in which they are used.
%
% Associated scriptfiles:
%
% (for all stuctures)
% indat.m (input data file)
% idrasmbl.m
% elmcoord.m
% draw.m
%
% (3 - for 2D-frames)
% length3.m
% stiffl3.m
% trans3.m
% assembl3.m
% loads3.m
% disp3.m
```

```
% react3.m
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% COMMENT CARDS ARE IN ALL CAPITALS

% SET NUMERIC FORMAT

format short e

% CLEAR MEMORY OF ALL VARIABLES

clear

%  INITIALIZE OUTPUT FILE
fid = fopen('casap.out', 'wt');

% SET ISTRTP EQUAL TO THE NUMBER CORRESPONDING TO THE TYPE OF STRUCTURE:
% 3 = 2DFRAME

istrtp=3;

% READ INPUT DATA SUPPLIED BY THE USER

indat

% REASSAMBLE THE ID MATRIX AND CALCULATE THE LM VECTORS
% CALL SCRIPTFILE IDRASMBL

idrasmbl

% ASSEMBLE THE ELEMENT COORDINATE MATRIX

elmcoord

% 2DFRAME CALCULATIONS

% CALCULATE THE LENGTH AND ORIENTATION ANGLE, ALPHA FOR EACH ELEMENT
% CALL SCRIPTFILE LENGTH3.M

length3

% CALCULATE THE 2DFRAME ELEMENT STIFFNESS MATRIX IN LOCAL COORDINATES
% CALL SCRIPTFILE STIFFL3.M
```

```
stiffl3

% CALCULATE THE 2DFRAME ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
% CALL SCRIPTFILE TRANS3.M

trans3

% ASSEMBLE THE GLOBAL STRUCTURAL STIFFNESS MATRIX
% CALL SCRIPTFILE ASSEMBL3.M

assembl3

%  PRINT STRUCTURAL INFO

print_general_info

% LOOP TO PERFORM ANALYSIS FOR EACH LOAD CASE
for iload=1:nload

   print_loads

   % DETERMINE THE LOAD VECTOR IN GLOBAL COORDINATES
   % CALL SCRIPTFILE LOADS3.M

   loads3

   % CALCULATE THE DISPLACEMENTS
   % CALL SCRIPTFILE DISP3.M

   disp3

   % CALCULATE THE REACTIONS AT THE RESTRAINED DEGREES OF FREEDOM
   % CALL SCRIPTFILE REACT3.M

   react3

   % CALCULATE THE INTERNAL FORCES FOR EACH ELEMENT

   intern3

   % END LOOP FOR EACH LOAD CASE

end

% DRAW THE STRUCTURE, IF USER HAS REQUESTED (DRAWFLAG=1)
% CALL SCRIPTFILE DRAW.M

draw

st=fclose('all');
% END OF MAIN PROGRAM (CASAP.M)
```

```
disp('Program completed! - See "casap.out" for complete output');
```

5.8.2.2 Assembly of ID Matrix

```
%*********************************************************************************************
%SCRIPTFILE NAME: IDRASMBL.M
%
%MAIN FILE : CASAP
%
%Description : This file re-assambles the ID matrix such that the restrained
% degrees of freedom are given negative values and the unrestrained
% degrees of freedom are given incremental values begining with one
% and ending with the total number of unrestrained degrees of freedom.
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% TAKE CARE OF SOME INITIAL BUSINESS: TRANSPOSE THE PNODS ARRAY

Pnods=Pnods.';

% SET THE COUNTER TO ZERO

count=1;
negcount=-1;

% REASSEMBLE THE ID MATRIX

if istrtp==3
ndofpn=3;
nterm=6;
else
error('Incorrect structure type specified')
end

% SET THE ORIGINAL ID MATRIX TO TEMP MATRIX

orig_ID=ID;

% REASSEMBLE THE ID MATRIX, SUBSTITUTING RESTRAINED DEGREES OF FREEDOM WITH NEGATIVES,
%  AND NUMBERING GLOBAL DEGREES OF FREEDOM

for inode=1:npoin
for icoord=1:ndofpn
if ID(inode,icoord)==0
```

```
ID(inode,icoord)=count;
count=count+1;
elseif ID(inode,icoord)==1
         ID(inode,icoord)=negcount;
         negcount=negcount-1;
else
error('ID input matrix incorrect')
end
end
end

% CREATE THE LM VECTORS FOR EACH ELEMENT

for ielem=1:nelem
LM(ielem,1:ndofpn)=ID(lnods(ielem,1),1:ndofpn);
LM(ielem,(ndofpn+1):(2*ndofpn))=ID(lnods(ielem,2),1:ndofpn);
end

% END OF IDRASMBL.M SCRIPTFILE
```

5.8.2.3 Element Nodal Coordinates

```
%**********************************************************************************************
%SCRIPTFILE NAME: ELEMCOORD.M
%
%MAIN FILE : CASAP
%
%Description : This file assembles a matrix, elemcoor which contains the coordinates
% of the first and second nodes on each element, respectively.
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%**********************************************************************************************

% ASSEMBLE THE ELEMENT COORDINATE MATRIX, ELEMCOOR FROM NODECOOR AND LNODS

for ielem=1:nelem

elemcoor(ielem,1)=nodecoor(lnods(ielem,1),1);
elemcoor(ielem,2)=nodecoor(lnods(ielem,1),2);
%elemcoor(ielem,3)=nodecoor(lnods(ielem,1),3);
elemcoor(ielem,3)=nodecoor(lnods(ielem,2),1);
elemcoor(ielem,4)=nodecoor(lnods(ielem,2),2);
%elemcoor(ielem,6)=nodecoor(lnods(ielem,2),3);
end
% END OF ELMCOORD.M SCRIPTFILE
```

5.8.2.4 Element Lengths

```
%*********************************************************************************************
% Scriptfile name : length3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it computes the length of each element and the
% angle alpha between the local and global x-axes.  This file can be used
% for 2-dimensional elements such as 2-D truss, 2-D frame, and grid elements.
% This information will be useful for transformation between local and global
% variables.
%
% Variable descriptions: (in the order in which they appear)
%
% nelem =  number of elements in the structure
% ielem =  counter for loop
% L(ielem) =  length of element ielem
% elemcoor(ielem,4) =  xj-coordinate of element ielem
% elemcoor(ielem,1) =  xi-coordinate of element ielem
% elemcoor(ielem,5) =  yj-coordinate of element ielem
% elemcoor(ielem,2) =  yi-coordinate of element ielem
% alpha(ielem) =  angle between local and global x-axes
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% COMPUTE THE LENGTH AND ANGLE BETWEEN LOCAL AND GLOBAL X-AXES FOR EACH ELEMENT

for ielem=1:nelem
L(ielem)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
alpha(ielem)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

% END OF LENGTH3.M SCRIPTFILE
```

5.8.2.5 Element Stiffness Matrices

```
%*********************************************************************************************
% Scriptfile name: stiffl3.m (for 2d-frame structures)
%
% Main program: casap.m
%
% When this file is called, it computes the element stiffenss matrix
% of a 2-D frame element in local coordinates.  The element stiffness
```

```
% matrix is calculated for each element in the structure.
%
% The matrices are stored in a single matrix of dimensions 6x6*i and
% can be recalled individually later in the program.
%
% Variable descriptions: (in the order in which the appear)
%
% ielem = counter for loop
% nelem = number of element in the structure
% k(ielem,6,6)= element stiffness matrix in local coordinates
% E(ielem) = modulus of elasticity of element ielem
% A(ielem) = cross-sectional area of element ielem
% L(ielem) = lenght of element ielem
% Iz(ielem) = moment of inertia with respect to the local z-axis of element ielem
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%**********************************************************************************************

for ielem=1:nelem
  k(1:6,1:6,ielem)=...
  XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end

% END OF STIFFL3.M SCRIPTFILE
```

5.8.2.6 Transformation Matrices

```
%**********************************************************************************************
% Scriptfile name : trans3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% This file calculates the rotation matrix and the element stiffness
% matrices for each element in a 2D frame.
%
% Variable descriptions: (in the order in which they appear)
%
% ielem = counter for the loop
% nelem = number of elements in the structure
% rotation = rotation matrix containing all elements info
% Rot = rotational matrix for 2d-frame element
% alpha(ielem) = angle between local and global x-axes
% K = element stiffness matrix in global coordinates
% k = element stiffness matrix in local coordinates
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
```

```
%*********************************************************************************************

% CALCULATE THE ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
% FOR EACH ELEMENT IN THE STRUCTURE

for ielem=1:nelem

% SET UP THE ROTATION MATRIX, ROTATAION

rotation(1:6,1:6,ielem)=...
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
   ktemp=k(1:6,1:6,ielem);
% CALCULATE THE ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
Rot=rotation(1:6,1:6,ielem);
K(1:6,1:6,ielem)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end
% END OF TRANS3.M SCRIPTFILE
```

5.8.2.7 Assembly of the Augmented Stiffness Matrix

```
%*********************************************************************************************
% Scriptfile name : assembl3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% This file assembles the global structural stiffness matrix from the
% element stiffness matrices in global coordinates using the LM vectors.
% In addition, this file assembles the augmented stiffness matrix.
%
% Variable Descritpions (in order of appearance):
%
% ielem = Row counter for element number
% nelem = Number of elements in the structure
% iterm = Counter for term number in LM matrix
% LM(a,b) = LM matrix
% jterm = Column counter for element number
% temp1 = Temporary variable
% temp2 = Temporary variable
% temp3 = Temporary variable
% temp4 = Temporary variable
% number_gdofs = Number of global dofs
% new_LM = LM matrix used in assembling the augmented stiffness matrix
% aug_total_dofs = Total number of structure dofs
% K_aug        = Augmented structural stiffness matrix
% Ktt          = Structural Stiffness Matrix (Upper left part of Augmented structural stiffness matrix)
% Ktu = Upper right part of Augmented structural stiffness matrix
% Kut = Lower left part of Augmented structural stiffness matrix
% Kuu = Lower rigth part of Augmented structural stiffness matrix
%
%
% By Dean A. Frank
```

```
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%**********************************************************************************************


%  RENUMBER DOF INCLUDE ALL DOF, FREE DOF FIRST, RESTRAINED NEXT
new_LM=LM;
number_gdofs=max(LM(:));
new_LM(find(LM<0))=number_gdofs-LM(find(LM<0));
aug_total_dofs=max(new_LM(:));

% ASSEMBLE THE AUGMENTED STRUCTURAL STIFFNESS MATRIX
K_aug=zeros(aug_total_dofs);
for ielem=1:nelem
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Tough one!
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end

% SET UP SUBMATRICES FROM THE AUGMENTED STIFFNESS MATRIX

Ktt=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Ktu=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Kut=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Kuu=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
% END OF ASSEMBL3.M SCRIPTFILE
```

5.8.2.8 Print General Information

```
%**********************************************************************************************
% Scriptfile name : print_general_info.m
%
% Main program : casap.m
%
% Prints the general structure info to the output file
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%**********************************************************************************************

fprintf(fid,'\n\nNumber of Nodes: %d\n',npoin);
fprintf(fid,'Number of Elements: %d\n',nelem);
fprintf(fid,'Number of Load Cases: %d\n',nload);
```

```
fprintf(fid,'Number of Restrained dofs: %d\n',abs(min(LM(:))));
fprintf(fid,'Number of Free dofs: %d\n',max(LM(:)));

fprintf(fid,'\nNode Info:\n');
for inode=1:npoin
   fprintf(fid,'     Node %d (%d,%d)\n',inode,nodecoor(inode,1),nodecoor(inode,2));
   freedof=' ';
   if(ID(inode,1))>0
      freedof=strcat(freedof,' X ');
   end
   if(ID(inode,2))>0
      freedof=strcat(freedof,' Y ');
   end
   if(ID(inode,3))>0
      freedof=strcat(freedof,' Rot');
   end
   if freedof==' '
      freedof=' none; node is fixed';
   end
   fprintf(fid,'        Free dofs:%s\n',freedof);
end

fprintf(fid,'\nElement Info:\n');
for ielem=1:nelem
   fprintf(fid,'     Element %d (%d->%d)',ielem,lnods(ielem,1),lnods(ielem,2));
   fprintf(fid,'   E=%d A=%d Iz=%d \n',E(ielem),A(ielem),Iz(ielem));
end
```

5.8.2.9 Print Load

```
%*********************************************************************************************
% Scriptfile name : print_loads.m
%
% Main program : casap.m
%
% Prints the current load case data to the output file
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

Load_case=iload
if iload==1
   fprintf(fid,'\n________________________________________________________________________\n\n');
end

fprintf(fid,'Load Case: %d\n\n',iload);
fprintf(fid,'   Nodal Loads:\n');
for k=1:max(LM(:));
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==k);
```

```
   elem=fix(LM_spot(1)/(nterm+1))+1;
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
   switch(dof)
   case {1,4}, dof='Fx';
   case {2,5}, dof='Fy';
   otherwise, dof=' M';
   end
   %PRINT THE DISPLACEMENTS
   if Pnods(k)~=0
      fprintf(fid,'     Node: %2d %s = %14d\n',node, dof, Pnods(k));
   end
end

fprintf(fid,'\n   Elemental Loads:\n');
for k=1:nelem
   fprintf(fid,'     Element: %d   Point load = %d at %d from left\n',k,Pelem(k),a(k));
   fprintf(fid,'                   Distributed load = %d\n',w(k));
end
fprintf(fid,'\n');
```

5.8.2.10 Load Vector

```
%*********************************************************************************************
% Scriptfile name: loads3.m (for 2d-frame structures)
%
% Main program: casap.m
%
% When this file is called, it computes the fixed end actions for elements which
% carry distributed loads for a 2-D frame.
%
% Variable descriptions: (in the order in which they appear)
%
% ielem = counter for loop
% nelem = number of elements in the structure
% b(ielem) = distance from the right end of the element to the point load
% L(ielem) = length of the element
% a(ielem) = distance from the left end of the element to the point load
% Ffl = fixed end force (reaction) at the left end due to the point load
% w(ielem) = distributed load on element ielem
% L(ielem) = length of element ielem
% Pelem(ielem) = element point load on element ielem
% Mfl = fixed end moment (reaction) at the left end due to the point load
% Ffr = fixed end force (reaction) at the right end due to the point load
% Mfr = fixed end moment (reaction) at the right end due to the point load
% feamatrix_local = matrix containing resulting fixed end actions in local coordinates
% feamatrix_global = matrix containing resulting fixed end actions in global coordinates
% fea_vector = vector of fea's in global dofs, used to calc displacements
% fea_vector_abs = vector of fea's in every structure dof
% dispflag = flag indicating initial displacements
% Ffld = fea (vert force) on left end of element due to initial disp
% Mfld = fea (moment) on left end of element due to initial disp
```

```
% Ffrd = fea (vert force) on right end of element due to initial disp
% Mfrd = fea (moment) on right end of element due to initial disp
% fea_vector_disp = vector of fea's due to initial disp, used to calc displacements
% fea_vector_react = vector of fea's due to initial disp, used to calc reactions
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%*******************************************************************************************

% CALCULATE THE FIXED END ACTIONS AND INSERT INTO A MATRIX IN WHICH THE ROWS CORRESPOND
% WITH THE ELEMENT NUMBER AND THE COLUMNS CORRESPOND WITH THE ELEMENT LOCAL DEGREES
% OF FREEDOM

for ielem=1:nelem

b(ielem)=L(ielem)-a(ielem);

Ffl=((w(ielem)*L(ielem))/2)+((Pelem(ielem)*(b(ielem))^2)/(L(ielem))^3)*(3*a(ielem)+b(ielem));
Mfl=((w(ielem)*(L(ielem))^2))/12+(Pelem(ielem)*a(ielem)*(b(ielem))^2)/(L(ielem))^2;
Ffr=((w(ielem)*L(ielem))/2)+((Pelem(ielem)*(a(ielem))^2)/(L(ielem))^3)*(a(ielem)+3*b(ielem));
Mfr=-((w(ielem)*(L(ielem))^2))/12+(Pelem(ielem)*a(ielem)*(b(ielem))^2)/(L(ielem))^2;

feamatrix_local(ielem,1:6)=[0 Ffl Mfl 0 Ffr Mfr];

% ROTATE THE LOCAL FEA MATRIX TO GLOBAL

feamatrix_global=...
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

end

% CREATE A LOAD VECTOR USING THE LM MATRIX

% INITIALIZE FEA VECTOR TO ALL ZEROS

for idofpn=1:ndofpn
fea_vector(idofpn,1)=0;
end

for ielem=1:nelem
for idof=1:6
if ielem==1
if LM(ielem,idof)>0
fea_vector(LM(ielem,idof),1)=feamatrix_global(idof,ielem);
end

elseif ielem>1
if LM(ielem,idof)>0
fea_vector(LM(ielem,idof),1)=fea_vector(LM(ielem,1))+feamatrix_global(idof,ielem);
end
```

```
end
end
end

for ielem=1:nelem
for iterm=1:nterm
if feamatrix_global(iterm,ielem)==0
else
if new_LM(ielem,iterm)>number_gdofs
fea_vector_react(iterm,1)=feamatrix_global(iterm,ielem);
end
end
end
end

% END OF LOADS3.M SCRIPTFILE
```

5.8.2.11 Nodal Displacements

```
%*********************************************************************************************
% Scriptfile name : disp3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it computes the displacements in the global
% degrees of freedom.
%
% Variable descriptions: (in the order in which they appear)
%
% Ksinv = inverse of the structural stiffness matrix
% Ktt = structural stiffness matrix
% Delta = vector of displacements for the global degrees of freedom
% Pnods = vector of nodal loads in the global degrees of freedom
% fea_vector = vector of fixed end actions in the global degrees of freedom
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% CREATE A TEMPORARY VARIABLE EQUAL TO THE INVERSE OF THE STRUCTURAL STIFFNESS MATRIX

Ksinv=inv(Ktt);

% CALCULATE THE DISPLACEMENTS IN GLOBAL COORDINATES

Delta=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
```

```
%  PRINT DISPLACEMENTS WITH NODE INFO

fprintf(fid,'   Displacements:\n');
for k=1:size(Delta,1)
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==k);
   elem=fix(LM_spot(1)/(nterm+1))+1;
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
   switch(dof)
   case {1,4}, dof='delta X';
   case {2,5}, dof='delta Y';
   otherwise, dof='rotate ';
   end
   %PRINT THE DISPLACEMENTS
   fprintf(fid,'     (Node: %2d %s) %14d\n',node, dof, Delta(k));
end
fprintf(fid,'\n');

% END OF DISP3.M SCRIPTFILE
```

5.8.2.12 Reactions

```
%**********************************************************************************************
% Scriptfile name : react3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it calculates the reactions at the restrained degrees of
% freedom.
%
% Variable Descriptions:
%
% Reactions = Reactions at restrained degrees of freedom
% Kut = Upper left part of aug stiffness matrix, normal structure stiff matrix
% Delta = vector of displacements
% fea_vector_react = vector of fea's in restrained dofs
%
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%**********************************************************************************************

% CALCULATE THE REACTIONS FROM THE AUGMENTED STIFFNESS MATRIX
```

```
Reactions=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

fprintf(fid,'   Reactions:\n');
for k=1:size(Reactions,1)
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==-k);
   elem=fix(LM_spot(1)/(nterm+1))+1;
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
   switch(dof)
   case {1,4}, dof='Fx';
   case {2,5}, dof='Fy';
   otherwise, dof='M ';
   end
   %PRINT THE REACTIONS
   fprintf(fid,'     (Node: %2d %s) %14d\n',node, dof, Reactions(k));
end
fprintf(fid,'\n');

% END OF REACT3.M SCRIPTFILE
```

5.8.2.13 Internal Forces

```
%*********************************************************************************************
% Scriptfile name : intern3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it calculates the internal forces in all elements
% freedom.
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

Pglobe=zeros(6,nelem);
Plocal=Pglobe;

fprintf(fid,'   Internal Forces:');
%LOOP FOR EACH ELEMENT
for ielem=1:nelem
   %FIND ALL 6 LOCAL DISPLACEMENTS
   elem_delta=zeros(6,1);
   for idof=1:6
      gdof=LM(ielem,idof);
      if gdof<0
         elem_delta(idof)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      else
         elem_delta(idof)=
```

```
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      end
   end

   %SOLVE FOR ELEMENT FORCES (GLOBAL)
   Pglobe(:,ielem)=K(:,:,ielem)*elem_delta+feamatrix_global(:,ielem);
   %ROTATE FORCES FROM GLOBAL TO LOCAL COORDINATES

   %ROTATE FORCES TO LOCAL COORDINATES
   Plocal(:,ielem)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

   %PRINT RESULTS
   fprintf(fid,'\n      Element: %2d\n',ielem);
   for idof=1:6
      if idof==1
         fprintf(fid,'       At Node: %d\n',lnods(ielem,1));
      end
      if idof==4
         fprintf(fid,'       At Node: %d\n',lnods(ielem,2));
      end
      switch(idof)
      case {1,4}, dof='Fx';
      case {2,5}, dof='Fy';
      otherwise, dof='M ';
      end
      fprintf(fid,'          (Global : %s ) %14d',dof, Pglobe(idof,ielem));
      fprintf(fid,'    (Local : %s ) %14d\n',dof, Plocal(idof,ielem));
   end
end
fprintf(fid,'\n_________________________________________________________________________\n\n');
```

5.8.2.14 Sample Output File

CASAP will display figure 5.16.

```
Number of Nodes: 3
Number of Elements: 2
Number of Load Cases: 1
Number of Restrained dofs: 6
Number of Free dofs: 3

Node Info:
     Node 1 (0,0)
        Free dofs: none; node is fixed
     Node 2 (7416,3000)
        Free dofs: X Y Rot
     Node 3 (15416,3000)
        Free dofs: none; node is fixed

Element Info:
     Element 1 (1->2)   E=200 A=6000 Iz=200000000
```

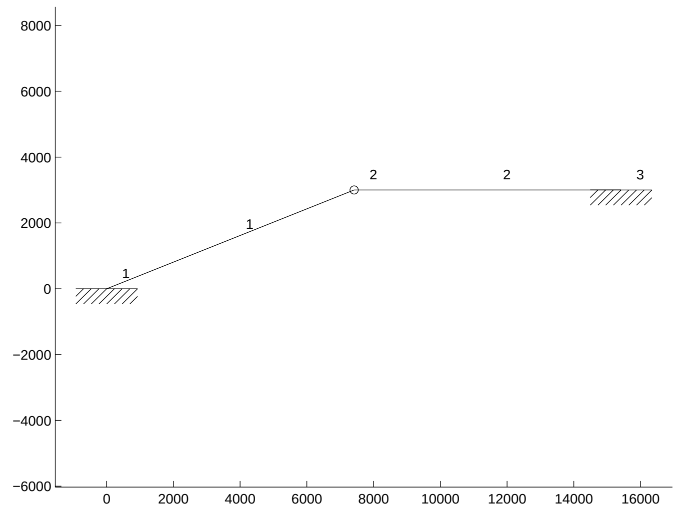


Figure 5.16: Structure Plotted with CASAP

```
    Element 2 (2->3)   E=200 A=6000 Iz=200000000

-------------------------------------------------------------------------

Load Case: 1

   Nodal Loads:
     Node:  2 Fx =  1.875000e+001
     Node:  2 Fy = -4.635000e+001

   Elemental Loads:
     Element: 1   Point load = 0 at 0 from left
                  Distributed load = 0
     Element: 2   Point load = 0 at 0 from left
                  Distributed load = 4.000000e-003

   Displacements:
     (Node:  2 delta X)  9.949820e-001
     (Node:  2 delta Y) -4.981310e+000
     (Node:  2 rotate ) -5.342485e-004

   Reactions:
     (Node:  1 Fx)  1.304973e+002
     (Node:  1 Fy)  5.567659e+001
     (Node:  1 M )  1.337416e+004
     (Node:  3 Fx) -1.492473e+002
     (Node:  3 Fy)  2.267341e+001
     (Node:  3 M ) -4.535573e+004
```

```
 Internal Forces:
    Element:  1
     At Node: 1
        (Global : Fx )  1.304973e+002    (Local : Fx )  1.418530e+002
        (Global : Fy )  5.567659e+001    (Local : Fy )  2.675775e+000
        (Global : M  )  1.337416e+004    (Local : M  )  1.337416e+004
     At Node: 2
        (Global : Fx ) -1.304973e+002    (Local : Fx ) -1.418530e+002
        (Global : Fy ) -5.567659e+001    (Local : Fy ) -2.675775e+000
        (Global : M  )  8.031549e+003    (Local : M  )  8.031549e+003

    Element:  2
     At Node: 2
        (Global : Fx )  1.492473e+002    (Local : Fx )  1.492473e+002
        (Global : Fy )  9.326590e+000    (Local : Fy )  9.326590e+000
        (Global : M  ) -8.031549e+003    (Local : M  ) -8.031549e+003
     At Node: 3
        (Global : Fx ) -1.492473e+002    (Local : Fx ) -1.492473e+002
        (Global : Fy )  2.267341e+001    (Local : Fy )  2.267341e+001
        (Global : M  ) -4.535573e+004    (Local : M  ) -4.535573e+004

------------------------------------------------------------------------
```

Chapter 6

EQUATIONS OF STATICS and KINEMATICS

Note: This section is largely based on chapter 6 of Mc-Guire and Gallagher, *Matrix Structural Analysis*, John Wiley

1 Having developed the stiffness method in great details, and prior to the introduction of energy based methods (which will culminate with the finite element formulation), we ought to revisit the flexibility method. This will be done by first introducing some basic statics and kinematics relationship.

2 Those relations will eventually enable us not only to formulate the flexibility/stiffness relation, but also other "by-products" such as algorithms for: 1) the extraction of a statically determinate structure from a statically indeterminate one; 2) checking prior to analysis whether a structure is kinematically unstable; 3) providing an alternative method of assembling the global stiffness matrix.

6.1 Statics Matrix $[\mathcal{B}]$

3 The statics matrix $[\mathcal{B}]$ relates the vector of all the structure's $\{\mathbf{P}\}$ nodal forces in global coordinates to all the unknown forces (element internal forces in their local coordinate system and structure's external reactions) $\{\mathbf{F}\}$, through equilibrium relationships and is defined as:

$$\boxed{\{\mathbf{P}\} \equiv [\mathcal{B}]\,\{\mathbf{F}\}} \tag{6.1}$$

4 $[\mathcal{B}]$ would have as many rows as the total number of independent equations of equilibrium; and as many columns as independent internal forces. This is reminiscent of the equilibrium matrix obtained in analyzing trusses by the "method of joints".

5 Depending on the type of structure, the internal element forces, and the equilibrium forces will vary according to Table 6.1. As with the flexibility method, there is more than one combination

Type	Internal Forces	Equations of Equilibrium
Truss	Axial force at one end	$\Sigma F_X = 0$, $\Sigma F_Y = 0$
Beam 1	Shear and moment at one end	$\Sigma F_y = 0$, $\Sigma M_z = 0$
Beam 2	Shear at each end	$\Sigma F_y = 0$, $\Sigma M_z = 0$
Beam 3	Moment at each end	$\Sigma F_y = 0$, $\Sigma M_z = 0$
2D Frame 1	Axial, Shear, Moment at each end	$\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$

Table 6.1: Internal Element Force Definition for the Statics Matrix

of independent element internal forces which can be selected.

6 Matrix $[\mathcal{B}]$ will be a square matrix for a statically determinate structure, and rectangular (more columns than rows) otherwise.

■ Example 6-1: Statically Determinate Truss Statics Matrix

Considering the truss shown in Fig. 6.1, it has 8 unknown forces (4 internal member forces and 4 external reactions), and 8 equations of equilibrium (2 at each of the 4 nodes). Assuming all the element forces to be tensile, and the reactions as shown in the figure, the equilibrium equations are:

Node	$\Sigma F_X = 0$	$\Sigma F_Y = 0$
Node 1	$\underbrace{P_{x1}}_{0} + F_3 C - R_{x1} = 0$	$\underbrace{P_{y1}}_{0} + F_1 + F_3 S - R_{y1} = 0$
Node 2	$P_{x2} + F_2 = 0$	$P_{y2} - F_1 = 0$
Node 3	$\underbrace{P_{x3}}_{0} - F_2 - F_3 C = 0$	$\underbrace{P_{y3}}_{0} - F_2 - F_3 S = 0$
Node 4	$\underbrace{P_{x4}}_{0} + R_{x4} = 0$	$\underbrace{P_{y4}}_{0} + F_4 - R_{y4} = 0$

where $\cos\alpha = \frac{L}{\sqrt{L^2+H^2}} = C$ and $\sin\alpha = \frac{H}{\sqrt{L^2+H^2}} = S$ Those equations of equilibrium can be cast in matrix form

$$
\begin{array}{r}
\Sigma F_x^1 : \\ \Sigma F_y^1 : \\ \Sigma F_x^2 : \\ \Sigma F_y^2 : \\ \Sigma F_x^3 : \\ \Sigma F_y^3 : \\ \Sigma F_x^4 : \\ \Sigma F_y^4 :
\end{array}
\underbrace{\left\{\begin{array}{c} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{array}\right\}}_{\{\mathbf{P}\}}
=
\underbrace{\left[\begin{array}{cccc|cccc}
0 & 0 & -C & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & -S & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}\right]}_{[\mathcal{B}]}
\underbrace{\left\{\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array}\right\}}_{\{\mathbf{F}\}}
\tag{6.2}
$$

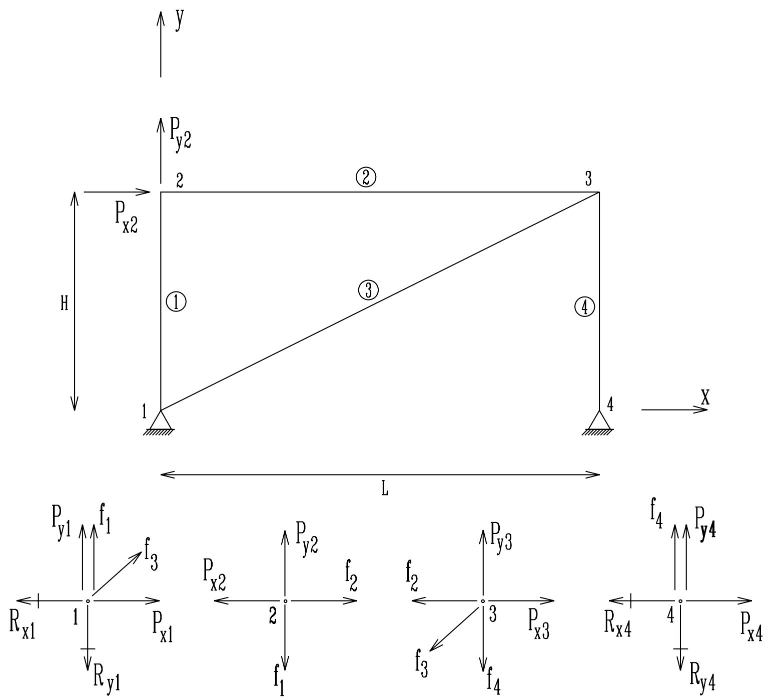


Figure 6.1: Example of $[\mathcal{B}]$ Matrix for a Statically Determinate Truss

the unknown forces and reactions can be determined through inversion of $[\mathcal{B}]$:

$$\underbrace{\left\{\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array}\right\}}_{\{\mathbf{F}\}} = \underbrace{\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{array}\right]}_{[\mathcal{B}]^{-1}} \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}}_{\{\mathbf{P}\}} = \left\{\begin{array}{c} P_{y2} \\ -P_{x2} \\ \frac{P_{x2}}{C} \\ -\frac{S}{C}P_{x2} \\ \hline P_{x2} \\ \frac{S}{C}P_{x2} + P_{y2} \\ 0 \\ -\frac{S}{C}P_{x2} \end{array}\right\} \tag{6.3}$$

We observe that the matrix $[\mathcal{B}]$ is totally independent of the external load, and once inverted can be used for multiple load cases with minimal computational efforts. ■

■ Example 6-2: Beam Statics Matrix

Considering the beam shown in Fig. 6.2, we have 3 elements, each with 2 unknowns (v and m) plus two unknown reactions, for a total of 8 unknowns. To solve for those unknowns we have 2 equations of equilibrium at each of the 4 nodes. Note that in this problem we have selected as primary unknowns the shear and moment at the right end of each element. The left components can be recovered from equilibrium. From equilibrium we thus have:

$$\underbrace{\left\{\begin{array}{c} P_1 \\ M_1 \\ P_2 \\ M_2 \\ P_3 \\ M_3 \\ P_4 \\ M_4 \end{array}\right\}}_{\{\mathbf{P}\}} = \underbrace{\left[\begin{array}{cccccc|cc} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]}_{[\mathcal{B}]} \underbrace{\left\{\begin{array}{c} v_1 \\ m_1 \\ v_2 \\ m_2 \\ v_3 \\ m_3 \\ \hline R_1 \\ R_2 \end{array}\right\}}_{\{\mathbf{F}\}} \tag{6.4}$$

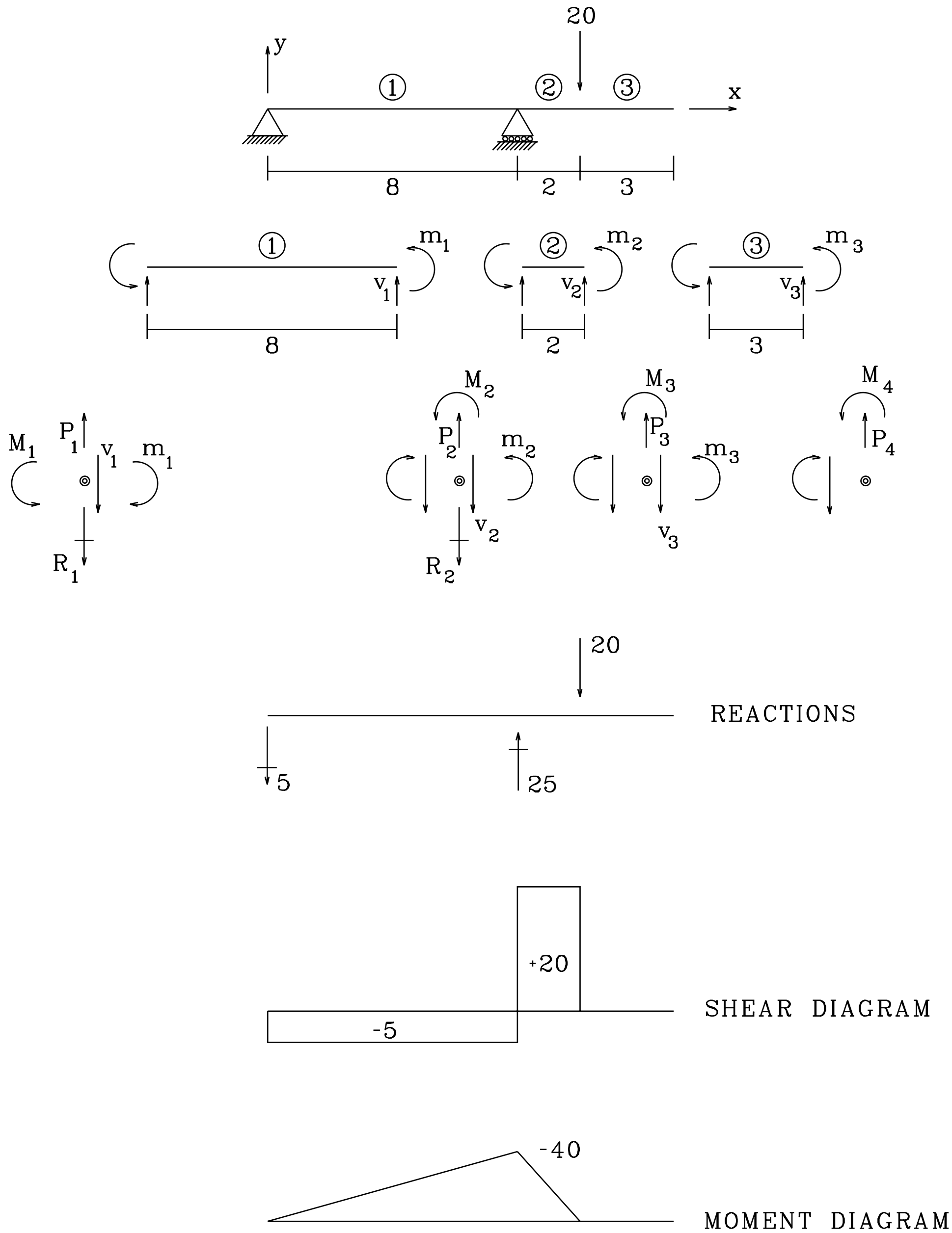


Figure 6.2: Example of [$\mathcal{B}$] Matrix for a Statically Determinate Beam

Inverting this 8 by 8 matrix would yield

$$
\underbrace{\left\{\begin{array}{c} v_1 \\ m_1 \\ v_2 \\ m_2 \\ v_3 \\ m_3 \\ \hline R_1 \\ R_2 \end{array}\right\}}_{\{\mathbf{F}\}} = \underbrace{\left[\begin{array}{cccccccc} 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} & -\frac{5}{8} & -\frac{1}{8} \\ 0 & 0 & 0 & 1 & 2 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & -\frac{1}{8} & 0 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} & -\frac{5}{8} & -\frac{1}{8} \\ 0 & \frac{1}{8} & 1 & \frac{1}{8} & \frac{5}{4} & \frac{1}{8} & \frac{13}{8} & \frac{1}{8} \end{array}\right]}_{[\mathcal{B}]^{-1}} \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \end{array}\right\}}_{\{\mathbf{P}\}} = \left\{\begin{array}{c} 5 \\ -40 \\ -20 \\ 0 \\ 0 \\ 0 \\ \hline 5 \\ -25 \end{array}\right\} \tag{6.5}
$$

■

7 For the case of a statically indeterminate structure, Eq. 6.1 can be generalized as:

$$
\boxed{\{\mathbf{P}\}_{2n\times 1} = [\ [\mathcal{B}_0]_{2n\times 2n} \mid [\mathcal{B}_x]_{2n\times r}\ ] \left\{\begin{array}{c} \mathbf{F}_0 \\ \hline \mathbf{F}_x \end{array}\right\}_{(2n+r)\times 1}} \tag{6.6}
$$

where $[\mathcal{B}_0]$ is a square matrix, $\{\mathbf{F}_0\}$ the vector of unknown internal element forces or external reactions, and $\{\mathbf{F}_x\}$ the vector of unknown *redundant* internal forces or reactions.

8 Hence, we can determine $\{\mathbf{F}_0\}$ from

$$
\begin{aligned}
\{\mathbf{F}_0\} &= [\mathcal{B}_0]^{-1}\{\mathbf{P}\} - [\mathcal{B}_0]^{-1}[\mathcal{B}_x]\{\mathbf{F}_x\} && (6.7) \\
&= [\mathcal{C}_1]_{2n\times 2n}\{\mathbf{P}\} - [\mathcal{C}_2]_{2n\times r}\{\mathbf{F}_x\} && (6.8)
\end{aligned}
$$

9 Note the following definitions which will be used later:

$$
\begin{aligned}
[\mathcal{B}_0]^{-1} &\equiv [\mathcal{C}_1] && (6.9) \\
-[\mathcal{B}_0]^{-1}[\mathcal{B}_x] &\equiv [\mathcal{C}_2] && (6.10)
\end{aligned}
$$

■ Example 6-3: Statically Indeterminate Truss Statics Matrix

Revisiting the first example problem, but with an additional member which makes it statically indeterminate, Fig. 6.3, it now has 9 unknown forces (5 internal member forces and 4 external reactions), and only 8 equations of equilibrium. Selecting the fifth element force as the redundant force, and with $r = 1$, we write Eq. 6.6

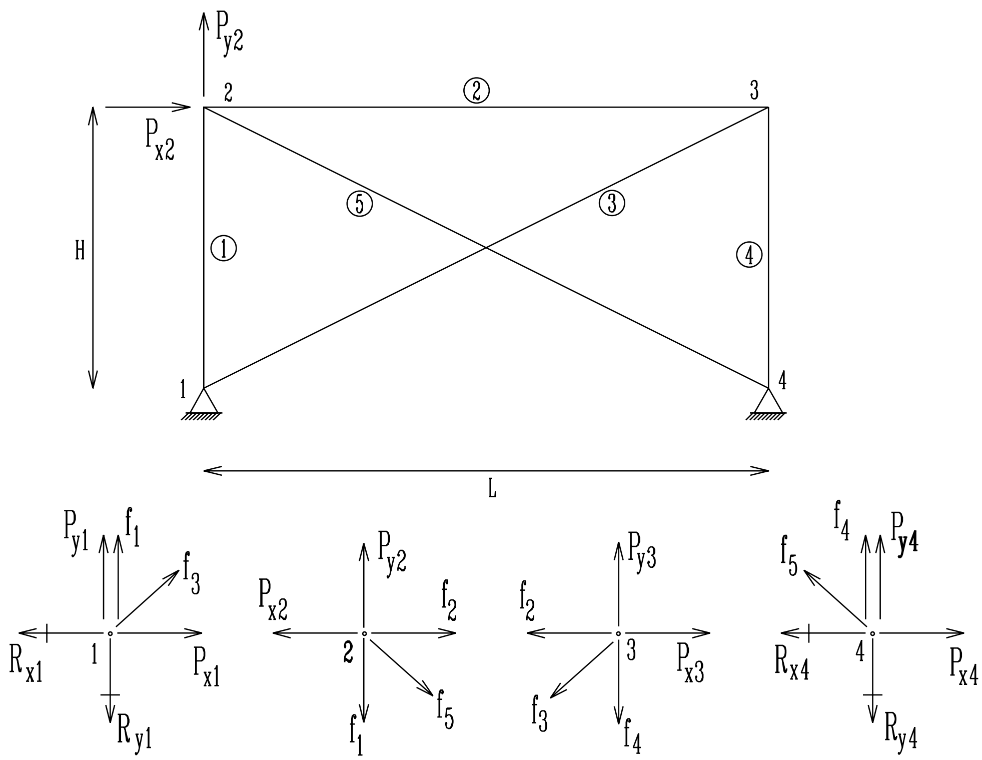


Figure 6.3: Example of $[\mathcal{B}]$ Matrix for a Statically Indeterminate Truss

$$
\begin{aligned}
\{\mathbf{P}\}_{2n\times 1} &= \left[\ [\mathcal{B}_0]_{2n\times 2n} \;\middle|\; [\mathcal{B}_x]_{2n\times r}\ \right] \left\{ \begin{array}{c} \mathbf{F}_0 \\ \hline \mathbf{F}_x \end{array} \right\}_{(2n+r)\times 1} && \text{(6.11-a)} \\
\{\mathbf{P}\} &= [\mathcal{B}_0]\{\mathbf{F}_0\} + [\mathcal{B}_x]\{\mathbf{F}_x\}
\end{aligned}
$$

$$
\underbrace{\left\{ \begin{array}{c} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{array} \right\}}_{\{\mathbf{P}\}} = \underbrace{\left[\begin{array}{cccc|cccc} 0 & 0 & -C & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -S & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & S & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array} \right]}_{[\mathcal{B}_0]} \underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array} \right\}}_{\{\mathbf{F}_0\}} + \underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ -C \\ S \\ 0 \\ 0 \\ C \\ -S \end{array} \right\}}_{[\mathcal{B}_x]} \underbrace{\{F_5\}}_{\{\mathbf{F}_x\}} \quad \text{(6.11-b)}
$$

We can solve for the internal forces in terms of the (still unknown) redundant force

$$
\{\mathbf{F}_0\} = [\mathcal{B}_0]^{-1}\{\mathbf{P}\} - [\mathcal{B}_0]^{-1}[\mathcal{B}_x]\{\mathbf{F}_x\} \quad \text{(6.12-a)}
$$

$$
\underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array} \right\}}_{\{\mathbf{F}_0\}} = \underbrace{\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{array} \right]}_{[\mathcal{B}_0]^{-1}\equiv[\mathcal{C}_1]} \left[\underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}}_{\{\mathbf{P}\}} - \underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ -C \\ S \\ 0 \\ 0 \\ C \\ -S \end{array} \right\}}_{[\mathcal{B}_x]} \underbrace{\{F_5\}}_{\{\mathbf{F}_x\}} \right] \quad \text{(6.12-b)}
$$

Or using the following relations $[\mathcal{B}_0]^{-1} \equiv [\mathcal{C}_1]$ and $-[\mathcal{B}_0]^{-1}[\mathcal{B}_x] \equiv [\mathcal{C}_2]$ we obtain

$$
\underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array} \right\}}_{\{\mathbf{F}_0\}} = \underbrace{\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1 \end{array} \right]}_{[\mathcal{C}_1]} \underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}}_{\{\mathbf{P}\}} + \underbrace{\left\{ \begin{array}{c} -S \\ -C \\ 1 \\ -S \\ C \\ 0 \\ -C \\ 0 \end{array} \right\}}_{[\mathcal{C}_2]} \underbrace{\{F_5\}}_{\{\mathbf{F}_x\}} \quad \text{(6.13)}
$$

Note, that this equation is not sufficient to solve for the unknown forces, as $\{\mathbf{F}_x\}$ must be obtained through force displacement relations ($[\mathbf{D}]$ or $[\mathbf{K}]$). ■

6.1.1 Identification of Redundant Forces

10 Whereas the identification of redundant forces was done by mere inspection of the structure in hand based analysis of structure, this identification process can be automated.

11 Starting with

$$\begin{aligned}\{\mathbf{P}\}_{2n\times 1} &= [\mathcal{B}]_{2n\times(2n+r)}\{\mathbf{F}\}_{(2n+r)\times 1}\\ [\mathcal{B}]_{2n\times(2n+r)}\{\mathbf{F}\}_{2n+r\times 1}-[\mathbf{I}]_{2n\times 2n}\{\mathbf{P}\}_{2n\times 1} &= \{\mathbf{0}\}\\ \left[\ \mathcal{B}\ \middle|\ -\mathbf{I}\ \right]_{2n\times(4n+r)}\left\{\begin{array}{c}\mathbf{F}\\ \hline \mathbf{P}\end{array}\right\} &= \{\mathbf{0}\}\end{aligned} \tag{6.14}$$

where $\left[\ \mathcal{B}\ \middle|\ -\mathbf{I}\ \right]$ corresponds to the *augmented matrix*.

12 If we apply a Gauss-Jordan elimination process to the augmented matrix, Eq. 6.14 is then transformed into:

$$\left[\ \mathbf{I}_{2n\times 2n}\ \middle|\ -\mathcal{C}_{2_{2n\times r}}\ \middle|\ -\mathcal{C}_{1_{2n\times 2n}}\ \right]\left\{\begin{array}{c}\mathbf{F}_0\\ \hline \mathbf{F}_x\\ \hline \mathbf{P}\end{array}\right\}_{4n+r\times 1} = \{\mathbf{0}\} \tag{6.15}$$

or:

$$\boxed{\{\mathbf{F}_0\}_{2n\times 1} = [\mathcal{C}_1]_{2n\times 2n}\{\mathbf{P}\}_{2n\times 1} + [\mathcal{C}_2]_{2n\times r}\{\mathbf{F}_x\}_{r\times 1}} \tag{6.16}$$

which is identical to Eq. 6.8; As before, $\mathbf{F}_x$ are the redundant forces and their solution obviously would depend on the elastic element properties.

■ **Example 6-4: Selection of Redundant Forces**

Revisiting the statically determined truss of Example 1, but with the addition of a fifth element, the truss would now be statically indeterminate to the first degree. The equation of

equilibrium 6.2 will then be written as:

$$\begin{array}{l} \Sigma F_x^1 : A \\ \Sigma F_y^1 : B \\ \Sigma F_x^2 : C \\ \Sigma F_y^2 : D \\ \Sigma F_x^3 : E \\ \Sigma F_y^3 : F \\ \Sigma F_x^4 : G \\ \Sigma F_y^4 : H \end{array} \underbrace{\left[\begin{array}{ccccc|cccc||cc} 0 & 0 & -C & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -S & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -C & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & S & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -S & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]}_{[\ \mathcal{B}\ |\ -\mathbf{I}\ ]} \underbrace{\left\{\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \\ \hline\hline P_{x2} \\ P_{y2} \end{array}\right\}}_{\{\mathbf{F}\}} \tag{6.17}$$

Note that since load is applied only on node 2, we have considered a subset of the identity matrix $[\mathbf{I}]$.

1. We start with the following matrix

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{ccccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & R_{x1} & R_{y1} & R_{x4} & R_{y4} & P_{x2} & P_{y2} \end{array} \\ \left[\begin{array}{ccccccccccc} 0 & 0 & -C & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -S & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -C & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & S & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -S & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \tag{6.18}$$

2. Interchange columns

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{ccccccccccc} R_{x1} & R_{y1} & F_2 & F_1 & F_3 & F_4 & F_5 & R_{x4} & R_{y4} & P_{x2} & P_{y2} \end{array} \\ \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -S & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -S & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \tag{6.19}$$

3. Operate as indicates

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc} R_{x1} & R_{y1} & F_2 & F_1 & F_3 & F_4 & F_5 & R_{x4} & R_{y4} & P_{x2} & P_{y2} \end{array} \\
\begin{array}{r} A \\ B' = B + DB \\ C \\ D \\ E' = \frac{E+C}{C} \\ F \\ G \\ H \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -S & 0 & S & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1/C & 0 \\
0 & 0 & 0 & 0 & S & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -S & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
\tag{6.20}
$$

4. Operate as indicated

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc} R_{x1} & R_{y1} & F_2 & F_1 & F_3 & F_4 & F_5 & R_{x4} & R_{y4} & P_{x2} & P_{y2} \end{array} \\
\begin{array}{r} A' = A + CE' \\ B'' = B' + SE' \\ C' = -C \\ D \\ E' \\ F' = F - SE' \\ G \\ H' = H + F' \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -C & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S/C & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & C & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & S & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1/C & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & S & 0 & 0 & S/C & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & S/C & 0
\end{bmatrix}
\end{array}
\tag{6.21}
$$

5. Interchange columns and observe that F_5 is the selected redundant.

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc} R_{x1} & R_{y1} & F_2 & F_1 & F_3 & F_4 & R_{x4} & R_{y4} & F_5 & P_{x2} & P_{y2} \end{array} \\
\begin{array}{r} A' \\ B'' \\ C' \\ D \\ E' \\ F' \\ G \\ H' \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S/C & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & C & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & S & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1/C & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & S & S/C & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & S/C & 0
\end{bmatrix}
\end{array}
\tag{6.22}
$$

From Eq. 6.8 we have

$$\underbrace{\left\{\begin{array}{c} R_{x1} \\ R_{y1} \\ F_2 \\ F_1 \\ F_3 \\ F_4 \\ R_{x4} \\ R_{y4} \end{array}\right\}}_{\{\mathbf{F}_0\}} = \underbrace{\left[\begin{array}{cc} 1 & 0 \\ S/C & 1 \\ -1 & 0 \\ 0 & 1 \\ 1/C & 0 \\ -S/C & 0 \\ 0 & 0 \\ -S/C & 0 \end{array}\right]}_{[\mathcal{C}_1]} \underbrace{\left\{\begin{array}{c} P_{x2} \\ P_{y2} \end{array}\right\}}_{\{\mathbf{P}\}} + \underbrace{\left\{\begin{array}{c} C \\ 0 \\ -C \\ -S \\ 1 \\ -S \\ -C \\ 0 \end{array}\right\}}_{[\mathcal{C}_2]} \underbrace{\{F_5\}}_{\{\mathbf{F}_x\}} \tag{6.23}$$

which is identical to the results in Eq. 6.13 except for the order of the terms.

■

6.1.2 Kinematic Instability

13 Kinematic instability results from a structure with inadequate restraint in which rigid body motion can occur.

14 For example in Fig. 6.4, there is no adequate restraint for the frame against displacement in the horizontal direction, and the truss may rotate with respect to point O. Kinematic instability will result in a matrix which is singular, and decomposition of this matrix will result in a division by zero causing a computer program to "crash". Hence, it is often desirable for "large" structures to determine *a priori* whether a structure is kinematically instable before the analysis is performed.

15 Conditions for static determinacy and instability can be stated as a function of the rank of $[\mathcal{B}]$. If $[\mathcal{B}]$ has n rows (corresponding to the number of equilibrium equations), u columns (corresponding to the number of internal forces and reactions), and is of rank r, then conditions of kinematic instability are summarized in Table 6.2

16 Note that kinematic instability is not always synonymous with structure collapse. In some cases equilibrium will be recovered only after geometry would have been completely altered (such as with a flexible cable structures) and equations of equilibrium would have to be completely rewritten with the new geometry.

6.2 Kinematics Matrix $[\mathcal{A}]$

17 The kinematics matrix $[\mathcal{A}]$ relates all the structure's $\{\boldsymbol{\Delta}\}$ nodal displacements in global coordinates to the element relative displacements in their local coordinate system and the

Figure 6.4: *Examples of Kinematic Instability

$n > u$ Kinematically Instable	
$n = u$ Statically Determinate	
$n = u = r$	Stable
$n = u > r$	Instable with $n - r$ modes of kinematic instability
$n < r$ Statically Indeterminate (degree $u - n$)	
$n = r$	Stable
$n > r$	Instable with $n - r$ modes of kinematic instability

Table 6.2: Conditions for Static Determinacy, and Kinematic Instability

support displacement (which may not be zero if settlement occurs) $\{\boldsymbol{\Upsilon}\}$, through kinematic relationships and is defined as:

$$\boxed{\{\boldsymbol{\Upsilon}\} \equiv [\mathcal{A}]\{\boldsymbol{\Delta}\}} \tag{6.24}$$

18 $[\mathcal{A}]$ is a rectangular matrix which number of rows is equal to the number of the element internal displacements, and the number of columns is equal to the number of nodal displacements. Contrarily to the rotation matrix introduced earlier and which transforms the displacements from global to local coordinate for one *single* element, the kinematics matrix applies to the entire structure.

19 It can be easily shown that for trusses:

$$\Upsilon^e = (u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha \tag{6.25}$$

where α is the angle between the element and the X axis. whereas for flexural members:

$$v_{21} = v_2 - v_1 - \theta_{z1} L \tag{6.26}$$

$$\theta_{z21} = \theta_{z2} - \theta_{z1} \tag{6.27}$$

■ Example 6-5: Kinematics Matrix of a Truss

Considering again the statically indeterminate truss of the previous example, the kinematic matrix will be given by:

$$\left\{\begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \hline u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}\right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -C & S & 0 & 0 & C & -S \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathcal{A}]} \left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}\right\} \tag{6.28}$$

Applying the constraints: $u_1 = 0$; $v_1 = 0$; $u_4 = 0$; and $v_4 = 0$ we obtain:

$$\left\{ \begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \Delta_5^e \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -C & S & 0 & 0 & C & -S \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]}_{[\mathcal{A}]} \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \right\} \tag{6.29}$$

We should observe that $[\mathcal{A}]$ is indeed the transpose of the $[\mathcal{B}]$ matrix in Eq. 6.17 ■

6.3 Statics-Kinematics Matrix Relationship

20 Having defined both the statics $[\mathcal{B}]$ and kinematics $[\mathcal{A}]$ matrices, it is intuitive that those two matrices must be related. In this section we seek to determine this relationship for both the statically determinate and statically indeterminate cases.

6.3.1 Statically Determinate

21 The external work being defined as

$$\left. \begin{array}{l} W_{ext} = \frac{1}{2}\lfloor \mathbf{P} \rfloor \{\mathbf{\Delta}\} \\ \{\mathbf{P}\} = [\mathcal{B}]\{\mathbf{F}\} \end{array} \right\} W_{ext} = \frac{1}{2}\lfloor \mathbf{F} \rfloor [\mathcal{B}]^T \{\mathbf{\Delta}\} \tag{6.30}$$

22 Alternatively, the internal work is given by:

$$\left. \begin{array}{l} W_{int} = \frac{1}{2}\lfloor \mathbf{F} \rfloor \{\mathbf{\Upsilon}\} \\ \{\mathbf{\Upsilon}\} = [\mathcal{A}]\{\mathbf{\Delta}\} \end{array} \right\} W_{int} = \frac{1}{2}\lfloor \mathbf{F} \rfloor [\mathcal{A}] \{\mathbf{\Delta}\} \tag{6.31}$$

23 Equating the external to the internal work $W_{ext} = W_{int}$ we obtain:

$$\frac{1}{2}\lfloor \mathbf{F} \rfloor [\mathcal{B}]^T \{\mathbf{\Delta}\} = \frac{1}{2}\lfloor \mathbf{F} \rfloor [\mathcal{A}] \{\mathbf{\Delta}\} \tag{6.32}$$

$$\boxed{[\mathcal{B}]^T = [\mathcal{A}]} \tag{6.33}$$

6.3.2 Statically Indeterminate

24 Whereas in the preceding case we used Eq. 6.1 for $[\mathcal{B}]$, for the most general case of statically indeterminate structures we can start from Eq. 6.6 and write:

$$\{\mathbf{P}\} = [\ \mathcal{B}_0 \mid \mathcal{B}_x\ ] \left\{ \frac{\mathbf{F}_0}{\mathbf{F}_x} \right\} \tag{6.34}$$

where $\mathbf{F}_x$ correspond to the redundant forces. The external work will then be

$$W_{ext} = \frac{1}{2} \lfloor\ \mathbf{F}_0 \mid \mathbf{F}_x\ \rfloor \left[\frac{[\mathcal{B}_0]^t}{[\mathcal{B}_x]^t} \right] \{\boldsymbol{\Delta}\} \tag{6.35}$$

25 Again, we can generalize Eq. 6.24 and write

$$\left\{ \frac{\boldsymbol{\Upsilon}_0}{\boldsymbol{\Upsilon}_x} \right\} = \left[\frac{\mathcal{A}_0}{\mathcal{A}_x} \right] \{\boldsymbol{\Delta}\} \tag{6.36}$$

where $\{\boldsymbol{\Upsilon}_0\}$ and $\{\boldsymbol{\Upsilon}_x\}$ are relative displacements corresponding to $\{\mathbf{F}_0\}$ and $\{\mathbf{F}_x\}$ respectively.

26 Consequently the internal work would be given by:

$$W_{int} = \frac{1}{2} \lfloor\ \mathbf{F}_0 \mid \mathbf{F}_x\ \rfloor \left[\frac{[\mathcal{A}_0]}{[\mathcal{A}_x]} \right] \{\boldsymbol{\Delta}\} \tag{6.37}$$

27 As before, equating the external to the internal work $W_{ext} = W_{int}$ and simplifying, we obtain:

$$\boxed{\begin{array}{lcll} [\mathcal{B}_0]^T & = & [\mathcal{A}_0] & (6.38) \\ [\mathcal{B}_x]^T & = & [\mathcal{A}_x] & (6.39) \end{array}}$$

6.4 Kinematic Relations through Inverse of Statics Matrix

28 We now seek to derive some additional relations between the displacements through the inverse of the statics matrix. Those relations will be used later in the flexibility methods, and have no immediate applications.

29 Rewriting Eq. 6.36 as

$$\{\boldsymbol{\Delta}\} = [\mathcal{A}_0]^{-1} \{\boldsymbol{\Upsilon}_0\} = \left[[\mathcal{B}_0]^t \right]^{-1} \{\boldsymbol{\Upsilon}_0\} = \left[[\mathcal{B}_0]^{-1} \right]^t \{\boldsymbol{\Upsilon}_0\} \tag{6.40}$$

we can solve for $\{\mathbf{F}_0\}$ from Eq. 6.8

$$\{\mathbf{F}_0\} = \underbrace{[\mathcal{B}_0]^{-1}}_{[\mathcal{C}_1]} \{\mathbf{P}\} - \underbrace{[\mathcal{B}_0]^{-1} [\mathcal{B}_x]}_{[\mathcal{C}_2]} \{\mathbf{F}_x\} \tag{6.41}$$

30 Combining this equation with $[\mathcal{B}_0]^{-1} = [\mathcal{C}_1]$ from Eq. 6.41, and with Eq. 6.40 we obtain

$$\boxed{\{\boldsymbol{\Delta}\} = [\mathcal{C}_1]^t \{\boldsymbol{\Upsilon}_0\}} \tag{6.42}$$

31 Similarly, we can revisit Eq. 6.36 and write

$$\{\boldsymbol{\Upsilon}_x\} = [\mathcal{A}_x] \{\boldsymbol{\Delta}\} \tag{6.43}$$

When the previous equation is combined with the rightmost side of Eq. 6.40 and 6.39 we obtain

$$\{\boldsymbol{\Upsilon}_x\} = [\mathcal{B}_x]^t \left[[\mathcal{B}_0]^{-1}\right]^t \{\boldsymbol{\Upsilon}_0\} \tag{6.44}$$

32 Thus, with $[\mathcal{B}_0]^{-1}[\mathcal{B}_x] = -[\mathcal{C}_2]$ from Eq. 6.41

$$\boxed{\{\boldsymbol{\Upsilon}_x\} = -[\mathcal{C}_2]^t \{\boldsymbol{\Upsilon}_0\}} \tag{6.45}$$

This equation relates the unknown relative displacements to the relative known ones.

6.5 Congruent Transformation Approach to [K]

Note: This section is largely based on section 3.3 of Gallagher, *Finite Element Analysys*, Prentice Hall.

33 For an arbitrary structure composed of n elements, we can define the *unconnected* nodal load and displacement vectors in global coordinate as

$$\{\mathbf{P}^e\} = \lfloor \; \lfloor \mathbf{P}^1 \rfloor \;\; \lfloor \mathbf{P}^2 \rfloor \;\; ... \;\; \lfloor \mathbf{P}^n \rfloor \; \rfloor^T \tag{6.46}$$

$$\{\boldsymbol{\Upsilon}^e\} = \lfloor \; \lfloor \boldsymbol{\Upsilon}^1 \rfloor \;\; \lfloor \boldsymbol{\Upsilon}^2 \rfloor \;\; ... \;\; \lfloor \boldsymbol{\Upsilon}^n \rfloor \; \rfloor^T \tag{6.47}$$

where $\{\mathbf{P}^i\}$ and $\{\boldsymbol{\Delta}^i\}$ are the nodal load and displacements arrays of element i. The size of each submatrix (or more precisely of each subarray) is equal to the total number of d.o.f. in global coordinate for element i.

34 Similarly, we can define the unconnected (or unassembled) global stiffness matrix of the structure as $[\mathbf{K}^e]$:

$$\{\mathbf{F}\} = [\mathbf{K}^e] \{\boldsymbol{\Upsilon}\} \tag{6.48}$$

$$[\mathbf{K}^e] = \begin{bmatrix} [\mathbf{K}^1] & & & & \\ & [\mathbf{K}^2] & & & \\ & & [\mathbf{K}^3] & & \\ & & & \ddots & \\ & & & & [\mathbf{K}^n] \end{bmatrix} \tag{6.49}$$

35 Note that all other terms of this matrix are equal to zero, and that there is no intersection between the various submatrices. Hence, this matrix does not reflect the connectivity among all the elements.

36 We recall the following relations (Eq. 6.1, 6.24, and 6.33 respectively)

$$\{\boldsymbol{\Upsilon}\} = [\mathcal{A}]\{\boldsymbol{\Delta}\} \tag{6.50}$$
$$\{\mathbf{P}\} = [\mathcal{B}]\{\mathbf{F}\} \tag{6.51}$$
$$[\mathcal{B}]^T = [\mathcal{A}] \tag{6.52}$$

We now combine those matrices with the definition of the stiffness matrix:

$$\left.\begin{array}{rcl}\{\mathbf{P}\} & = & [\mathcal{B}]\{\mathbf{F}\} \\ \{\mathbf{P}\} & = & [\mathbf{K}]\{\boldsymbol{\Delta}\} \\ [\mathcal{B}] & = & [\mathcal{A}]^T\end{array}\right\} \left.\begin{array}{rcl}[\mathcal{A}]^T\{\mathbf{F}\} & = & [\mathbf{K}]\{\boldsymbol{\Delta}\} \\ \{\mathbf{F}\} & = & [\mathbf{K}^e]\{\boldsymbol{\Upsilon}\} \\ \{\boldsymbol{\Upsilon}\} & = & [\mathcal{A}]\{\boldsymbol{\Delta}\}\end{array}\right\} \boxed{[\mathbf{K}] = [\mathcal{A}]^T[\mathbf{K}^e][\mathcal{A}]} \tag{6.53}$$

37 Thus, we have just defined a congruent transformation on the unconnected global stiffness matrix written in terms of $[\mathbf{K}^e]$ to obtain the structure stiffness matrix. We shall note that:

1. If $[\mathbf{K}^e]$ is expressed in global coordinates, then $[\mathcal{A}]$ is a boolean matrix.
2. If $[\mathbf{K}^e]$ is in local coordinates, then $[\mathcal{A}]$ must include transformation from element to global coordinate systems, and is no longer boolean.
3. $[\mathbf{K}]$ accounts for the B.C. as those terms associated with the restrained d.o.f. are not included.
4. Note the similarity between the direct stiffness method: $[\mathbf{K}] = \sum[\boldsymbol{\Gamma}]^T[\mathbf{k}][\boldsymbol{\Gamma}]$ and the congruent transformation approach: $[\mathbf{K}] = [\mathcal{A}]^T[\mathbf{K}^e][\mathcal{A}]$.
5. If the structure is a frame with n elements, then we would have $[\mathbf{K}]_{neq\times neq} = \sum_1^n[\boldsymbol{\Gamma}]^T_{6\times6}[\mathbf{k}]_{6\times6}[\boldsymbol{\Gamma}]_{6\times6}$ and the congruent transformation approach: $[\mathbf{K}]_{neq\times neq} = [\mathcal{A}]^T_{neq\times 6n}[\mathbf{K}^e]_{6n\times 6n}[\mathcal{A}]_{6n\times neq}$.
6. Congruent approach appears to be less efficient than the direct stiffness method as both $[\mathbf{K}^e]$ and $[\mathcal{A}]$ are larger than $[\mathbf{K}]$.

■ Example 6-6: Congruent Transformation

Assemble the global stiffness matrix of the grid shown in Fig. 6.5 using the direct stiffness method and the congruent transformation method.

Solution:

The 2 element stiffness matrices in global coordinate system are given by:

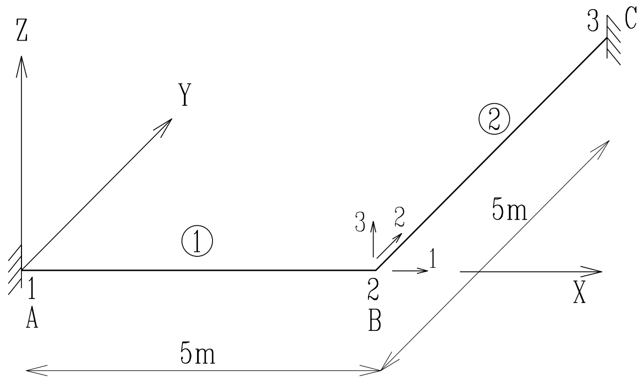


Figure 6.5: Example 1, Congruent Transfer

$$[\mathbf{K}_{AB}] = \left[\begin{array}{ccc|ccc} 7.692 & 0. & 0. & -7.692 & 0. & 0. \\ & .4\times10^5 & -12. & 0. & .2\times10^5 & 12. \\ & & .0048 & 0. & -12. & -.0048 \\ \hline & & & 7.692 & 0. & 0. \\ & & & & .4\times10^5 & 12. \\ & & & & & .0048 \end{array}\right] \quad (6.54)$$

$$[\mathbf{K}_{BC}] = \left[\begin{array}{ccc|ccc} 1\times10^5 & 0. & 18.75 & .5\times10^5 & 0. & -18.75 \\ & 14.423 & 0. & 0. & -14.423 & 0. \\ & & .00469 & 18.75 & 0. & -.00469 \\ \hline & & & 1\times10^5 & 0. & -18.75 \\ & & & & 14.423 & 0. \\ & & & & & .00469 \end{array}\right] \quad (6.55)$$

We shall determine the global stiffness matrix using the two approaches:

Direct Stiffness

$$[\mathbf{ID}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad (6.56)$$

$$\{\mathbf{LM}^1\} = \lfloor\ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \rfloor^T \quad (6.57)$$

$$\{\mathbf{LM}^2\} = \lfloor\ 1\ \ 2\ \ 3\ \ 0\ \ 0\ \ 0\ \rfloor^T \quad (6.58)$$

$$[\mathbf{K}] = \begin{bmatrix} (7.692+1\times10^5) & (0.+0.) & (18.75+0.) \\ (0.+0.) & (.4\times10^5+14.423) & (12.+0.) \\ (0.+18.75) & (12.+0.) & (.0048+.00469) \end{bmatrix} \quad (6.59)$$

$$= \begin{bmatrix} 1\times 10^5 & 0. & 18.75 \\ 0. & .4\times 10^5 & 12. \\ 18.75 & 12. & .00949 \end{bmatrix}$$

Congruent Transformation

1. The unassembled stiffness matrix $[\mathbf{K}^e]$, for node 2, is given by:

$$\{\mathbf{F}\} = [\mathbf{K}^e]\{\boldsymbol{\Upsilon}\} \tag{6.60}$$

$$\left\{\begin{array}{c} M_x^1 \\ M_y^1 \\ F_z^1 \\ \hline M_x^2 \\ M_y^2 \\ F_z^2 \end{array}\right\} = \left[\begin{array}{ccc|ccc} 7.692 & 0 & 0 & & & \\ & .4\times 10^5 & 12. & & 0. & \\ \text{sym} & & .0048 & & & \\ \hline & & & 1\times 10^5 & 0. & -18.75 \\ & 0. & & & 14.423 & 0. \\ & & & \text{sym} & & .00469 \end{array}\right] \left\{\begin{array}{c} \Theta_x^1 \\ \Theta_y^1 \\ W_z^1 \\ \hline \Theta_x^2 \\ \Theta_y^2 \\ W_z^2 \end{array}\right\} \begin{array}{l} \text{element 1} \\ \\ \text{element 2} \end{array} \tag{6.61}$$

   Note that the B.C. are implicitely accounted for by ignoring the restrained d.o.f. however the connectivity of the elements is not reflected by this matrix.

2. The kinematics matrix is given by:

$$\{\boldsymbol{\Upsilon}\} = [\mathcal{A}]\{\boldsymbol{\Delta}\} \tag{6.62}$$

$$\left\{\begin{array}{c} \theta_x^1 \\ \theta_y^1 \\ w_z^1 \\ \hline \theta_x^2 \\ \theta_y^2 \\ w_z^2 \end{array}\right\} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} \theta_x \\ \theta_y \\ w_z \end{array}\right\} \tag{6.63}$$

   As for the kinematics matrix, we are relating the local displacements of each element to the global ones. Hence this matrix is analogous to the connectivity matrix. Whereas the connectivity matrix defined earlier reflected the element connection, this one reflects the connectivity among all the unrestrained degrees of freedom.

3. If we take the product: $[\mathcal{A}]^T[\mathbf{K}^e][\mathcal{A}]$ then we will recover $[\mathbf{K}]$ as shown above.

■

■ **Example 6-7: Congruent Transformation of a Frame**

Assemble the stiffness matrix of the frame shown in Fig. 6.6 using the direct stiffness method, and the two congruent approaches.

Solution:

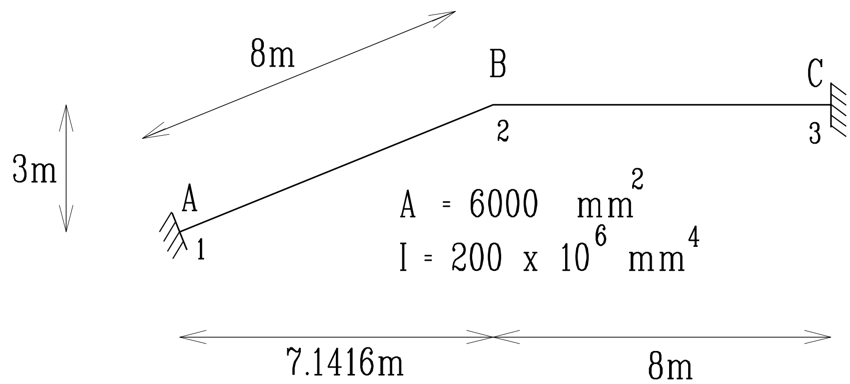


Figure 6.6: Example 2

The stiffness matrices of elements AB and BC in local coordinate system are given by:

$$[\mathbf{k}]_{AB} = [\mathbf{k}]_{BC} = 200 \begin{bmatrix} .75 & 0. & 0. & -.75 & 0. & 0. \\ & .00469 & 18.75 & 0 & -.0048 & 18.75 \\ & & 1\times 10^5 & 0 & -18.75 & .5\times 10^5 \\ & & & .75 & 0. & 0. \\ \text{sym} & & & & .00469 & -18.75 \\ & & & & & 1\times 10^5 \end{bmatrix} \tag{6.64}$$

while the rotation matrix is given by:

$$[\boldsymbol{\Gamma}]_{AB} = \left[\begin{array}{ccc|ccc} .9272 & .375 & 0. & 0. & 0. & 0. \\ -.375 & .927 & 0. & 0. & 0. & 0. \\ 0. & 0. & 1 & 0. & 0. & 0. \\ \hline 0. & 0. & 0. & .9272 & .375 & 0. \\ 0. & 0. & 0. & -.375 & .9272 & 0. \\ 0. & 0. & 0. & 0. & 0. & 1 \end{array}\right] \tag{6.65}$$

$$[\boldsymbol{\Gamma}]_{BC} = [\mathbf{I}] \tag{6.66}$$

The element stiffness matrices in global coordinates will then be given by:

$$[\mathbf{K}]_{AB} = [\boldsymbol{\Gamma}]^T_{AB}[\mathbf{k}]_{AB}[\boldsymbol{\Gamma}]_{AB} \tag{6.67-a}$$

$$= 200 \begin{bmatrix} .645 & .259 & -7.031 & -.645 & -.259 & -7.031 \\ & .109 & 17.381 & -.259 & -.109 & 17.381 \\ & & 1\times 10^5 & 7.031 & -17.381 & .5\times 10^5 \\ & & & .645 & .259 & 7.031 \\ & \text{sym} & & & .109 & -17.381 \\ & & & & & 1\times 10^5 \end{bmatrix} \tag{6.67-b}$$

and $[\mathbf{K}]_{BC} = [\mathbf{k}]_{BC}$

Direct Stiffness: We can readily assemble the global stiffness matrix:

$$
[\mathbf{K}] = 200 \begin{bmatrix} (.645+.75) & (.259+0.) & (7.031+0.) \\ & (.109+.00469) & (-17.38+18.75) \\ \text{sym} & & (1+1)\times 10^5 \end{bmatrix} \tag{6.68}
$$

$$
= 200 \begin{bmatrix} 1.395 & .259 & 7.031 \\ & .1137 & 1.37 \\ \text{sym} & & 2\times 10^5 \end{bmatrix} \tag{6.69}
$$

Congruent Transformation, global axis, Boolean $[\mathcal{A}]$

1. We start with the unconnected global stiffness matrix in global coordinate system:

$$
\{\mathbf{F}\} = [\mathbf{K}^e]\{\boldsymbol{\Upsilon}\} \tag{6.70}
$$

$$
\left\{ \begin{array}{c} P_X^1 \\ P_Y^1 \\ M_Z^1 \\ \hline P_X^2 \\ P_Y^2 \\ M_Z^2 \end{array} \right\} = 200 \left[\begin{array}{ccc|ccc} .645 & .259 & 7.031 & & & \\ & .109 & -17.381 & & 0 & \\ \text{sym} & & 1\times 10^5 & & & \\ \hline & & & .75 & 0. & 0. \\ & 0 & & & .00469 & 18.75 \\ & & & \text{sym} & & 1\times 10^5 \end{array} \right] \left\{ \begin{array}{c} U_X^1 \\ V_Y^1 \\ \Theta_Z^1 \\ \hline U_X^2 \\ V_Y^2 \\ \Theta_Z^2 \end{array} \right\} \tag{6.71}
$$

2. Next we determine the kinematics matrix $\mathcal{A}$:

$$
\{\boldsymbol{\Upsilon}\} = [\mathcal{A}]\{\boldsymbol{\Delta}\} \tag{6.72}
$$

$$
\left\{ \begin{array}{c} u^1 \\ v^1 \\ \theta^1 \\ u^2 \\ v^2 \\ \theta^2 \end{array} \right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} u \\ v \\ \theta \end{array} \right\} \tag{6.73}
$$

3. Finally, if we take the product $\underbrace{[\mathcal{A}]^T}_{3\times 6}\underbrace{[\mathbf{K}^e]}_{6\times 6}\underbrace{[\mathcal{A}]}_{6\times 3}$ we obtain the structure global stiffness matrix $[\mathbf{K}]$ in Eq. 6.69

Congruent Transformation (local axis):

1. Unconnected stiffness matrix in local coordinates:

$$
\{\mathbf{p}^e\} = [\mathbf{k}^e]\{\boldsymbol{\delta}^e\} \tag{6.74}
$$

$$
\left\{ \begin{array}{c} P_X^1 \\ P_Y^1 \\ M_Z^1 \\ \hline P_X^2 \\ P_Y^2 \\ M_Z^2 \end{array} \right\} = 200 \left[\begin{array}{ccc|ccc} .75 & 0. & 0. & & & \\ & .00469 & -18.75 & & & \\ \text{sym} & & 1\times 10^5 & & & \\ \hline & & & .75 & 0. & 0. \\ & & & & .00469 & 18.75 \\ & & & \text{sym} & & 1\times 10^5 \end{array} \right] \left\{ \begin{array}{c} u_x^1 \\ v_y^1 \\ \theta_z^1 \\ \hline u_x^2 \\ v_y^2 \\ \theta_z^2 \end{array} \right\} \tag{6.75}
$$

2. The kinematics matrix $[\mathcal{A}]$ is now given by:

$$\{\boldsymbol{\delta}^e\} = [\mathcal{A}]\{\boldsymbol{\Delta}\} \tag{6.76}$$

$$\left\{\begin{array}{c} u_x^1 \\ v_y^1 \\ \theta_z^1 \\ \hline u_x^2 \\ v_y^2 \\ \theta_z^2 \end{array}\right\} = \left[\begin{array}{ccc} .9272 & .375 & 0. \\ -.375 & .9272 & 0. \\ 0. & 0. & 1. \\ \hline 1. & 0. & 0. \\ 0. & 1. & 0. \\ 0. & 0. & 1. \end{array}\right] \left\{\begin{array}{c} U_X \\ V_Y \\ \Theta_Z \end{array}\right\} \tag{6.77}$$

3. When the product: $[\mathcal{A}]^T[\mathbf{k}^e][\mathcal{A}]$ we recover the structure global stiffness matrix

■

Chapter 7

FLEXIBILITY METHOD

7.1 Introduction

1 Recall the definition of the flexibility matrix

$$\boxed{\{\mathbf{\Upsilon}\} \equiv [\mathbf{d}]\{\mathbf{p}\}} \tag{7.1}$$

where $\{\mathbf{\Upsilon}\}$, $[\mathbf{d}]$, and $\{\mathbf{p}\}$ are the element relative displacements, element flexibility matrix, and forces at the element degrees of freedom free to displace.

2 As with the congruent approach for the stiffness matrix, we define:

$$\{\mathbf{F}^e\} = \lfloor\ \lfloor \mathbf{F}^1 \rfloor\ \ \lfloor \mathbf{F}^2 \rfloor\ \ \ldots\ \ \lfloor \mathbf{F}^n \rfloor\ \rfloor^T \tag{7.2}$$

$$\{\mathbf{\Upsilon}^e\} = \lfloor\ \lfloor \mathbf{\Upsilon}^1 \rfloor\ \ \lfloor \mathbf{\Upsilon}^2 \rfloor\ \ \ldots\ \ \lfloor \mathbf{\Upsilon}^n \rfloor\ \rfloor^T \tag{7.3}$$

for n elements, and where $\{\mathbf{F}^i\}$ and $\{\mathbf{\Upsilon}^i\}$ are the nodal load and displacements vectors for element i. The size of these vectors is equal to the total number of global dof for element i.

3 Denoting by $\{\mathbf{R}\}$ the reaction vector, and by $\{\mathbf{\Upsilon}^R\}$ the corresponding displacements, we define the unassembled structure flexibility matrix as:

$$\left\{\begin{array}{c} \mathbf{\Upsilon}^e \\ \hline \mathbf{\Upsilon}^R \end{array}\right\} = \left[\begin{array}{c|c} [\mathbf{d}^e] & \\ \hline & [\mathbf{0}] \end{array}\right] \left\{\begin{array}{c} \mathbf{F}^e \\ \hline \mathbf{R} \end{array}\right\} \tag{7.4}$$

where $[\mathbf{d}^e]$ is the *unassembled global flexibility matrix*.

4 In its present form, Eq. 7.4 is of no help as the element forces $\{\mathbf{F}^e\}$ and reactions $\{\mathbf{R}\}$ are not yet known.

7.2 Flexibility Matrix

5 We recall from Sect. 6.1.1 that we can automatically identify the redundant forces $[\mathbf{F}_x]$ and rewrite Eq. 7.4 as:

$$\left\{\begin{array}{c}\boldsymbol{\Upsilon}_0\\ \hline \boldsymbol{\Upsilon}_x\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{d}^e_{00} & [\mathbf{0}]\\ \hline [\mathbf{0}] & \mathbf{d}^e_{xx}\end{array}\right]\left\{\begin{array}{c}\mathbf{F}_0\\ \mathbf{F}_x\end{array}\right\} \tag{7.5}$$

where $[\mathbf{d}^e_{00}]$ and $[\mathbf{d}^e_{xx}]$ correspond to the unassembled global flexibility matrix, and $\{\mathbf{F}_0\}$ and $\{\mathbf{F}_x\}$ are the corresponding forces.

6 Next we must relate the redundant and nonredundant forces (which together constitute the unknown element forces and reactions) to the externally applied load $\{\mathbf{P}\}$. Hence we recall from Eq. 6.16:

$$\{\mathbf{F}_0\} = [\mathcal{C}_1]\{\mathbf{P}\} + [\mathcal{C}_2]\{\mathbf{F}_x\} \tag{7.6}$$

which can be rewritten (for convenience:) as:

$$\left\{\begin{array}{c}\mathbf{F}_0\\ \hline \mathbf{F}_x\end{array}\right\} = \left[\begin{array}{c|c}\mathcal{C}_1 & \mathcal{C}_2\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right]\left\{\begin{array}{c}\mathbf{P}\\ \hline \mathbf{F}_x\end{array}\right\} \tag{7.7}$$

7 From Eq. 6.42 we had: $\{\boldsymbol{\Delta}\} = [\mathcal{C}_1]^t\{\boldsymbol{\Upsilon}_0\}$ and from Eq. 6.45: $\{\boldsymbol{\Upsilon}_x\} = -[\mathcal{C}_2]^t\{\boldsymbol{\Upsilon}_0\}$ which lead to

$$\left\{\begin{array}{c}\boldsymbol{\Delta}_p\\ \hline \mathbf{0}\end{array}\right\} = \left[\begin{array}{c|c}\mathcal{C}_1^t & \mathbf{0}\\ \hline \mathcal{C}_2^t & \mathbf{I}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{\Upsilon}_0\\ \hline \boldsymbol{\Upsilon}_x\end{array}\right\} \tag{7.8}$$

where the subscript p in $\{\boldsymbol{\Delta}_p\}$ has been added to emphasize that we are referring only to the global displacements corresponding to $\{\mathbf{P}\}$.

8 Finally we substitute Eq. 7.7 into Eq. 7.5 and the results into Eq. 7.8 to obtain:

$$\left\{\begin{array}{c}\boldsymbol{\Delta}_p\\ \hline \mathbf{0}\end{array}\right\} = \left[\begin{array}{c|c}\mathcal{C}_1^t & \mathbf{0}\\ \hline \mathcal{C}_2^t & \mathbf{I}\end{array}\right]\left[\begin{array}{c|c}\mathbf{d}^e_{00} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{d}^e_{xx}\end{array}\right]\left[\begin{array}{c|c}\mathcal{C}_1 & \mathcal{C}_2\\ \hline \mathbf{0} & \mathbf{I}\end{array}\right]\left\{\begin{array}{c}\mathbf{P}\\ \hline \mathbf{F}_x\end{array}\right\} \tag{7.9}$$

or:

$$\boxed{\left\{\begin{array}{c}\boldsymbol{\Delta}_p\\ \hline \mathbf{0}\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{D}_{pp} & \mathbf{D}_{px}\\ \hline \mathbf{D}_{xp} & \mathbf{D}_{xx}\end{array}\right]\left\{\begin{array}{c}\mathbf{P}\\ \hline \mathbf{F}_x\end{array}\right\}} \tag{7.10}$$

where:

$$[\mathbf{D}_{pp}] = [\mathcal{C}_1]^t[\mathbf{d}^e_{00}][\mathcal{C}_1] \tag{7.11}$$
$$[\mathbf{D}_{px}] = [\mathbf{D}_{xp}]^t = [\mathcal{C}_1]^t[\mathbf{d}^e_{00}][\mathcal{C}_2] \tag{7.12}$$
$$[\mathbf{D}_{xx}] = [\mathcal{C}_2]^t[\mathbf{d}^e_{00}][\mathcal{C}_2] + [\mathbf{d}^e_{xx}] \tag{7.13}$$

9 This equation should be compared with Eq. 7.1 and will be referred to as the *unsolved global assembled flexibility equation.*

7.2.1 Solution of Redundant Forces

10 We can solve for the redundant forces (recall that in the flexibility method, redundant forces are the primary unknowns as opposed to displacements in the stiffness method) by solving the lower partition of Eq. 7.10:

$$\boxed{\{\mathbf{F}_x\} = -\left[\mathbf{D}_{xx}\right]^{-1}\left[\mathbf{D}_{xp}\right]\{\mathbf{P}\}} \tag{7.14}$$

7.2.2 Solution of Internal Forces and Reactions

11 The internal forces and reactions can in turn be obtained through Eq. 6.16:

$$\{\mathbf{F}_0\} = [\mathcal{C}_1]\{\mathbf{P}\} + [\mathcal{C}_2]\{\mathbf{F}_x\} \tag{7.15}$$

which is combined with Eq. 7.14 to yield:

$$\boxed{\{\mathbf{F}_0\} = \left[[\mathcal{C}_1] - [\mathcal{C}_2]\left[\mathbf{D}_{xx}\right]^{-1}\left[\mathbf{D}_{xp}\right]\right]\{\mathbf{P}\}} \tag{7.16}$$

7.2.3 Solution of Joint Displacements

12 Joint displacements are in turn obtained by considering the top partition of Eq. 7.10:

$$\boxed{\{\mathbf{\Delta}_p\} = \underbrace{\left[[\mathbf{D}_{pp}] - [\mathbf{D}_{px}]\left[\mathbf{D}_{xx}\right]^{-1}\left[\mathbf{D}_{xp}\right]\right]}_{[\mathbf{D}]}\{\mathbf{P}\}} \tag{7.17}$$

■ Example 7-1: Flexibility Method

Solve for the internal forces and displacements of joint 2 of the truss in example 6.1.1. Let $H = 0.75L$ and assign area A to members 3 and 5, and $0.5A$ to members 1, 2, and 4. Let f_5 be the redundant force, and use the $[\mathcal{C}_1]$ and $[\mathcal{C}_2]$ matrices previously derived.

Solution:

$C = \frac{L}{\sqrt{L^2+H^2}} = 0.8$ and $S = \frac{H}{\sqrt{L^2+H^2}} = 0.6$

From Eq. 7.5 we obtain

$$\left\{\begin{array}{c}\mathbf{\Upsilon}_0\\ \hline \mathbf{\Upsilon}_x\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{d}^e_{00} & 0\\ \hline 0 & \mathbf{d}^e_{xx}\end{array}\right]\left\{\begin{array}{c}\mathbf{F}_0\\ \mathbf{F}_x\end{array}\right\} \tag{7.18-a}$$

$$\left\{\begin{array}{c} u_1 \\ v_1 \\ \Delta_2^e \\ \Delta_1^e \\ \Delta_3^e \\ \Delta_4^e \\ u_4 \\ v_4 \\ \hline \Delta_5^e \end{array}\right\} = \frac{L}{AE}\left[\begin{array}{cccccccc|c} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 2 &&&&&& \\ &&& 1.5 &&&&& \\ &&&& 1.25 &&&& \\ &&&&& 1.5 &&& \\ &&&&&& 0 && \\ &&&&&&& 0 & \\ \hline &&&&&&&& 1.25 \end{array}\right]\left\{\begin{array}{c} R_{x1} \\ R_{y1} \\ f_2 \\ f_1 \\ f_3 \\ f_4 \\ R_{x4} \\ R_{y4} \\ \hline f_5 \end{array}\right\} \tag{7.18-b}$$

From Example 6.1.1 we have

$$[\mathcal{C}_1] = \begin{bmatrix} 1 & 0 \\ S/C & 1 \\ -1 & 0 \\ 0 & 1 \\ 1/C & 0 \\ -S/C & 0 \\ 0 & 0 \\ -S/C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.75 & 1 \\ -1 & 0 \\ 0 & 1 \\ 1.25 & 0 \\ -0.75 & 0 \\ 0 & 0 \\ -0.75 & 0 \end{bmatrix} \qquad [\mathcal{C}_2] = \left\{\begin{array}{c} C \\ 0 \\ -C \\ -S \\ 1 \\ -S \\ -C \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 0.8 \\ 0 \\ -0.8 \\ -0.6 \\ 1 \\ -0.6 \\ -0.8 \\ 0 \end{array}\right\} \tag{7.19}$$

From Eq. 7.11

$$[\mathbf{D}_{pp}] = [\mathcal{C}_1]^t \, [\mathbf{d}_{00}^e] \, [\mathcal{C}_1] = \frac{L}{AE}\begin{bmatrix} 4.797 & 0 \\ 0 & 1.50 \end{bmatrix} \tag{7.20}$$

From Eq. 7.12

$$[\mathbf{D}_{px}] = [\mathcal{C}_1]^t \, [\mathbf{d}_{00}^e] \, [\mathcal{C}_2] = \frac{L}{AE}\left\{\begin{array}{c} 3.838 \\ -0.900 \end{array}\right\} \tag{7.21}$$

From Eq. 7.13

$$[\mathbf{D}_{xx}] = [\mathcal{C}_2]^t \, [\mathbf{d}_{00}^e] \, [\mathcal{C}_2] + [\mathbf{d}_{xx}^e] = \frac{L}{AE}(4.860) \tag{7.22}$$

We can now solve for the redundant force f_5 from Eq. 7.14

$$\{\mathbf{F}_x\} = -[\mathbf{D}_{xx}]^{-1} [\mathbf{D}_{xp}] \{\mathbf{P}\} = -\frac{1}{4.860}\lfloor\, 3.8387 \; -0.90 \,\rfloor \left\{\begin{array}{c} P_{x2} \\ P_{y2} \end{array}\right\} \tag{7.23-a}$$

$$f_5 = -0.790 P_{x2} + 0.185 P_{y2} \tag{7.23-b}$$

The nonredundant forces are now obtained from Eq. 7.16

$$\{\mathbf{F}_0\} = \left[[\mathcal{C}_1] - [\mathcal{C}_2] \, [\mathbf{D}_{xx}]^{-1} [\mathbf{D}_{xp}]\right] \{\mathbf{P}\} \tag{7.24-a}$$

$$
\left\{ \begin{array}{c} R_{x1} \\ R_{y1} \\ f_2 \\ f_1 \\ f_3 \\ f_4 \\ R_{x4} \\ R_{y4} \\ \hline f_5 \end{array} \right\} = \left[\left[\begin{array}{cc} 1 & 0 \\ 0.75 & 1 \\ -1 & 0 \\ 0 & 1 \\ 1.25 & 0 \\ -0.75 & 0 \\ 0 & 0 \\ -0.75 & 0 \end{array} \right] - \left[\begin{array}{cc} 0.632 & -0.148 \\ 0 & 0 \\ -0.632 & 0.148 \\ -0.474 & 0.111 \\ 0.790 & -0.185 \\ -0.474 & 0.111 \\ -0.632 & 0.148 \\ 0 & 0 \end{array} \right] \right] \left\{ \begin{array}{c} P_{x2} \\ P_{y2} \end{array} \right\} \tag{7.24-b}
$$

$$
= \left[\begin{array}{cc} 0.368 & 0.148 \\ 0.75 & 1.000 \\ -0.368 & -0.148 \\ 0.474 & 0.889 \\ 0.460 & 0.185 \\ -0.276 & -0.111 \\ 0.632 & -0.148 \\ -0.750 & 0 \end{array} \right] \left\{ \begin{array}{c} P_{x2} \\ P_{y2} \end{array} \right\} \tag{7.24-c}
$$

Finally, the displacements are obtained from Eq. 7.17

$$
\{\boldsymbol{\Delta}_p\} = \underbrace{\left[[\mathbf{D}_{pp}] - [\mathbf{D}_{px}] [\mathbf{D}_{xx}]^{-1} [\mathbf{D}_{xp}] \right]}_{[\mathbf{D}]} \{\mathbf{P}\} \tag{7.25-a}
$$

$$
\left\{ \begin{array}{c} u_2 \\ v_2 \end{array} \right\} = \frac{L}{AE} \left[\left[\begin{array}{cc} 4.797 & 0 \\ 0 & 1.500 \end{array} \right] - \frac{1}{4.860} \left[\begin{array}{cc} 14.730 & -3.454 \\ -3.454 & 0.810 \end{array} \right] \right] \left\{ \begin{array}{c} P_{x2} \\ P_{y2} \end{array} \right\} \tag{7.25-b}
$$

$$
= \frac{L}{AE} \left[\begin{array}{cc} 1.766 & 0.711 \\ 0.711 & 1.333 \end{array} \right] \left\{ \begin{array}{c} P_{x2} \\ P_{y2} \end{array} \right\} \tag{7.25-c}
$$

It should be noted that whereas we have used the flexibility method in its algorithmic implementation (as it would lead itself to computer implementation) to analyse this simple problem, the solution requires a formidable amount of matrix operations in comparaison with the "classical" (hand based) flexibility method.

■

7.3 Stiffness Flexibility Relations

13 Having introduced both the stiffness and flexibility methods, we shall rigorously consider the relationship among the two matrices [**k**] and [**d**] at the structure level.

14 Let us generalize the stiffness relation by partitioning it into two groups: 1) subscript 's' for

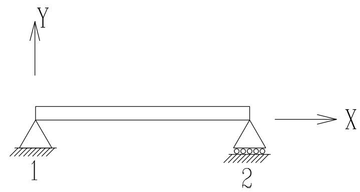


Figure 7.1: Stable and Statically Determinate Element

those d.o.f.'s which are supported, and 2) subscript 'f' for those dof which are free.

$$\boxed{\left\{\begin{array}{c}\mathbf{P}_f\\ \hline \mathbf{P}_s\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{k}_{ff} & \mathbf{k}_{fs}\\ \hline \mathbf{k}_{sf} & \mathbf{k}_{ss}\end{array}\right]\left\{\begin{array}{c}\mathbf{\Delta}_f\\ \hline \mathbf{\Delta}_s\end{array}\right\}} \tag{7.26}$$

7.3.1 From Stiffness to Flexibility

15 To obtain $[\mathbf{d}]$ the structure must be supported in a stable and statically determinate way, as for the beam in Fig. 7.1. for which we would have:

$$\{\mathbf{\Delta}_f\} = \left\{\begin{array}{c}\theta_1\\ \theta_2\end{array}\right\} \tag{7.27}$$

$$\{\mathbf{\Delta}_s\} = \left\{\begin{array}{c}v_1\\ v_2\end{array}\right\} \tag{7.28}$$

$$\{\mathbf{P}_f\} = \left\{\begin{array}{c}M_1\\ M_2\end{array}\right\} \tag{7.29}$$

$$\{\mathbf{P}_s\} = \left\{\begin{array}{c}V_1\\ V_2\end{array}\right\} \tag{7.30}$$

Since $\{\mathbf{\Delta}_s\} = \{\mathbf{0}\}$ the above equation reduces to:

$$\left\{\begin{array}{c}\mathbf{P}_f\\ \mathbf{P}_s\end{array}\right\} = \left[\begin{array}{c}\mathbf{k}_{ff}\\ \mathbf{k}_{sf}\end{array}\right]\{\mathbf{\Delta}_f\} \tag{7.31}$$

and we would have:

$$\{\mathbf{P}_f\} = [\mathbf{k}_{ff}]\{\mathbf{\Delta}_f\} \tag{7.32}$$

$$\boxed{[\mathbf{d}] = [\mathbf{k}_{ff}]^{-1}} \tag{7.33}$$

■ **Example 7-2: Flexibility Matrix**

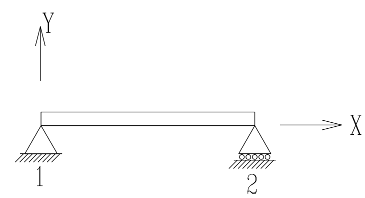


Figure 7.2: Example 1, $[\mathbf{k}] \rightarrow [\mathbf{d}]$

From Fig. 7.2

$$\left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} = \underbrace{\frac{EI}{L} \left[\begin{array}{cc} 4 & 2 \\ 2 & 4 \end{array} \right]}_{[\mathbf{k}_{ff}]} \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} \tag{7.34}$$

$$[\mathbf{k}_{ff}]^{-1} = [\mathbf{d}] = \frac{L}{EI}\frac{1}{12} \left[\begin{array}{rr} 4 & -2 \\ -2 & 4 \end{array} \right] = \frac{L}{6EI} \left[\begin{array}{rr} 2 & -1 \\ -1 & 2 \end{array} \right] \tag{7.35}$$

■

7.3.2 From Flexibility to Stiffness

16 $[\mathbf{k}_{ff}]$: From Eq. 7.26, $[\mathbf{k}]$ was subdivided into free and supported d.o.f.'s, and we have shown that $[\mathbf{k}_{ff}] = [\mathbf{d}]^{-1}$, or $\{\mathbf{P}_f\} = [\mathbf{k}_{ff}]\{\mathbf{\Delta}_f\}$ but we still have to determine $[\mathbf{k}_{fs}]$, $[\mathbf{k}_{sf}]$, and $[\mathbf{k}_{ss}]$.

17 $[\mathbf{k}_{sf}]$: Since $[\mathbf{d}]$ is obtained for a stable statically determinate structure, we have:

$$\{\mathbf{P}_s\} = [\mathcal{B}]\{\mathbf{P}_f\} \tag{7.36}$$

$$\{\mathbf{P}_s\} = \underbrace{[\mathcal{B}][\mathbf{k}_{ff}]}_{[\mathbf{k}_{sf}]}\{\mathbf{\Delta}_f\} \tag{7.37}$$

$$[\mathbf{k}_{sf}] = [\mathcal{B}][\mathbf{d}]^{-1} \tag{7.38}$$

18 $[\mathbf{k}_{fs}]$: Equating the external to the internal work:

1. External work: $W_{ext} = \frac{1}{2}\lfloor\mathbf{\Delta}_f\rfloor\{\mathbf{P}_f\}$
2. Internal work: $W_{int} = \frac{1}{2}\lfloor\mathbf{P}_s\rfloor\{\mathbf{\Delta}_s\}$

Equating W_{ext} to W_{int} and combining with

$$\lfloor\mathbf{P}_s\rfloor = \lfloor\mathbf{\Delta}_f\rfloor[\mathbf{k}_{sf}]^T \tag{7.39}$$

from Eq. 7.26 with $\{\mathbf{\Delta}_s\} = \{\mathbf{0}\}$ (zero support displacements) we obtain:

$$[\mathbf{k}_{fs}] = [\mathbf{k}_{sf}]^T = [\mathbf{d}]^{-1}[\mathcal{B}]^T \tag{7.40}$$

19 $[\mathbf{k}_{ss}]$: This last term is obtained from

$$\begin{aligned}\{\mathbf{P}_s\} &= [\mathcal{B}]\{\mathbf{P}_f\} && (7.41)\\ \{\mathbf{P}_f\} &= [\mathbf{k}_{fs}]\{\boldsymbol{\Delta}_s\} && (7.42)\\ [\mathbf{k}_{fs}] &= [\mathbf{d}]^{-1}[\mathcal{B}]^T && (7.43)\end{aligned}$$

Combining Eqns. 7.42, 7.41, and 7.43 we obtain:

$$\{\mathbf{P}_s\} = \underbrace{[\mathcal{B}][\mathbf{d}]^{-1}[\mathcal{B}]^T}_{[\mathbf{k}_{ss}]}\{\boldsymbol{\Delta}_s\} \tag{7.44}$$

20 In summary we have:

$$\boxed{[\mathbf{k}] = \left[\begin{array}{c|c}[\mathbf{d}]^{-1} & [\mathbf{d}]^{-1}[\mathcal{B}]^T\\ \hline [\mathcal{B}][\mathbf{d}]^{-1} & [\mathcal{B}][\mathbf{d}]^{-1}[\mathcal{B}]^T\end{array}\right]} \tag{7.45}$$

21 A very important observation, is that the stiffness matrix is obviously singular, since the second "row" is linearly dependent on the first one (through $[\mathcal{B}]$) and thus, its determinent is equal to zero.

■ Example 7-3: Flexibility to Stiffness

With reference to Fig. 7.2, and with both M_1 and M_2 assumed to be positive (ccw):

1. The flexibility matrix is given by:

$$\left\{\begin{array}{c}\theta_1\\ \theta_2\end{array}\right\} = \underbrace{\frac{L}{6EI}\left[\begin{array}{rr}2 & -1\\ -1 & 2\end{array}\right]}_{[\mathbf{d}]}\left\{\begin{array}{c}M_1\\ M_2\end{array}\right\} \tag{7.46}$$

2. The statics matrix $[\mathcal{B}]$ relating external to internal forces is given by:

$$\left\{\begin{array}{c}V_1\\ V_2\end{array}\right\} = \underbrace{\frac{1}{L}\left[\begin{array}{rr}1 & 1\\ -1 & -1\end{array}\right]}_{[\mathcal{B}]}\left\{\begin{array}{c}M_1\\ M_2\end{array}\right\} \tag{7.47}$$

3. $[\mathbf{k}_{ff}]$: would simply be given by:

$$[\mathbf{k}_{ff}] = [\mathbf{d}]^{-1} = \frac{EI}{L}\left[\begin{array}{cc}4 & 2\\ 2 & 4\end{array}\right] \tag{7.48}$$

4. $[\mathbf{k}_{fs}]$: The upper off-diagonal

$$[\mathbf{k}_{fs}] = [\mathbf{d}]^{-1}[\mathcal{B}]^T = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{EI}{L^2}\begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix} \tag{7.49}$$

5. $[\mathbf{k}_{sf}]$: Lower off-diagonal term

$$[\mathbf{k}_{sf}] = [\mathcal{B}][\mathbf{d}]^{-1} = \frac{1}{L}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{L^2}\begin{bmatrix} 6 & 6 \\ -6 & -6 \end{bmatrix} \tag{7.50}$$

6. $[\mathbf{k}_{ss}]$:

$$[\mathbf{k}_{ss}] = [\mathcal{B}]\,[\mathbf{d}]^{-1}\,[\mathcal{B}]^T = [\mathbf{k}_{sf}]\,[\mathcal{B}]^T \tag{7.51}$$

$$= \frac{EI}{L^2}\frac{1}{L}\begin{bmatrix} 6 & 6 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix} \tag{7.52}$$

Let us observe that we can rewrite:

$$\left\{\begin{array}{c} M_1 \\ M_2 \\ \hline V_1 \\ V_2 \end{array}\right\} = \frac{EI}{L^3}\left[\begin{array}{cc|cc} 4L^2 & 2L^2 & 6l & -6l \\ 2L^2 & 4L^2 & 6l & -6l \\ \hline 6l & 6l & 12 & -12 \\ -6l & -6l & -12 & 12 \end{array}\right]\left\{\begin{array}{c} \theta_1 \\ \theta_2 \\ \hline v_1 \\ v_2 \end{array}\right\} \tag{7.53}$$

If we rearrange the stiffness matrix we would get:

$$\left\{\begin{array}{c} V_1 \\ M_1 \\ \hline V_2 \\ M_2 \end{array}\right\} = \underbrace{\frac{EI}{L}\left[\begin{array}{cc|cc} \frac{12}{L^2} & \frac{6}{L} & \frac{-12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & \frac{-6}{L} & 2 \\ \hline \frac{-12}{L^2} & \frac{-6}{L} & \frac{12}{L} & \frac{-6}{L} \\ \frac{6}{L} & 2 & \frac{-6}{L} & 4 \end{array}\right]}_{[\mathbf{k}]}\left\{\begin{array}{c} v_1 \\ \theta_1 \\ \hline v_2 \\ \theta_2 \end{array}\right\} \tag{7.54}$$

and is the same stiffness matrix earlier derived. ■

7.4 Stiffness Matrix of a Curved Element

22 We seek to determine the stiffness matrix of a circular arc of radius R and sustaining an angle θ.

23 First, we determine the flexibility matrix of a cantilevered arc from

$$\delta U = \int_s \delta\overline{M}\frac{M}{EI}dx = \frac{R}{EI}\int_0^\theta \delta\overline{M}Md\phi \tag{7.55}$$

where M is the real moment at arbitrary point A caused by loads and $\delta\overline{M}$ is the virtual moment at A caused by unit load

24 The flexibility matrix will thus be given by:

$$\left\{\begin{array}{c} u \\ v \\ \theta \end{array}\right\} = \left[\begin{array}{ccc} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{array}\right] \left\{\begin{array}{c} N \\ V \\ M \end{array}\right\} \tag{7.56}$$

and

$$M = M_3 + N(R - R\cos\phi) + V(R_2 \sin\phi) \tag{7.57-a}$$

$$\delta\overline{M} + 1_3 = R(1 - \cos_1\phi) + R\sin_2\phi \tag{7.57-b}$$

$$f_{ij} = \text{Disp.in}DOFi\text{causedbyunitloadin}DOFj \tag{7.57-c}$$

$$f_{11} = \frac{R}{EI}\int_o^\theta \delta M_{p1}\cdot M_{D1}d\phi \tag{7.57-d}$$

$$= \frac{R}{EI}\int_o^\theta R^2(1-\cos\phi)^2 d\phi \tag{7.57-e}$$

$$= \frac{R^3}{EI}\int_o^\theta (1 - 2\cos\phi + \cos^2\phi)d\phi \tag{7.57-f}$$

$$= \frac{R^3}{EI}\left[\phi - 2\sin\phi + \phi/2 = 1/4\sin 2\phi\right]_o^\theta \tag{7.57-g}$$

$$= \boxed{\frac{R^3}{EI}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]} \tag{7.57-h}$$

$$f_{12} = f_{21} = \frac{R}{EI}\int_o^\theta R^2\sin\phi(1-\cos\phi)d\phi \tag{7.57-i}$$

$$= \frac{R^3}{EI}\int_o^\theta (\sin\phi - cos\phi sin\phi)d\phi \tag{7.57-j}$$

$$= \frac{R^3}{EI}\left[-\cos\phi - \frac{1}{2}\sin^2\phi\right]_o^\theta \tag{7.57-k}$$

$$= \frac{R^3}{EI}\left[(-\cos\theta - \frac{1}{2}\sin^2\theta) - (-1 - 0)\right] \tag{7.57-l}$$

$$= \boxed{f_{21} = \frac{R^3}{EI}\left[1 - \cos\theta - \frac{1}{2}\sin^2\theta\right]} \tag{7.57-m}$$

$$f_{13} = f_{31} = \frac{R}{EI}\int_o^\theta R(1-\cos\phi)d\phi \tag{7.57-n}$$

$$= \frac{R^2}{EI}\left[\phi - \sin\phi\right]_o^\theta \tag{7.57-o}$$

$$= \boxed{\frac{R^2}{EI}\left[\theta - \sin\theta\right]} \tag{7.57-p}$$

$$f_{22} = \frac{R}{EI}\int_o^\theta R^2\sin^2\phi d\phi \tag{7.57-q}$$

$$= \frac{R^3}{EI}\left[\frac{\phi}{\partial} - \frac{1}{4}\sin 2\theta\right]_o^\theta \tag{7.57-r}$$

$$= \boxed{f_{22}\frac{R^3}{EI}\left[\frac{\theta}{\partial} - \frac{1}{4}\sin 2\theta\right]} \tag{7.57-s}$$

$$f_{23} = f_{32} = \frac{R}{EI}\int_o^\theta R\sin\phi d\phi \tag{7.57-t}$$

$$= \frac{R^2}{EI}\left[-\cos\phi\right]_o^\theta \tag{7.57-u}$$

$$= \boxed{\frac{R^2}{EI}\left[-\cos\theta + 1\right]} \tag{7.57-v}$$

$$f_{33} = \frac{R}{EI}\int_o^\theta d\theta \tag{7.57-w}$$

$$= \boxed{\frac{R\theta}{EI}} \tag{7.57-x}$$

7.5 Duality between the Flexibility and the Stiffness Methods

	FLEXIBILITY	**STIFFNESS**
Indeterminancy	Static	Kinematic
Primary Unknows	Nodal Forces	Nodal Displacements
Variational Principle	Virtual Force	Virtual Displacement
	$\{\mathbf{p}\} = [\mathcal{B}']\{\mathbf{P}\}$	$\{\boldsymbol{\delta}\} = [\boldsymbol{\Gamma}]\{\boldsymbol{\Delta}\}$
	$[\mathbf{d}]$	$[\mathbf{k}]$
	$[D] = [\mathcal{B}]^T[\mathbf{d}][\mathcal{B}]$	$[\mathbf{k}] = \Sigma[\boldsymbol{\Gamma}]^T[\mathbf{k}][\boldsymbol{\Gamma}]$
	$\left\{\begin{array}{c}\boldsymbol{\Delta}_P\\ \boldsymbol{\Delta}_R\end{array}\right\} = \left[\begin{array}{cc}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{array}\right]\left\{\begin{array}{c}\mathbf{P}\\ \mathbf{R}\end{array}\right\}$	$\left\{\begin{array}{c}\mathbf{P}\\ \mathbf{R}\end{array}\right\} = \left[\begin{array}{cc}\mathbf{k}_{ff} & \mathbf{D}_{fr}\\ \mathbf{D}_{rf} & \mathbf{D}_{rr}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{\Delta}_f\\ \boldsymbol{\Delta}_r\end{array}\right\}$
	$\{\mathbf{R}\} = [D_{22}]^{-1}(\{\boldsymbol{\Delta}_R\} - [\mathbf{D}_{21}]\{\mathbf{p}\})$	$\{\boldsymbol{\Delta}_f\} = [K_{ff}]^{-1}(\{\mathbf{P}\} - [\mathbf{k}_{fr}]\{\boldsymbol{\Delta}_r\})$
	$\{\boldsymbol{\Delta}_P\} = [D_{11}]^{-1}(\{\mathbf{P}\} + [\mathbf{D}_{12}]\{\mathbf{R}\})$	$\{\mathbf{R}\} = [K_{rf}]\{\boldsymbol{\Delta}_f\} + [\mathbf{k}_{rr}]\{\boldsymbol{\Delta}_r\}$
	$\{\mathbf{p}\} = [\mathcal{B}]\{\mathbf{P}\}$	$\{\mathbf{p}\} = [\mathbf{k}][\boldsymbol{\Gamma}]\{\boldsymbol{\Delta}\}$

Part II

Introduction to Finite Elements

Chapter 8

REVIEW OF ELASTICITY

8.1 Stress

1 A stress, Fig 8.1 is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to

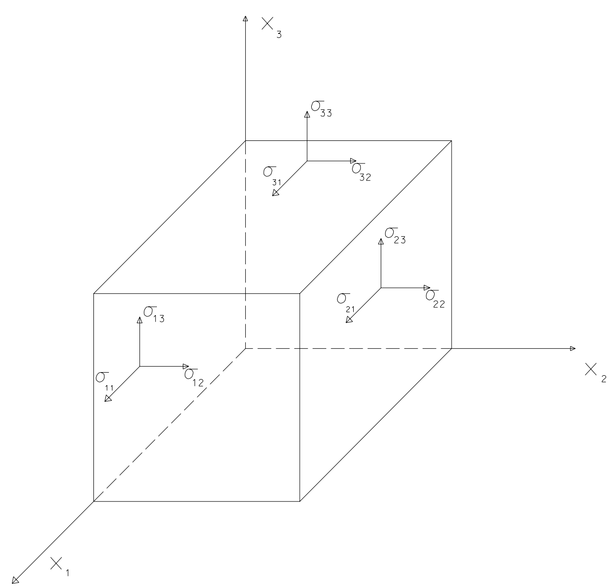


Figure 8.1: Stress Components on an Infinitesimal Element

the direction of outward facing normal, and the second one (j) to the direction of component force.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{8.1}$$

2 The stress tensor is symmetric $\sigma_{ij} = \sigma_{ji}$; this can easily be proved through rotational equilibrium.

8.1.1 Stress Traction Relation

3 The relation between stress tensor σ_{ij} at a point and the stress vector t_i (or traction) on a plane of arbitrary orientation, can be established through the following, Fig. 8.2.

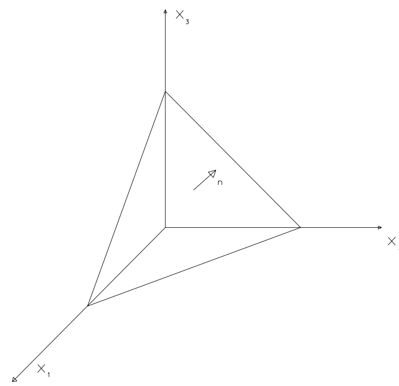


Figure 8.2: Stress Traction Relations

$$\boxed{t_i = n_j \sigma_{ij}} \tag{8.2}$$

$$\lfloor\ t_1 \quad t_2 \quad t_3\ \rfloor = \underbrace{\lfloor\ n_1 \quad n_2 \quad n_3\ \rfloor}_{\text{direction cosines}} \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{\text{stress tensor}} \tag{8.3}$$

where **n** is a unit outward vector normal to the plane.

4 Note that the stress is defined at a point, and a traction is defined at a point and with respect to a given plane orientation.

5 When expanded in cartesian coordinates,, the previous equation yields

$$\begin{array}{rcl} t_x & = & \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z \\ t_y & = & \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z \\ t_z & = & \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z \end{array} \tag{8.4}$$

8.2 Strain

6 Given the displacement u_i of a point, the strain ε_{ij} is defined as

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)} \tag{8.5}$$

or

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{8.6}$$

7 When expanded in 2D, this equation yields:

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1} \tag{8.7-a}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\gamma_{12}}{2} \tag{8.7-b}$$

$$\varepsilon_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2} \tag{8.7-c}$$

$$\varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = \frac{\gamma_{21}}{2} \tag{8.7-d}$$

8 Initial (or thermal strain)

$$\varepsilon_{ij} = \underbrace{\left\{\begin{array}{c} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{array}\right\}}_{\text{Plane Stress}} = \underbrace{(1+\nu)\left\{\begin{array}{c} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{array}\right\}}_{\text{Plane Strain}} \tag{8.8}$$

note there is no shear strains caused by thermal expansion.

9 The strain may also be expressed as

$$\boxed{\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}} \tag{8.9}$$

or

$$\left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{array}\right\} = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{array}\right] \left\{\begin{array}{c} u_x \\ u_y \\ u_z \end{array}\right\} \tag{8.10}$$

8.3 Fundamental Relations in Elasticity

8.3.1 Equation of Equilibrium

10 Starting with the set of forces acting on an infinitesimal element of dimensions $dx_1 \times dx_2 \times dx_3$, Fig. 8.3 and writing the summation of forces, will yield

$$\boxed{\sigma_{ij,j} + \rho b_i = 0} \tag{8.11}$$

where ρ is the density, b_i is the body force (including inertia).

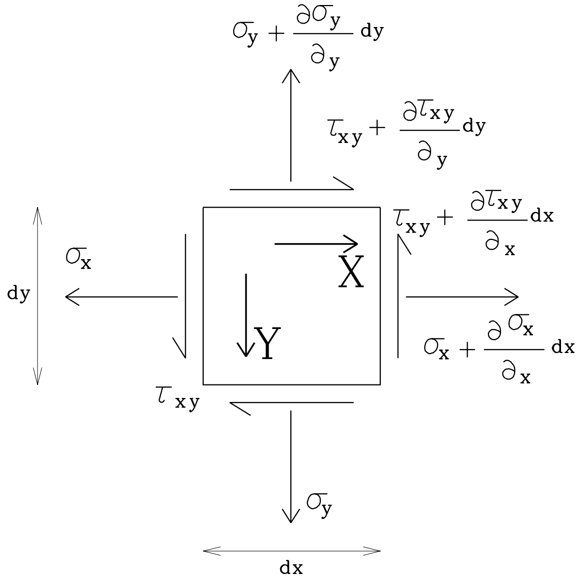


Figure 8.3: Equilibrium of Stresses, Cartesian Coordinates

11 When expanded in 3D, this equation yields:

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 &= 0 \\
\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 &= 0 \\
\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 &= 0
\end{aligned} \tag{8.12-a}$$

12 Alternatively, the equation of equilibrium can be written as

$$\boxed{\mathbf{L}^T\boldsymbol{\sigma}+\mathbf{b}=0} \tag{8.13}$$

or

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{array} \right\} + \left\{ \begin{array}{c} b_x \\ b_y \\ b_z \end{array} \right\} = 0 \tag{8.14}$$

13 Expanding

$$\begin{array}{rcl} \frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x & = & 0 \\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y & = & 0 \\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z & = & 0 \end{array} \tag{8.15}$$

8.3.2 Compatibility Equation

14 If $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ then we have six differential equations (in 3D the strain tensor has a total of 9 terms, but due to symmetry, there are 6 independent ones) for determining (upon integration) three unknowns displacements u_i. Hence the system is overdetermined, and there must be some linear relations between the strains.

15 It can be shown (through appropriate successive differentiation of Eq. **??**) that the compatibility relation for strain reduces to:

$$\boxed{\frac{\partial^2\varepsilon_{ik}}{\partial x_j\partial x_j}+\frac{\partial^2\varepsilon_{jj}}{\partial x_i\partial x_k}-\frac{\partial^2\varepsilon_{jk}}{\partial x_i\partial x_j}-\frac{\partial^2\varepsilon_{ij}}{\partial x_j\partial x_k}=0.} \tag{8.16}$$

16 In 3D, this would yield 9 equations in total, however only six are distinct. In 2D, this results in (by setting $i=2$, $j=1$ and $l=2$):

$$\boxed{\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=\frac{\partial^2\gamma_{12}}{\partial x_1\partial x_2}} \tag{8.17}$$

(recall that $2\varepsilon_{12}=\gamma_{12}$.

17 When he compatibility equation is written in term of the stresses, it yields:

$$\frac{\partial^2\sigma_{11}}{\partial x_2^2}-\nu\frac{\partial\sigma_{22}{}^2}{\partial x_2^2}+\frac{\partial^2\sigma_{22}}{\partial x_1^2}-\nu\frac{\partial^2\sigma_{11}}{\partial x_1^2}=2(1+\nu)\frac{\partial^2\sigma_{21}}{\partial x_1\partial x_2} \tag{8.18}$$

8.4 Stress-Strain Relations in Elasticity

18 The Generalized Hooke's Law can be written as:

$$\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \qquad i,j,k,l = 1,2,3} \tag{8.19}$$

19 The (fourth order) tensor of elastic constants C_{ijkl} has 81 (3^4) components however, due to the symmetry of both $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, there are at most 36 $\left(\frac{9(9-1)}{2}\right)$ distinct elastic terms.

20 For the purpose of writing Hooke's Law, the double indexed system is often replaced by a simple indexed system with a range of six:

$$\sigma_k = \overbrace{D_{km}}^{6^2=36}\varepsilon_m \qquad k,m = 1,2,3,4,5,6 \tag{8.20}$$

21 For isotropic bodies (elastic properties independent of reference system used to describe it), it can be shown that the number of independent elastic constants is two. The stress strain relations can be written in terms of E and ν as:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \tag{8.21}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \tag{8.22}$$

where δ_{ij} is the kroneker delta and is equal to 1 if $i = j$ and to 0 if $i \neq j$. When the strain equation is expanded in 3D cartesian coordinates it would yield:

$$\begin{aligned}
\varepsilon_{xy} &= \tfrac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \tfrac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] \\
\varepsilon_{zz} &= \tfrac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \\
\varepsilon_{xy} &= \tfrac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{yz} &= \tfrac{1+\nu}{E}\sigma_{yz} \\
\varepsilon_{zx} &= \tfrac{1+\nu}{E}\sigma_{zx}
\end{aligned} \tag{8.23}$$

22 When the stress equation is expanded in 3D cartesian coordinates, it will yield:

$$\begin{aligned}
\sigma_{xx} &= \tfrac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})\right] \\
\sigma_{yy} &= \tfrac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{zz} + \varepsilon_{xx})\right] \\
\sigma_{zz} &= \tfrac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})\right] \\
\sigma_{xy} &= \mu\gamma_{xy} \\
\sigma_{yz} &= \mu\gamma_{yz} \\
\sigma_{zx} &= \mu\gamma_{zx}
\end{aligned} \tag{8.24}$$

where μ is the shear modulus and $\gamma_{xy} = 2\varepsilon_{xy}$.

23 In terms of Lamé constant we would have

$$\boxed{\begin{aligned} \sigma_{ij} &= \lambda\varepsilon_{ii}\delta_{ij} + 2\mu\varepsilon_{ij} && (8.25)\\ \varepsilon_{ij} &= \frac{1}{2\mu}\left(\sigma_{ij} - \lambda\delta_{ij}\frac{\sigma_{kk}}{3\lambda+2\mu}\right) && (8.26)\end{aligned}}$$

where

$$\boxed{\begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} && (8.27)\\ \mu &= \frac{E}{2(1+\nu)} && (8.28)\end{aligned}}$$

24 In terms of initial stresses and strains

$$\boxed{\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^0) + \sigma_{ij}^0} \tag{8.29}$$

8.5 Strain Energy Density

25 Any elastically deforming body possesses a uniquely defined strain energy[1] density which can be expressed as:

$$\boxed{U_0 = \int_0^{\varepsilon}\sigma_{ij}d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}D_{ijkl}\varepsilon_{ij}\varepsilon_{kl}} \tag{8.30}$$

8.6 Summary

26 Fig. 8.4 illustrates the fundamental equations in solid mechanics.

[1]Used in the evaluation of Energy release rate later on.

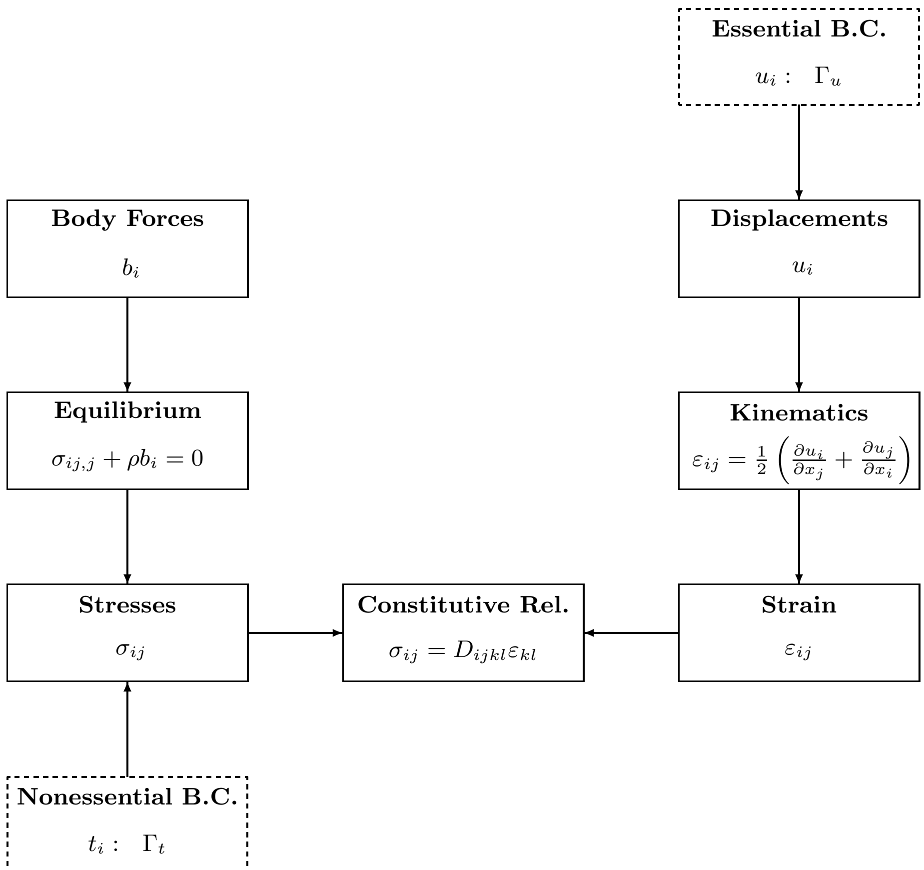


Figure 8.4: Fundamental Equations in Solid Mechanics

Chapter 9

VARIATIONAL AND ENERGY METHODS

9.1 † Variational Calculus; Preliminaries

`From Pilkey and Wunderlich book`

9.1.1 Euler Equation

1 The fundamental problem of the calculus of variation[1] is to find a function $u(x)$ such that

$$\Pi = \int_a^b F(x, u, u')dx \tag{9.1}$$

is stationary. Or,

$$\boxed{\delta\Pi = 0} \tag{9.2}$$

where δ indicates the *variation*

2 We define $u(x)$ to be a function of x in the interval (a, b), and F to be a known function (such as the energy density).

3 We define the *domain* of a functional as the collection of admissible functions belonging to a class of functions in function space rather than a region in coordinate space (as is the case for a function).

4 We seek the function $u(x)$ which extremizes Π.

[1]Differential calculus involves a function of one or more variable, whereas variational calculus involves a function of a function, or a functional.

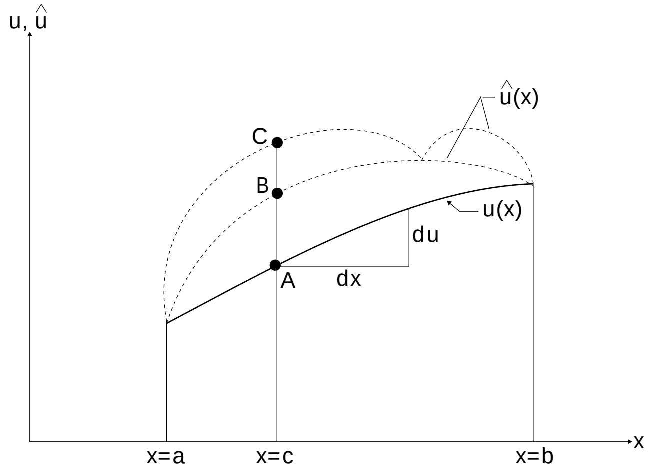


Figure 9.1: Variational and Differential Operators

5 Letting $\tilde{u}$ to be a family of neighbouring paths of the extremizing function $u(x)$ and we assume that at the end points $x = a, b$ they coincide. We define $\tilde{u}$ as the sum of the extremizing path and some arbitrary variation, Fig. 9.1.

$$\tilde{u}(x, \varepsilon) = u(x) + \varepsilon\eta(x) = u(x) + \delta u(x) \tag{9.3}$$

where ε is a small parameter, and $\delta u(x)$ is the *variation* of $u(x)$

$$\delta u = \tilde{u}(x, \varepsilon) - u(x) \tag{9.4-a}$$
$$= \varepsilon\eta(x) \tag{9.4-b}$$

and $\eta(x)$ is twice differentiable, has undefined amplitude, and $\eta(a) = \eta(b) = 0$. We note that $\tilde{u}$ coincides with u if $\varepsilon = 0$

6 It can be shown that the variation and derivation operators are commutative

$$\left.\begin{array}{rcl} \frac{d}{dx}(\delta u) & = & \tilde{u}'(x, \varepsilon) - u'(x) \\ \delta u' & = & \tilde{u}'(x, \varepsilon) - u'(x) \end{array}\right\} \boxed{\frac{d}{dx}(\delta u) = \delta\left(\frac{du}{dx}\right)} \tag{9.5}$$

7 Furthermore, the variational operator δ and the differential calculus operator d can be similarly used, i.e.

$$\delta(u')^2 = 2u'\delta u' \tag{9.6-a}$$
$$\delta(u + v) = \delta u + \delta v \tag{9.6-b}$$
$$\delta\left(\int u dx\right) = \int (\delta u) dx \tag{9.6-c}$$
$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \tag{9.6-d}$$

however, they have clearly different meanings. du is associated with a neighbouring point at a distance dx, however δu is a small *arbitrary* change in u for a given x (there is no associated δx).

8 For boundaries where u is specified, its variation must be zero, and it is arbitrary elsewhere. The variation δu of u is said to undergo a *virtual* change.

9 To solve the variational problem of extremizing Π, we consider

$$\Pi(u+\varepsilon\eta) = \Phi(\varepsilon) = \int_a^b F(x, u+\varepsilon\eta, u'+\varepsilon\eta')dx \tag{9.7}$$

10 Since $\tilde{u} \to u$ as $\varepsilon \to 0$, the necessary condition for Π to be an extremum is

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{9.8}$$

11 From Eq. 9.3 $\tilde{u} = u + \varepsilon\eta$, and $\tilde{u}' = u' + \varepsilon\eta'$, and applying the chain rule

$$\frac{d\Phi(\varepsilon)}{d\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial \tilde{u}}\frac{d\tilde{u}}{d\varepsilon} + \frac{\partial F}{\partial \tilde{u}'}\frac{d\tilde{u}'}{d\varepsilon}\right)dx = \int_a^b \left(\eta\frac{\partial F}{\partial \tilde{u}} + \eta'\frac{\partial F}{\partial \tilde{u}'}\right)dx \tag{9.9}$$

for $\varepsilon = 0$, $\tilde{u} = u$, thus

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left(\eta\frac{\partial F}{\partial u} + \eta'\frac{\partial F}{\partial u'}\right)dx = 0 \tag{9.10}$$

Integration by part (Eq. 5.1 and 5.1) of the second term leads to

$$\int_a^b \left(\eta'\frac{\partial F}{\partial u'}\right)dx = \left.\eta\frac{\partial F}{\partial u'}\right|_a^b - \int_a^b \eta(x)\left(\frac{d}{dx}\frac{\partial F}{\partial u'}\right)dx \tag{9.11}$$

Using the end conditions $\eta(a) = \eta(b) = 0$, Eq. 9.10 leads to

$$\int_a^b \eta(x)\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right)dx = 0 \tag{9.12}$$

12 The fundamental lemma of the calculus of variation states that for continuous $\Psi(x)$ in $a \leq x \leq b$, and with arbitrary continuous function $\eta(x)$ which vanishes at a and b, then

$$\boxed{\int_a^b \eta(x)\Psi(x)dx = 0 \Leftrightarrow \Psi(x) = 0} \tag{9.13}$$

Thus,

$$\boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0} \tag{9.14}$$

13 This differential equation is called the **Euler equation** associated with Π and is a necessary condition for $u(x)$ to extremize Π.

14 Generalizing for a functional Π which depends on two field variables, $u = u(x, y)$ and $v = v(x, y)$

$$\Pi = \int\int F(x, y, u, v, u_{,x}, u_{,y}, v_{,x}, v_{,y}, \cdots, v_{,yy})dxdy \tag{9.15}$$

There would be as many Euler equations as dependent field variables

$$\left\{ \begin{array}{lcl} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial u_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial u_{,yy}} & = & 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial v_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{,yy}} & = & 0 \end{array} \right. \tag{9.16}$$

15 We note that the Functional and the corresponding Euler Equations, Eq. 9.1 and 9.14, or Eq. 9.15 and 9.16 describe the same problem.

16 The Euler equations usually correspond to the governing differential equation and are referred to as the **strong form** (or classical form).

17 The functional is referred to as the **weak form** (or generalized solution). This classification stems from the fact that equilibrium is enforced in an average sense over the body (and the field variable is differentiated m times in the weak form, and $2m$ times in the strong form).

18 It can be shown that in the principle of virtual displacements, the Euler equations are the equilibrium equations, whereas in the principle of virtual forces, they are the compatibility equations.

19 Euler equations are differential equations which can not always be solved by exact methods. An alternative method consists in bypassing the Euler equations and go directly to the variational statement of the problem to the solution of the Euler equations.

20 Finite Element formulation are based on the weak form, whereas the formulation of Finite Differences are based on the strong form.

21 We still have to define $\delta\Pi$. The first variation of a functional expression is

$$\left. \begin{array}{lcl} \delta F & = & \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \\ \delta \Pi & = & \int_a^b \delta F dx \end{array} \right\} \delta\Pi = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx \tag{9.17}$$

As above, integration by parts of the second term yields

$$\boxed{\delta\Pi = \int_a^b \delta u \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} \right) dx} \tag{9.18}$$

22 We have just shown that finding the stationary value of Π by setting $\delta\Pi = 0$ is equivalent to finding the extremal value of Π by setting $\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$ equal to zero.

23 Similarly, it can be shown that as with second derivatives in calculus, the second variation $\delta^2\Pi$ can be used to characterize the extremum as either a minimum or maximum.

9.1.2 Boundary Conditions

24 Revisiting the integration by parts of the second term in Eq. 9.10, we had

$$\boxed{\int_a^b \eta' \frac{\partial F}{\partial u'} dx = \eta \left. \frac{\partial F}{\partial u'} \right|_a^b - \int_a^b \eta \frac{d}{dx} \frac{\partial F}{\partial u'} dx} \tag{9.19}$$

We note that

1. Derivation of the Euler equation required $\eta(a) = \eta(b) = 0$, thus this equation is a statement of the *essential* (or forced) boundary conditions, where $u(a) = u(b) = 0$.
2. If we left η arbitrary, then it would have been necessary to use $\frac{\partial F}{\partial u'} = 0$ at $x = a$ and b. These are the *natural* boundary conditions.

25 For a problem with, one field variable, in which the highest derivative in the governing differential equation is of order $2m$ (or simply m in the corresponding functional), then we have

Essential (or Forced, or geometric) boundary conditions, involve derivatives of order zero (the field variable itself) through m-1. Trial displacement functions are explicitely required to satisfy this B.C. Mathematically, this corresponds to *Dirichlet boundary-value problems*.

Nonessential (or Natural, or static) boundary conditions, involve derivatives of order m and up. This B.C. is implied by the satisfaction of the variational statement but not explicitly stated in the functional itself. Mathematically, this corresponds to *Neuman boundary-value problems*.

26 Table 9.1 illustrates the boundary conditions associated with some problems

Problem	Axial Member Distributed load	Flexural Member Distributed load
Differential Equation	$AE\frac{d^2u}{dx^2} + q = 0$	$EI\frac{d^4w}{dx^4} - q = 0$
m	1	2
Essential B.C. $[0, m-1]$	u	w, $\frac{dw}{dx}$
Natural B.C. $[m, 2m-1]$	$\frac{du}{dx}$ or $\sigma_x = Eu_{,x}$	$\frac{d^2w}{dx^2}$ and $\frac{d^3w}{dx^3}$ or $M = EIw_{,xx}$ and $V = EIw_{,xxx}$

Table 9.1: Essential and Natural Boundary Conditions

■ **Example 9-1: Extension of a Bar**

The total potential energy Π of an axial member of length L, modulus of elasticity E, cross sectional area A, fixed at left end and subjected to an axial force P at the right one is given by

$$\Pi = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx - Pu(L) \tag{9.20}$$

Determine the Euler Equation by requiring that Π be a minimum.
Solution:

Solution I The first variation of Π is given by

$$\delta\Pi = \int_0^L \frac{EA}{2}2\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right)dx - P\delta u(L) \tag{9.21}$$

Integrating by parts we obtain

$$\begin{aligned}
\delta\Pi &= \int_0^L -\frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u dx + EA\frac{du}{dx}\delta u\Big|_0^L - P\delta u(L) && \text{(9.22-a)}\\
&= -\int_0^L \delta u \underbrace{\frac{d}{dx}\left(EA\frac{du}{dx}\right)}\, dx + \left[\underbrace{\left(EA\frac{du}{dx}\right)\Big|_{x=L} - P}\right]\delta u(L) \\
&= -\underbrace{\left(EA\frac{du}{dx}\right)}\Big|_{x=0} \delta u(0) && \text{(9.22-b)}
\end{aligned}$$

The last term is zero because of the specified essential boundary condition which implies that $\delta u(0) = 0$. Rcalling that δ in an arbitrary operator which can be assigned any value, we set the coefficients of δu between $(0, L)$ and those for δu at $x = L$ equal to zero separately, and obtain

Euler Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \quad 0 < x < L \tag{9.23}$$

Natural Boundary Condition:

$$EA\frac{du}{dx} - P = 0 \quad \text{at} \;\; x = L \tag{9.24}$$

Solution II We have

$$F(x, u, u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2 \tag{9.25}$$

(note that since P is an applied load at the end of the member, it does not appear as part of $F(x, u, u')$ To evaluate the Euler Equation from Eq. 9.14, we evaluate

$$\frac{\partial F}{\partial u} = 0 \;\;\&\;\; \frac{\partial F}{\partial u'} = EAu' \tag{9.26-a}$$

Thus, substituting, we obtain

$$\begin{aligned}\frac{\partial F}{\partial u}-\frac{d}{dx}\frac{\partial F}{\partial u'} &= 0 \text{ Euler Equation} &\text{(9.27-a)}\\ \frac{d}{dx}\left(EA\frac{du}{dx}\right) &= 0 \text{ B.C.} &\text{(9.27-b)}\end{aligned}$$

■

■ **Example 9-2: Flexure of a Beam**

The total potential energy of a beam is given by

$$\Pi=\int_0^L\left(\frac{1}{2}M\kappa-pw\right)dx=\int_0^L\left(\frac{1}{2}(EIw'')w''-pw\right)dx \tag{9.28}$$

Derive the first variational of Π.
Solution:
Extending Eq. 9.17, and integrating by part twice

$$\begin{aligned}\delta\Pi &= \int_0^L\delta F dx=\int_0^L\left(\frac{\partial F}{\partial w''}\delta w''+\frac{\partial F}{\partial w}\delta w\right)dx &\text{(9.29-a)}\\ &= =\int_0^L(EIw''\delta w''-p\delta w)dx &\text{(9.29-b)}\\ &= (EIw''\delta w')\big|_0^L-\int_0^L\left[(EIw'')'\delta w'-p\delta w\right]dx &\text{(9.29-c)}\\ &= (EIw''\delta w')\big|_0^L-\left[(EIw'')'\delta w\right]\big|_0^L+\int_0^L\left[(EIw'')''+p\right]\delta w dx=0 &\text{(9.29-d)}\end{aligned}$$

Or

$$(EIw'')''=-p \quad \text{for all x}$$

which is the governing differential equation of beams and

Essential		Natural
$\delta w'=0$	or	$EIw''=-M=0$
$\delta w=0$	or	$(EIw'')'=-V=0$

at $x=0$ and $x=L$ ■

9.2 Work, Energy & Potentials; Definitions

9.2.1 Introduction

27 Work is defined as the product of a force and displacement

$$W \stackrel{\text{def}}{=} \int_a^b \mathbf{F}.d\mathbf{s} \tag{9.30-a}$$

$$dW = F_x dx + F_y dy \tag{9.30-b}$$

28 Energy is a quantity representing the ability or capacity to perform work.

29 The change in energy is proportional to the amount of work performed. Since only the change of energy is involved, any datum can be used as a basis for measure of energy. Hence energy is neither created nor consumed.

30 The first law of thermodynamics states

> The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time:
>
> $$\boxed{\frac{d}{dt}(K+U) = W_e + H} \tag{9.31}$$

where K is the kinetic energy, U the internal strain energy, W the external work, and H the heat input to the system.

31 For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \tag{9.32}$$

9.2.2 Internal Strain Energy

32 The **strain energy density** of an arbitrary material is defined as, Fig. 9.2

$$\boxed{U_0 \stackrel{\text{def}}{=} \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma} \mathrm{d}\boldsymbol{\varepsilon}} \tag{9.33}$$

33 The **complementary strain energy** density is defined

$$\boxed{U_0^* \stackrel{\text{def}}{=} \int_0^{\boldsymbol{\sigma}} \boldsymbol{\varepsilon} \mathrm{d}\boldsymbol{\sigma}} \tag{9.34}$$

34 **The strain energy** itself is equal to

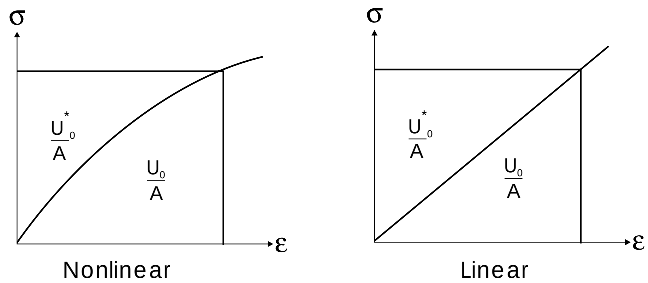


Figure 9.2: *Strain Energy and Complementary Strain Energy

$$\boxed{\begin{aligned} U &\stackrel{\text{def}}{=} \int_\Omega U_0 d\Omega \quad (9.35)\\ U^* &\stackrel{\text{def}}{=} \int_\Omega U_0^* d\Omega \quad (9.36)\end{aligned}}$$

35 To obtain a general form of the internal strain energy, we first define a stress-strain relationship accounting for both initial strains and stresses

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) + \boldsymbol{\sigma}_0 \tag{9.37}$$

where $\mathbf{D}$ is the constitutive matrix; $\boldsymbol{\epsilon}$ is the strain vector due to the displacements $\mathbf{u}$; $\boldsymbol{\epsilon}_0$ is the initial strain vector; $\boldsymbol{\sigma}_0$ is the initial stress vector; and $\boldsymbol{\sigma}$ is the stress vector.

36 The initial strains and stresses are the result of conditions such as heating or cooling of a system or the presence of pore pressures in a system.

37 The strain energy U for a linear elastic system is obtained by substituting

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{9.38}$$

with Eq. 9.33 and 9.37

$$\boxed{U = \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}\mathrm{d}\Omega - \int_\Omega \boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega + \int_\Omega \boldsymbol{\epsilon}^T\boldsymbol{\sigma}_0\mathrm{d}\Omega} \tag{9.39}$$

where Ω is the volume of the system.

38 Considering uniaxial stresses, in the absence of initial strains and stresses, and **for linear elastic systems**, Eq. 9.39 reduces to

$$\boxed{U = \frac{1}{2}\int_\Omega \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega} \tag{9.40}$$

39 When this relation is applied to various one dimensional structural elements it leads to

Axial Members:

$$\left.\begin{array}{l} U = \int_\Omega \frac{\varepsilon\sigma}{2} d\Omega \\ \sigma = \frac{P}{A} \\ \varepsilon = \frac{P}{AE} \\ d\Omega = Adx \end{array}\right\} \boxed{U = \frac{1}{2}\int_0^L \frac{P^2}{AE}dx} \tag{9.41}$$

Torsional Members:

$$\left.\begin{array}{l} U = \frac{1}{2}\int_\Omega \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega \\ U = \frac{1}{2}\int_\Omega \tau_{xy} \underbrace{G\tau_{xy}}_{\tau_{xy}} d\Omega \\ \tau_{xy} = \frac{Tr}{J} \\ \gamma_{xy} = \frac{\tau_{xy}}{G} \\ d\Omega = rd\theta drdx \\ \int_o^r \int_0^{2\pi} r^2 d\theta dr = J \end{array}\right\} \boxed{U = \frac{1}{2}\int_0^L \frac{T^2}{GJ}dx} \tag{9.42}$$

Flexural Members:

$$\left.\begin{array}{l} U = \frac{1}{2}\int_\Omega \varepsilon \underbrace{E\varepsilon}_{\sigma} \\ \sigma_x = \frac{M_z y}{I_z} \\ \varepsilon = \frac{M_z y}{EI_z} \\ d\Omega = dAdx \\ \int_A y^2 dA = I_z \end{array}\right\} \boxed{U = \frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx} \tag{9.43}$$

9.2.2.1 Internal Work versus Strain Energy

40 During strain increment, the work done by internal forces in a differential element will be the negative of that performed by the stresses acting upon it.

$$W_i = -\int_\Omega \boldsymbol{\sigma} d\boldsymbol{\epsilon} \mathrm{d}\Omega \tag{9.44}$$

41 If the strained elastic solid were permitted to slowly return to their unstrained state, then the solid would return the work performed by the external forces. This is due to the release of strain energy stored in the solid.

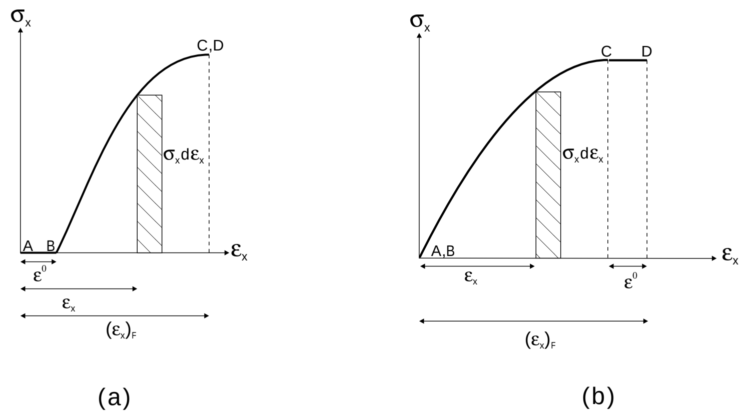


Figure 9.3: Effects of Load Histories on U and W_i

42 Thus, in the absence of initial strains,

$$\boxed{U = -W_i} \tag{9.45}$$

43 The internal work depends on the load history, this is illustrated by considering an axial member subjected to two load cases, Fig. 9.3: a) Initial thermal strains (with no corresponding stress increase), followed by an external force; and b) External force, followed by thermal strain. In both cases the internal work is equal to the area under the curve ABCD.

$$U_i = \int_0^L \left(\int_0^{(\varepsilon_x)_F} \sigma_x d\varepsilon_x \right) A dx \tag{9.46-a}$$

$$W_i^a = -U_i \tag{9.46-b}$$

$$W_i^b = -U_i - \int_0^L \left(\int_0^{\varepsilon^0} \sigma_x d\varepsilon_x \right) A dx \tag{9.46-c}$$

Hence, W_i is not always equal to $-U_i$.

9.2.3 External Work

44 External work W performed by the applied loads on an arbitrary system is defined as

$$\boxed{W_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \hat{\mathbf{t}} d\Gamma} \tag{9.47}$$

where $\mathbf{b}$ is the body force vector; $\hat{\mathbf{t}}$ is the applied surface traction vector; and Γ_t is that portion of the boundary where $\hat{\mathbf{t}}$ is applied, and $\mathbf{u}$ is the displacement.

45 For point loads and moments, the external work is

$$\boxed{W_e = \int_0^{\Delta_f} P d\Delta + \int_0^{\theta_f} M d\theta} \tag{9.48}$$

46 For **linear elastic systems**, we have for point loads

$$\left.\begin{array}{rcl} P & = & K\Delta \\ W_e & = & \int_0^{\Delta_f} P d\Delta \end{array}\right\} W_e = K\int_0^{\Delta_f} \Delta d\Delta = \frac{1}{2}K\Delta_f^2 \tag{9.49}$$

When this last equation is combined with $P_f = K\Delta_f$ we obtain

$$\boxed{W_e = \frac{1}{2}P_f\Delta_f} \tag{9.50}$$

where K is the **stiffness** of the structure.

47 Similarly for an applied moment we have

$$\boxed{W_e = \frac{1}{2}M_f\theta_f} \tag{9.51}$$

9.2.3.1 † Path Independence of External Work

48 In this section we seek to prove that the total work performed in going from state A to B is independent of the path.

49 In 2D the differential expression of the work is given by

$$dW = F_x dx + F_y dy \tag{9.52}$$

50 From calculus, a necessary and sufficient condition for dW to be an exact differential is that

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \tag{9.53}$$

51 If the force were to move along a closed contour (or from A to B and then back to A along any arbitrary path), corresponding to Γ, then from Green's theorem (Eq. 5.3) we have

$$\oint (Rdx + Sdy) = \int_\Gamma \left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right) dxdy \tag{9.54}$$

If we let $R = F_x$ and $S = F_y$, then

$$W = \oint (F_x dx + F_y dy) = \int_\Gamma \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{0} dxdy \tag{9.55}$$

52 Thus, from Eq. 9.53 the work is equal to zero,

53 If we decompose the path

$$W = \oint = \int_A^B + \int_B^A = 0 \Rightarrow \int_A^B = -\int_B^A \tag{9.56}$$

then, the integration for the work leads to

$$\boxed{W = \int_A^B (F_x dx + F_y dy)} \tag{9.57}$$

which is path independent.

54 Note that if no net work is done in moving around a closed path, the system is said to be **conservative**. This is the case for purely elastic systems.

55 When friction or plastic (or damped) deformations occur, then we would have a **nonconservative** system.

9.2.4 Virtual Work

56 We define the virtual work done by the load on a body during a small, admissible (continuous and satisfying the boundary conditions) change in displacements.

$$\text{Internal Virtual Work} \quad \delta W_i \stackrel{\text{def}}{=} -\int_\Omega \boldsymbol{\sigma} \delta \boldsymbol{\epsilon} d\Omega \tag{9.58}$$

$$\text{External Virtual Work} \quad \delta W_e \stackrel{\text{def}}{=} \int_{\Gamma_t} \hat{\mathbf{t}} \delta \mathbf{u} d\Gamma + \int_\Omega \mathbf{b} \delta \mathbf{u} d\Omega \tag{9.59}$$

where all the terms have been previously defined and **b** is the body force vector.

9.2.4.1 Internal Virtual Work

57 Next we shall derive a displacement based expression of δU for each type of one dimensional structural member. It should be noted that the Virtual Force method would yield analogous ones but based on forces rather than displacements.

58 Two sets of solutions will be given, the first one is independent of the material stress strain relations, and the other assumes a linear elastic stress strain relation.

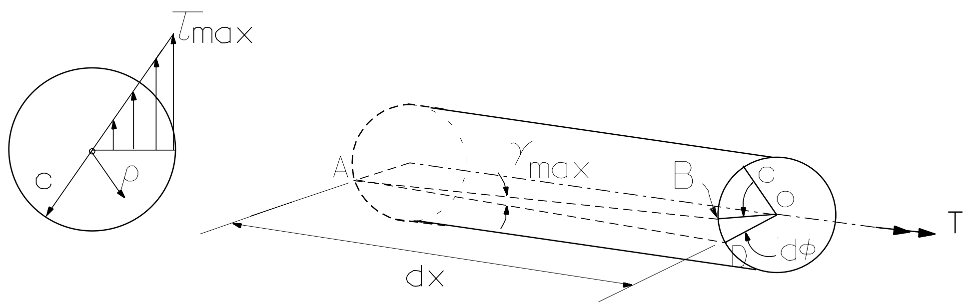


Figure 9.4: Torsion Rotation Relations

9.2.4.1.1 Elastic Systems

59 In this set of formulation, we derive expressions of the virtual strain energies which are independent of the material constitutive laws. Thus δU will be left in terms of forces and displacements.

Axial Members:

$$\left.\begin{array}{l} \delta U = \int_0^L \sigma\delta\varepsilon d\Omega \\ d\Omega = Adx \end{array}\right\} \boxed{\delta U = A\int_0^L \sigma\delta\varepsilon dx} \tag{9.60}$$

Torsional Members: With reference to Fig. 9.4

$$\left.\begin{array}{l} \delta U = \int_\Omega \tau_{xy}\delta\gamma_{xy}d\Omega \\ T = \int_A \tau_{xy} r dA \\ \delta\gamma_{xy} = r\delta\theta \\ d\Omega = dAdx \end{array}\right\} \delta U = \int_0^L \underbrace{(\int_A \tau_{xy} r dA)}_{T} \delta\theta dx \Rightarrow \boxed{\delta U = \int_0^L T\delta\theta dx} \tag{9.61}$$

Shear Members:

$$\left.\begin{array}{l} \delta U = \int_\Omega \tau_{xy}\delta\gamma_{xy}d\Omega \\ V = \int_A \tau_{xy} dA \\ d\Omega = dAdx \end{array}\right\} \delta U = \int_0^L \underbrace{(\int_A \tau_{xy} dA)}_{V} \delta\gamma_{xy} dx \Rightarrow \boxed{\delta U = \int_0^L V\delta\gamma_{xy} dx} \tag{9.62}$$

Flexural Members: With reference to Fig. 9.5.

Figure 9.5: Flexural Member

$$\left.\begin{array}{l}\delta U = \int \sigma_x \delta\varepsilon_x d\Omega \\ M = \int_A \sigma_x y dA \Rightarrow \frac{M}{y} = \int_A \sigma_x dA \\ \delta\phi = \frac{\delta\varepsilon}{y} \Rightarrow \delta\phi y = \delta\varepsilon \\ d\Omega = \int_0^L \int_A dA dx\end{array}\right\} \boxed{\delta U = \int_0^L M \delta\phi dx} \tag{9.63}$$

9.2.4.1.2 Linear Elastic Systems

60 Should we have a linear elastic material ($\sigma = E\varepsilon$) then:

Axial Members:

$$\left.\begin{array}{l}\delta U = \int \sigma\delta\varepsilon d\Omega \\ \sigma_x = E\varepsilon_x = E\frac{du}{dx} \\ \delta\varepsilon = \frac{d(\delta u)}{dx} \\ d\Omega = Adx\end{array}\right\} \boxed{\delta U = \int_0^L \underbrace{E\frac{du}{dx}}_{\text{“}\sigma\text{''}} \underbrace{\frac{d(\delta u)}{dx}}_{\text{“}\delta\varepsilon\text{''}} \underbrace{Adx}_{d\Omega}} \tag{9.64}$$

Torsional Members: With reference to Fig. 9.4

$$\delta U = \int_\Omega \tau_{xy}\delta\gamma_{xy} d\Omega = \int_\Omega \tau_r \delta\beta d\Omega \tag{9.65}$$

For an infinitesimal element:

$$\left.\begin{array}{l}\tau = \frac{Tr}{J} \\ d\delta\theta = \frac{Tdx}{GJ}\end{array}\right\} \tau = Gr\frac{d\delta\theta_x}{dx} \tag{9.66}$$

since the rate of change of rotation is the strain.

$$\left.\begin{array}{l}d\Omega = rd\theta drdx \\ \delta U = \int_0^L\int_0^r\int_0^{2\pi} \underbrace{\left[Gr\frac{d\theta_x}{dx}\right]}_{\text{real}} \underbrace{\left[\frac{d(\delta\theta_x)}{dx}\right]}_{\text{virtual}} rd\theta drdx \\ \int_o^r\int_0^{2\pi} r^2 d\theta dr = J\end{array}\right\} \boxed{\delta U = \int_0^L \underbrace{GJ\frac{d\theta_x}{dx}}_{\text{“}\sigma\text{''}} \underbrace{\frac{d(\delta\theta_x)}{dx}}_{\text{“}\delta\varepsilon\text{''}} dx} \tag{9.67}$$

Flexural Members: With reference to Fig. 9.5.

$$\left.\begin{array}{l}\delta U = \int \sigma_x \delta\varepsilon_x d\Omega \\ \left.\begin{array}{l}\sigma_x = \frac{My}{I_z} \\ M = \frac{d^2v}{dx^2}EI_z\end{array}\right\} \sigma_x = \underbrace{\frac{d^2v}{dx^2}}_{\kappa}Ey \\ \delta\varepsilon_x = \frac{\delta\sigma_x}{E} = \frac{d^2(\delta v)}{dx^2}y \\ d\Omega = dAdx\end{array}\right\} \delta U = \int_0^L \int_A \frac{d^2v}{dx^2}Ey\frac{d^2(\delta v)}{dx^2}ydAdx \tag{9.68}$$

or:

$$\left.\begin{array}{l}\text{Eq. 9.68} \\ \int_A y^2 dA = I_z\end{array}\right\} \boxed{\delta U = \int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{``\sigma"}\underbrace{\frac{d^2(\delta v)}{dx^2}}_{``\delta\varepsilon"}dx} \tag{9.69}$$

9.2.4.2 External Virtual Work δW

61 For concentrated forces (and moments):

$$\boxed{\delta W = \int \delta\Delta q dx + \sum_i (\delta\Delta_i)P_i + \sum_i (\delta\theta_i)M_i} \tag{9.70}$$

where: $\delta\Delta_i$ = virtual displacement.

9.2.5 Complementary Virtual Work

62 We define the complementary virtual work done by the load on a body during a small, admissible (continuous and satisfying the boundary conditions) change in displacements.

$$\text{Complementary Internal Virtual Work} \quad \delta W_i^* \stackrel{\text{def}}{=} -\int_\Omega \boldsymbol{\epsilon}\delta\boldsymbol{\sigma}d\Omega \tag{9.71}$$

$$\text{Complementary External Virtual Work} \quad \delta W_e^* \stackrel{\text{def}}{=} \int_{\Gamma_u} \hat{\mathbf{u}}\delta\mathbf{t}d\Gamma \tag{9.72}$$

9.2.5.1 Internal Complementary Virtual Strain Energy δU^*

63 Again we shall consider two separate cases.

9.2.5.1.1 Arbitrary System

64 In this set of formulation, we derive expressions of the complemetary virtual strain energies which are independent of the material constitutive laws. Thus δU^* will be left in terms of forces and displacements.

Axial Members:

$$\left.\begin{array}{l}\delta U^* = \int_0^L \delta\sigma\varepsilon d\Omega \\ d\Omega = Adx\end{array}\right\} \boxed{\delta U^* = A\int_0^L \delta\sigma\varepsilon dx} \tag{9.73}$$

Torsional Members: With reference to Fig. 9.4

$$\left.\begin{array}{l}\delta U^* = \int_\Omega \delta\tau_{xy}\gamma_{xy} d\Omega \\ \delta T = \int_A \delta\tau_{xy} r dA \\ \gamma_{xy} = r\theta \\ d\Omega = dAdx\end{array}\right\} \delta U^* = \int_0^L \underbrace{\left(\int_A \delta\tau_{xy} r dA\right)}_{\delta T}\theta dx \Rightarrow \boxed{\delta U^* = \int_0^L \delta T\theta dx} \tag{9.74}$$

Shear Members:

$$\left.\begin{array}{l}\delta U^* = \int_\Omega \delta\tau_{xy}\gamma_{xy} d\Omega \\ \delta V = \int_A \delta\tau_{xy} dA \\ d\Omega = dAdx\end{array}\right\} \delta U^* = \int_0^L \underbrace{\left(\int_A \delta\tau_{xy} dA\right)}_{\delta V}\gamma_{xy} dx \Rightarrow \boxed{\delta U^* = \int_0^L \delta V\gamma_{xy} dx} \tag{9.75}$$

Flexural Members: With reference to Fig. 9.5.

$$\left.\begin{array}{l}\delta U^* = \int \delta\sigma_x\varepsilon_x d\Omega \\ \delta M = \int_A \delta\sigma_x y dA \Rightarrow \dfrac{\delta M}{y} = \int_A \delta\sigma_x dA \\ \phi = \frac{\varepsilon}{y} \Rightarrow \phi y = \varepsilon \\ d\Omega = \int_0^L \int_A dAdx\end{array}\right\} \boxed{\delta U^* = \int_0^L \delta M\phi dx} \tag{9.76}$$

9.2.5.1.2 Linear Elastic Systems

65 Should we have a linear elastic material ($\sigma = E\varepsilon$) then:

Axial Members:

$$\left.\begin{array}{l}\delta U^* = \int_\Omega \varepsilon\delta\sigma d\Omega \\ \delta\sigma = \frac{\delta P}{A} \\ \varepsilon = \frac{P}{AE} \\ d\Omega = Adx\end{array}\right\} \boxed{\delta U^* = \int_0^L \underbrace{\delta P}_{\text{“}\delta\sigma\text{''}} \underbrace{\frac{P}{AE}}_{\text{“}\varepsilon\text{''}} dx} \tag{9.77}$$

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Torsional Members:

$$\left.\begin{array}{l} \delta U^* = \int_\Omega \varepsilon \underbrace{E\delta\varepsilon}_{\delta\sigma} d\text{vol} \\ \delta U^* = \int_\Omega \delta\tau_{xy} \underbrace{G\tau_{xy}}_{\gamma_{xy}} d\text{vol} \\ \tau_{xy} = \frac{Tr}{J} \\ \gamma_{xy} = \frac{\tau_{xy}}{G} \\ d\Omega = rd\theta dr dx \\ \int_o^r \int_0^{2\pi} r^2 d\theta dr = J \end{array}\right\} \boxed{\delta U^* = \int_0^L \underbrace{\delta T}_{\text{“}\delta\sigma\text{''}} \underbrace{\frac{T}{GJ}}_{\text{“}\varepsilon\text{''}} dx} \tag{9.78}$$

Flexural Members:

$$\left.\begin{array}{l} \delta U^* = \int_\Omega \varepsilon \underbrace{E\delta\varepsilon}_{\delta\sigma} \\ \sigma_x = \frac{M_z y}{I_z} \\ \varepsilon = \frac{M_z y}{EI_z} \\ d\Omega = dAdx \\ \int_A y^2 dA = I_z \end{array}\right\} \boxed{\delta U^* = \int_0^L \underbrace{\delta M}_{\text{“}\delta\sigma\text{''}} \underbrace{\frac{M}{EI_z}}_{\text{“}\varepsilon\text{''}} dx} \tag{9.79}$$

9.2.5.2 External Complementary Virtual Work δW^*

66 For concentrated forces (and moments):

$$\delta W^* = \sum_i (\Delta_i)\delta P_i \tag{9.80}$$

where: Δ_i = actual displacement. Or:

$$\delta W^* = \sum_i (\theta_i)\delta M_i \tag{9.81}$$

67 For distributed load:

$$\boxed{\delta W^* = \int \Delta\delta q dx + \sum_{i=1}^n (\Delta_i)\delta P_i + \sum_{i=1}^n (\theta_i)\delta M_i} \tag{9.82}$$

9.2.6 Potential Energy

9.2.6.1 Potential Functions

68 If during loading and unloading, U and U^* are independent of the path of deformation (i.e. no intial strains), but depend only on the initial and final states, then the differential dU_0 and dU_0^* are exact differentials and U_0 and U_0^* are then potential functions.

9.2.6.2 Potential of External Work

69 The potential of external work W in an arbitrary system is defined as

$$\boxed{\mathcal{W}_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T \mathbf{b} \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{u}^T \hat{\mathbf{t}} \mathrm{d}\Gamma + \mathbf{u}\mathbf{P}} \tag{9.83}$$

where $\mathbf{u}$ are the displacements, $\mathbf{b}$ is the body force vector; $\hat{\mathbf{t}}$ is the applied surface traction vector; Γ_t is that portion of the boundary where $\hat{\mathbf{t}}$ is applied, and $\mathbf{P}$ are the applied nodal forces.

70 Note that the potential of the external work is different from the external work itself (usually by a factor of 1/2)

9.2.6.3 Potential Energy

71 The potential energy of a system is defined as

$$\Pi \stackrel{\text{def}}{=} U - \mathcal{W}_e \tag{9.84}$$

$$= \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathbf{u}\mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}\hat{\mathbf{t}} d\Gamma + \mathbf{u}\mathbf{P} \right) \tag{9.85}$$

72 Note that in the potential the full load is always acting, and through the displacements of its points of application it does work but loses an equivalent amount of potential, this explains the negative sign.

9.3 Principle of Virtual Work and Complementary Virtual Work

73 The principles of Virtual Work and Complementary Virtual Work relate *force* systems which satisfy the requirements of *equilibrium*, and *deformation* systems which satisfy the requirement of *compatibility*:

1. In any application the force system could either be the actual set of *external* loads $d\mathbf{p}$ or some *virtual* force system which happens to satisfy the condition of *equilibrium* $\delta\overline{\mathbf{p}}$. This set of external forces will induce internal actual forces $d\boldsymbol{\sigma}$ or internal hypothetical forces $\delta\overline{\boldsymbol{\sigma}}$ compatible with the externally applied load.
2. Similarly the deformation could consist of either the actual joint deflections $d\mathbf{u}$ and compatible internal deformations $d\boldsymbol{\epsilon}$ of the structure, or some *hypothetical* external and internal deformation $\delta\overline{\mathbf{u}}$ and $\delta\overline{\boldsymbol{\varepsilon}}$ which satisfy the conditions of *compatibility.*

74 Thus we may have 4 possible combinations, Table 9.2: where: d corresponds to the actual, and δ (with an overbar) to the hypothetical values.

This table calls for the following observations

	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$d\mathbf{u}$	$d\boldsymbol{\epsilon}$		
2	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$d\mathbf{u}$	$d\boldsymbol{\epsilon}$	δU^*	CVW/Flexibility
3	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$	δU	VW/Stiffness
4	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$		

Table 9.2: Possible Combinations of Real and Hypothetical Formulations

1. The second approach is the same one on which the method of virtual or unit load is based. It is simpler to use than the third as a internal force distribution compatible with the assumed virtual force can be easily obtained for statically determinate structures. This approach will yield exact solutions for staticaally determinate structures.
2. The third approach is favoured for kinematically indeterminate problems or in conjunction with approximate solution. It requires a proper "guess" of a displacement shape and is the basis of the stifness method.

9.3.1 Principle of Virtual Work

9.3.1.1 † Derivation

75 Derivation of the principle of virtual work starts with the assumption of that forces are in equilibrium and satisfaction of the static boundary conditions.

76 The Equation of equilibrium (Eq. 8.11) which is rewritten as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \tag{9.86}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + b_y = 0 \tag{9.87}$$

where $\mathbf{b}$ representing the body force. In matrix form, this can be rewritten as

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = 0 \tag{9.88}$$

or

$$\boxed{\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = 0} \tag{9.89}$$

Note that this equation can be generalized to 3D.

77 The surface Γ of the solid can be decomposed into two parts Γ_t and Γ_u where tractions and displacements are respectively specified.

$$\Gamma = \Gamma_t + \Gamma_u \tag{9.90-a}$$
$$\mathbf{t} = \hat{\mathbf{t}} \quad \text{on } \Gamma_t \quad \text{Natural B.C.} \tag{9.90-b}$$
$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u \quad \text{Essential B.C.} \tag{9.90-c}$$

Equations 9.89 and 9.90-b constitute a statically admissible stress field.

78 We now express the local condition of equilibrium Eq. 9.89 and the static boundary condition Eq, 9.90-b in global (or integral) form. This is accomplished by multiplying both equations by a virtual displacement $\delta\mathbf{u}$ and integrating the first equation over Ω and the second one over Γ_t, and we then take the sum of these two integrals (each of which must be equal to zero)

$$-\int_\Omega \delta\mathbf{u}^T \left(\mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b}\right) d\Omega + \int_{\Gamma_t} \delta\mathbf{u}^T(\mathbf{t} - \hat{\mathbf{t}})d\Gamma = 0 \tag{9.91}$$

Note that since each term is equal to zero, the negative sign is introduced to maintain later on consistency with previous results. Furthermore, according to the fundamental lemma of the calculus of variation (Eq. 9.13), this equation is still equivalent to Eq. 9.89 and 9.89

79 Next, we will focus our attention on $\int_{\Gamma_t}\delta\mathbf{u}^T\mathbf{t}d\Gamma$ which will be replaced (from Eq. 9.90-a) by

$$\int_{\Gamma_t}\delta\mathbf{u}^T\mathbf{t}d\Gamma = \int_\Gamma \delta\mathbf{u}^T\mathbf{t}d\Gamma - \int_{\Gamma_u}\delta\mathbf{u}^T\mathbf{t}d\Gamma \tag{9.92}$$

and which we seek to convert into a volume integral through Gauss Theorem, Eq. 5.6 and 5.7.

80 But first let us recall the definition of the traction vector

$$\mathbf{t} = \boldsymbol{\sigma}.\mathbf{n} \qquad \text{or} \quad t_i = \sigma_{ij}n_j \tag{9.93-a}$$

applying Gauss theorem we obtain

$$\int_\Gamma \delta\mathbf{u}^T\mathbf{t}d\Gamma = \int_\Gamma (\delta\mathbf{u}^T\boldsymbol{\sigma})\mathbf{n}d\Gamma = \int_\Omega \operatorname{div}(\delta\mathbf{u}^T\boldsymbol{\sigma})d\Omega \tag{9.94-a}$$
$$= \int_\Omega \operatorname{div}\delta\mathbf{u}^T\boldsymbol{\sigma}d\Omega + \int_\Omega \delta\mathbf{u}^T\operatorname{div}\boldsymbol{\sigma}d\Omega \tag{9.94-b}$$

However, from Eq. 9.89 we have $\operatorname{div}\boldsymbol{\sigma} = \mathbf{L}^T\boldsymbol{\sigma}$ thus

$$\int_\Gamma \delta\mathbf{u}^T\mathbf{t}d\Gamma = \int_\Omega \operatorname{div}\delta\mathbf{u}^T\boldsymbol{\sigma}d\Omega + \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T\boldsymbol{\sigma}d\Omega \tag{9.95}$$

81 Combining Eq. 9.92 with the previous equation, leads to

$$\int_{\Gamma_t}\delta\mathbf{u}^T\mathbf{t}d\Gamma = \int_\Omega \operatorname{div}\delta\mathbf{u}^T\boldsymbol{\sigma}d\Omega + \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T\boldsymbol{\sigma}d\Omega - \int_{\Gamma_u}\delta\mathbf{u}^T\mathbf{t}d\Gamma \tag{9.96}$$

82 We next substitute this last equation into Eq. 9.91 and reduce

$$
\begin{aligned}
-\int_\Omega \delta\mathbf{u}^T\mathbf{L}^T\boldsymbol{\sigma}d\Omega - \int_\Omega \delta\mathbf{u}^T\mathbf{b}d\Omega + \int_\Omega \text{div}\delta\mathbf{u}^T\boldsymbol{\sigma}d\Omega + \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T\boldsymbol{\sigma}d\Omega \\
- \int_{\Gamma_u} \delta\mathbf{u}^T\mathbf{t}d\Gamma - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}d\Gamma &= 0 \qquad (9.97\text{-a}) \\
-\int_\Omega \delta\mathbf{u}^T\mathbf{b}d\Omega + \int_\Omega \text{div}\delta\mathbf{u}^T\boldsymbol{\sigma}d\Omega - \int_{\Gamma_u} \delta\mathbf{u}^T\mathbf{t}d\Gamma - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}d\Gamma &= 0 \qquad (9.97\text{-b})
\end{aligned}
$$

83 The strain displacement relation can be written as

$$\text{div}\delta\mathbf{u} = \delta\boldsymbol{\epsilon} \tag{9.98}$$

Substituting in the previous equation, we get

$$-\int_\Omega \delta\mathbf{u}^T\mathbf{b}d\Omega + \int_\Omega \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}d\Omega - \int_{\Gamma_u} \delta\mathbf{u}^T\mathbf{t}d\Gamma - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}d\Gamma = 0 \tag{9.99}$$

84 Virtual displacement must be kinematically admissible, i.e. $\delta\mathbf{u}$ must satisfy the essential boundary conditions $\mathbf{u} = 0$ on Γ_u, (note that the exact solution had to satisfy the natural B.C. instead), hence the previous equation reduces to

$$\underbrace{\int_\Omega \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}d\Omega}_{-\delta W_i=\delta U_i} \underbrace{- \int_\Omega \delta\mathbf{u}^T\mathbf{b}d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}d\Gamma}_{-\delta W_e} = 0 \tag{9.100}$$

Each of the preceding equations is a work expression, (Eq. 9.59). The first one corresponds to the internal virtual work, and the last two are expressions of the work done by the body forces and the surface tractions through the corresponding virtual displacement $\delta\mathbf{u}$, hence

$$\boxed{-\delta W_i = \delta W_e} \tag{9.101}$$

or

$$\boxed{\delta U_i = \delta W_e} \tag{9.102}$$

which is the expression of the *principle of virtual work* (or more specifically of virtual displacement) which can be stated as

> A deformable system is in equilibrium if the sum of the external virtual work and the internal virtual work is zero for virtual displacements $\delta\mathbf{u}$ which are kinematically admissible.

The major governing equations are summarized

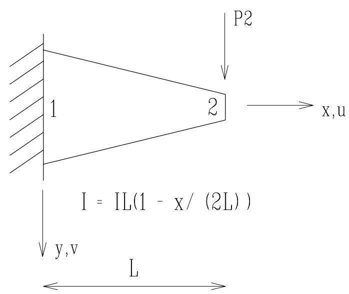

Figure 9.6: Tapered Cantilivered Beam Analysed by the Vitual Displacement Method

$$\underbrace{\int_\Omega \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}d\Omega}_{-\delta W_i} - \underbrace{\int_\Omega \delta\mathbf{u}^T\mathbf{b}d\Omega - \int_{\Gamma_t}\delta\mathbf{u}^T\hat{\mathbf{t}}d\Gamma}_{-\delta W_e} = 0 \quad (9.103)$$

$$\delta\boldsymbol{\epsilon} = \mathbf{L}\delta\mathbf{u} \quad \text{in} \quad \Omega \quad (9.104)$$

$$\delta\mathbf{u} = 0 \quad \text{on} \quad \Gamma_u \quad (9.105)$$

85 Note that the principle is independent of material properties, and that the primary unknowns are the displacements.

86 For one dimensional elements, with no initial strains ($U = -W_i$)

$$\boxed{\underbrace{\int \sigma\delta\varepsilon d\Omega}_{\delta U} = \underbrace{P\delta v}_{\delta W}} \tag{9.106}$$

■ Example 9-3: Tapered Cantiliver Beam, Virtual Displacement

Analyse the problem shown in Fig. 9.6, by the virtual displacement method.

Solution:

For this flexural problem, we must apply the expression of the virtual internal strain energy as derived for beams in Eq. 9.69. And the solutions must be expressed in terms of the displacements which in turn must satisfy the essential boundary conditions.

The *approximate* solutions proposed to this problem are

$$v = \left(1 - \cos\frac{\pi x}{2l}\right)v_2 \tag{9.107}$$

$$v = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]v_2 \tag{9.108}$$

Note that these equations do indeed satisfy the essential B.C.

Using the virtual displacement method we evaluate the displacements v_2 from three different combination of virtual and actual displacement:

Solution	Total	Virtual
1	Eqn. 9.107	Eqn. 9.108
2	Eqn. 9.107	Eqn. 9.107
3	Eqn. 9.108	Eqn. 9.108

Where actual and virtual values for the two assumed displacement fields are given below.

	Trigonometric (Eqn. 9.107)	Polynomial (Eqn. 9.108)
v	$\left(1-\cos\frac{\pi x}{2l}\right)v_2$	$\boxed{3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3}\ v_2$
δv	$\left(1-\cos\frac{\pi x}{2l}\right)\delta v_2$	$\boxed{3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3}\ \delta v_2$
v''	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2l}v_2$	$\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2$
$\delta v''$	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2l}\delta v_2$	$\boxed{\frac{6}{L^2}-\frac{12x}{L^3}}\ \delta v_2$

Note that both Eqn. 9.107 and Eqn. 9.108 satisfy the essential (geometric) B.C.

$$\delta U = \int_0^L \delta v'' EI_z v'' dx \tag{9.109}$$

$$\delta W = P_2 \delta v_2 \tag{9.110}$$

Solution 1:

$$\begin{aligned}\delta U &= \int_0^L \frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2l}\right)v_2\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)\delta v_2 EI_1\left(1-\frac{x}{2L}\right)dx \\ &= \frac{3\pi EI_1}{2L^3}\left[1-\frac{10}{\pi}+\frac{16}{\pi^2}\right]v_2\delta v_2 \\ &= P_2\delta v_2\end{aligned} \tag{9.111}$$

which yields:

$$v_2 = \frac{P_2 L^3}{2.648 EI_1} \tag{9.112}$$

Solution 2:

$$\begin{aligned}\delta U &= \int_0^L \frac{\pi^4}{16L^4}\cos^2\left(\frac{\pi x}{2l}\right)v_2\delta v_2 EI_1\left(1-\frac{x}{2l}\right)dx \\ &= \frac{\pi^4 EI_1}{32L^3}\left(\frac{3}{4}+\frac{1}{\pi^2}\right)v_2\delta v_2 \\ &= P_2\delta v_2\end{aligned} \tag{9.113}$$

which yields:

$$v_2 = \frac{P_2 L^3}{2.57 EI_1} \tag{9.114}$$

Solution 3:

$$\begin{aligned}\delta U &= \int_0^L \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)^2\left(1-\frac{x}{2l}\right)EI_1\delta v_2 v_2 dx \\ &= \frac{9EI}{L^3}v_2\delta v_2 \\ &= P_2\delta v_2\end{aligned} \tag{9.115}$$

which yields:

$$v_2 = \frac{P_2L^3}{9EI} \tag{9.116}$$

■

9.3.2 Principle of Complementary Virtual Work

9.3.2.1 † Derivation

87 Derivation of the principle of complementary virtual work starts from the assumption of a *kinematicaly admissible displacements* and satisfaction of the essential boundary conditions.

88 Whereas we have previously used the vector notation for the principle of virtual work, we will now use the tensor notation for this derivation.

89 The kinematic condition (strain-displacement) was given in Eq. 8.5.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{9.117}$$

90 The essential boundary conditions are expressed as

$$u_i = \hat{u} \quad \text{on} \quad \Gamma_u \tag{9.118}$$

Those two equations are rewritten as

$$\varepsilon_{ij} - u_{i,j} = 0 \tag{9.119-a}$$

$$u_i - \hat{u} = 0 \tag{9.119-b}$$

We premultiply the first equation by a virtual stress field $\partial\sigma_{ij}$ and integrate over the volume; and we premultiply the second by corresponding virtual tractions δt_i and integrate over the corresonding surface

$$\int_\Omega \left(\varepsilon_{ij}-u_{i,j}\right)\delta\sigma_{ij}d\Omega - \int_{\Gamma_u}\left(u_i-\hat{u}\right)\delta t_i d\Gamma = 0 \tag{9.120}$$

Note that since each term is equal to zero, the negative sign is introduced to maintain later on consistency with previous results. Furthermore, according to the fundamental lemma of the

calculus of variation (Eq. 9.13), this equation is still equivalent to the kinematic conditions 9.117 and 9.118.

91 Since the arbitrary stresses must be statically admissible, it follows that they must satisfy the equation of equilibrium

92 Next, we will focus our attention on $\int_{\Gamma_u} u_i \delta t_i d\Gamma$ which will be replaced (from Eq. 9.90-a) by

$$\int_{\Gamma_u} u_i \delta t_i d\Gamma = \int_{\Gamma} u_i \delta t_i d\Gamma - \int_{\Gamma_t} u_i \delta t_i d\Gamma \tag{9.121}$$

and note that the second term on the right hand side is zero since

$$\delta t_i = 0 \quad \text{on} \quad \Gamma_t \tag{9.122}$$

in order to satisfy the boundary conditions.

93 We now seek seek to convert the previous expression into a volume integral through Gauss Theorem, Eq. 5.6 and 5.7.

$$\int_{\Gamma_u} u_i \delta t_i d\Gamma = \int_{\Gamma} u_i \delta t_i d\Gamma = \int_{\Gamma} u_i \delta(\sigma_{ij} n_j) d\Gamma \tag{9.123-a}$$

$$= \int_{\Omega} u_i \delta \sigma_{ij,j} d\Omega + \int_{\Omega} u_{i,j} \delta \sigma_{ij} d\Omega \tag{9.123-b}$$

However,the virtual stresses must be in equilibrium within Ω, thus from Eq. 8.11, and in the absence of body forces

$$\delta \sigma_{ij,j} = 0 \;\; \text{in} \;\; \Omega \tag{9.124}$$

thus

$$\int_{\Gamma_u} u_i \delta t_i d\Gamma = \int_{\Omega} u_{i,j} \delta \sigma_{ij} d\Omega \tag{9.125}$$

94 Combinig this last equation with Eq. 9.120 leads to

$$\int_{\Omega} \varepsilon_{ij} \delta \sigma_{ij} d\Omega - \int_{\Omega} u_{i,j} \delta \sigma_{ij} d\Omega - \int_{\Gamma_u} \hat{u} \delta t_i d\Gamma + \int_{\Omega} u_{i,j} \delta \sigma_{ij} d\Omega = 0 \tag{9.126}$$

which simplifies into

$$\int_{\Omega} \varepsilon_{ij} \delta \sigma_{ij} d\Omega - \int_{\Gamma_u} \hat{u} \delta t_i d\Gamma = 0 \tag{9.127}$$

95 We note that each of the preceding term is a work expression, and that the first one corresponds to the internal complementary virtualwork, and the scond to the external complementary virtual work, Eq. 9.72

$$\boxed{-\delta W_i^* - \delta W_e^* = 0} \tag{9.128}$$

which is the expression of the *principle of virtual complementary work* (or more specifically of virtual force) which can be stated as

A deformable system satisfies all kinematical requirements if the sum of the external complementary virtual work and the internal complementary virtual work is zero for all statically admissible virtual stresses $\delta\sigma_{ij}$.

The major governing equations are summarized

$$\boxed{\begin{array}{rcll} \underbrace{\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}d\Omega}_{-\delta W_i^*} - \underbrace{\int_{\Gamma_u} \hat{u}_i\delta t_i d\Gamma}_{\delta W_e^*} & = & 0 & (9.129)\\ \delta\sigma_{ij,j} = 0 & \text{in} & \Omega & (9.130)\\ \delta t_i = 0 & \text{on} & \Gamma_t & (9.131) \end{array}}$$

96 Note that the principle is independent of material properties, and that the primary unknowns are the stresses.

97 The principle of virtual forces leads to the flexibility matrix.

■ Example 9-4: Tapered Cantilivered Beam; Virtual Force

"Exact" solution of previous problem using principle of virtual work with virtual force.

$$\boxed{\underbrace{\int_0^L \delta M \frac{M}{EI_z}dx}_{\text{Internal}} = \underbrace{\delta P\Delta}_{\text{External}}} \tag{9.132}$$

Note: This represents the internal virtual strain energy and external virtual work written in terms of *forces* and should be compared with the similar expression derived in Eq. 9.69 written in terms of displacements:

$$\delta U^* = \int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\sigma}\underbrace{\frac{d^2(\delta v)}{dx^2}}_{\delta\varepsilon}dx \tag{9.133}$$

Here: δM and δP are the virtual forces, and $\frac{M}{EI_z}$ and Δ are the actual displacements. See Fig. 9.7 If $\delta P = 1$, then $\delta M = x$ and $M = P_2x$ or:

$$\begin{aligned} (1)\Delta &= \int_0^L x\frac{P_2x}{EI_1(.5+\frac{x}{L})}dx\\ &= \frac{P_2}{EI_1}\int_0^L \frac{x^2}{\frac{L+x}{2l}}dx\\ &= \frac{P_22L}{EI_1}\int_0^L \frac{x^2}{L+x}dx \end{aligned} \tag{9.134}$$

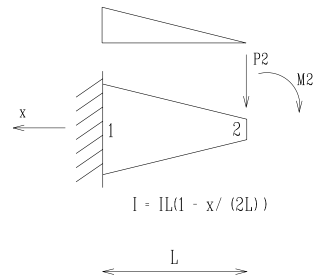

Figure 9.7: Tapered Cantilevered Beam Analysed by the Virtual Force Method

From *Mathematica* we note that:

$$\int_0^0 \frac{x^2}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln(a+bx)\right] \tag{9.135}$$

Thus substituting $a = L$ and $b = 1$ into Eqn. 9.135, we obtain:

$$\begin{aligned} \Delta &= \frac{2P_2L}{EI_1}\left[\frac{1}{2}(L+x)^2 - 2L(L+x) + L^2\ln(L+x)\right]\Big|_0^L \\ &= \frac{2P_2L}{EI_1}\left[2L^2 - 4L^2 + L^2\ln 2L - \frac{L^2}{2} + 2L^2 + L^2\log L\right] \\ &= \frac{2P_2L}{EI_1}\left[L^2(\ln 2 - \frac{1}{2})\right] \\ &= \frac{P_2L^3}{2.5887EI_1} \end{aligned} \tag{9.136}$$

This exact value should be compared with the approximate one obtained with the Virtual Displacement method in which a displacement field was *assumed* in Eq. 9.215 of $\frac{PL^3}{2.55EI_1}$.

Similarly:

$$\begin{aligned} \theta &= \int_0^L \frac{M(1)}{EI_1\left(.5 + \frac{x}{L}\right)} \\ &= \frac{2ML}{EI_1}\int_0^L \frac{1}{L+x} \\ &= \frac{2ML}{EI_1}\ln(L+x)\ |_0^L \\ &= \frac{2ML}{EI_1}(\ln 2L - \ln L) \\ &= \frac{2ML}{EI_1}\ln 2 \\ &= \frac{ML}{.721EI_1} \end{aligned} \tag{9.137}$$

■

■ Example 9-5: Three Hinged Semi-Circular Arch

We seek to determine the vertical deflection of the crown of the three hinged statically determined semi-circular arch under its own dead weight w. Fig. 9.8 We first seek to determine the analytical expression of the moment diagram. From statics, it can be shown that the vertical and horizontal reactions are $R_v = \frac{\pi}{2}wR$ and $R_h = \left(\frac{\pi}{2}-1\right)wR$.

Next considering the free body diagram of the arch, and summing the forces in the radial direction ($\Sigma F_R = 0$):

$$-\left(\tfrac{\pi}{2}-1\right)wR\cos\theta + \tfrac{\pi}{2}wR\sin\theta - \int_{\alpha=0}^{\theta} wRd\alpha\sin\theta + V = 0 \tag{9.138}$$

$$V = wR\left[\left(\tfrac{\pi}{2}-1\right)\cos\theta + \left(\theta-\tfrac{\pi}{2}\right)\sin\theta\right] \tag{9.139}$$

Similarly, if we consider the summation of forces in the axial direction ($\Sigma F_T = 0$):

$$\left(\tfrac{\pi}{2}-1\right)wR\sin\theta + \tfrac{\pi}{2}wR\cos\theta - \int_{\alpha=0}^{\theta} wRd\alpha\cos\theta + N = 0 \tag{9.140}$$

$$N = wR\left[\left(\theta-\tfrac{\pi}{2}\right)\cos\theta - \left(\tfrac{\pi}{2}-1\right)\sin\theta\right] \tag{9.141}$$

Now we can consider the third equation of equilibrium ($\Sigma M_\theta = 0$):

$$\begin{aligned}&\left(\tfrac{\pi}{2}-1\right)wR\cdot R\sin\theta - \tfrac{\pi}{2}wR^2\left(1-\cos\theta\right) + \\ &\int_{\alpha=0}^{\theta} wRd\alpha\cdot R(\cos\alpha-\cos\theta) + M = 0\end{aligned} \tag{9.142}$$

$$M = wR^2\left[\tfrac{\pi}{2}(1-\sin\theta) + \left(\theta-\tfrac{\pi}{2}\right)\cos\theta\right] \tag{9.143}$$

The real curvature ϕ is obtained by deviding the moment by EI

$$\phi = \frac{wR^2}{EI}\left[\frac{\pi}{2}(1-\sin\theta) + \left(\theta-\frac{\pi}{2}\right)\cos\theta\right] \tag{9.144}$$

The virtual force δP will be aa unit vertical point in the direction of the desired deflection, causing a virtual internal moment

$$\delta M = \frac{R}{2}\left[1-\cos\theta-\sin\theta\right] \qquad 0 \le \theta \le \frac{\pi}{2} \tag{9.145}$$

Hence, application of the virtual work equation yields:

$$\begin{aligned}\underbrace{1}_{\delta P}\cdot\Delta &= 2\int_{\theta=0}^{\frac{\pi}{2}} \underbrace{\frac{wR^2}{EI}\left[\frac{\pi}{2}(1-\sin\theta) + \left(\theta-\frac{\pi}{2}\right)\cos\theta\right]}_{\phi}\cdot\underbrace{\frac{R}{2}\cdot\left[1-\cos\theta-\sin\theta\right]}_{\delta M}\underbrace{Rd\theta}_{dx} \\ &= \frac{wR^4}{16EI}\left[7\pi^2-18\pi-12\right] \\ &= .0337\frac{wR^4}{EI}\end{aligned} \tag{9.146}$$

■

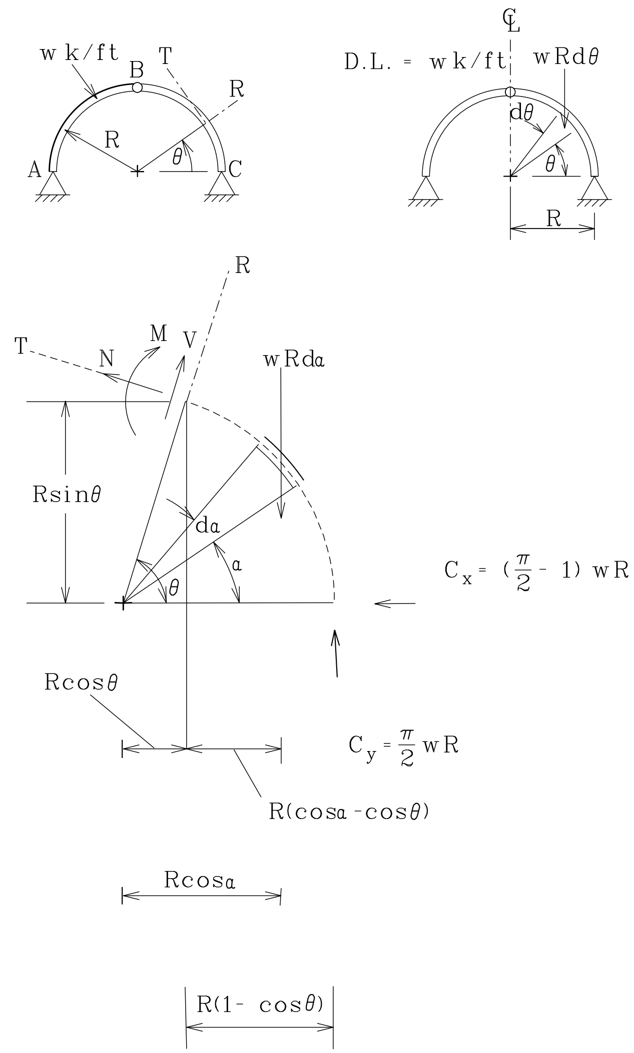

Figure 9.8: Three Hinge Semi-Circular Arch

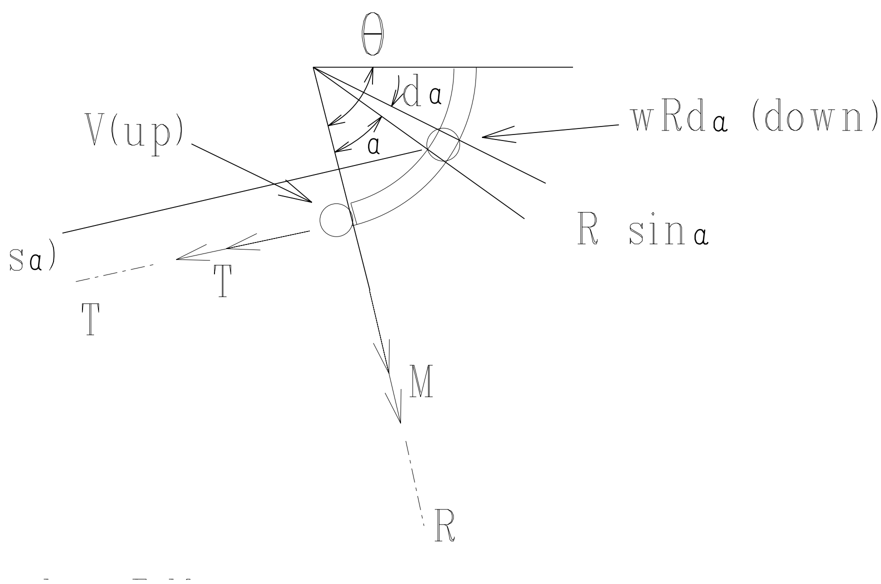

Figure 9.9: Semi-Circular Cantilevered Box Girder

■ Example 9-6: Cantilivered Semi-Circular Bow Girder

Considering the semi-circular cantilevered box girder shown in Fig. 9.9 subjected to its own weight w, and with a rectangular cross-section of width b and height $d = 2b$ and with Poisson's ratio $\nu = 0.3$.

First, we determine the internal forces by applying the three applicable equations of equilibrium:

$$\begin{array}{lll} \Sigma F_Z = 0 & V - \int_0^\theta wRd\alpha = 0 & V = wr\theta \\ \Sigma M_R = 0 & M - \int_0^\theta (wRd\alpha)(R\sin\alpha) = 0 & M = wR^2(1-\cos\theta) \\ \Sigma M_T = 0 & +\int_0^\theta (wRd\alpha)R(1-\cos\alpha) = 0 & T = -wR^2(\theta - \sin\theta) \end{array} \tag{9.147}$$

Noting that the member will be subjected to both flexural and torsional deformations, we seek to determine the two stiffnesses.

The flexural stiffness EI is given by $EI = E\frac{bd^3}{12} = E\frac{b(2b)^3}{12} = \frac{2Eb^4}{3} = .667Eb^4$.

The torsional stiffness of solid rectangular sections $J = kb^3d$ where b is the shorter side of the section, d the longer, and k a factor equal to .229 for $\frac{d}{b} = 2$. Hence $G = \frac{E}{2(1+\nu)} = \frac{E}{2(1+.3)} = .385E$, and $GJ = (.385E)(.229b^4) = .176Eb^4$.

Considering both flexural and torsional deformations, and replacing dx by $rd\theta$:

$$\underbrace{\delta P\Delta}_{\delta W^*} = \underbrace{\underbrace{\int_0^\pi \delta M\frac{M}{EI_z}Rd\theta}_{\text{flexural}} + \underbrace{\int_0^\pi \delta T\frac{T}{GJ}Rd\theta}_{\text{torsional}}}_{\delta U^*} \tag{9.148}$$

where the real moments were given above.

Assuming a unit virtual downward force $\delta P = 1$, we have

$$\delta M = R\sin\theta \tag{9.149}$$

$$\delta T = -R(1-\cos\theta) \tag{9.150}$$

Substituting these expression into Eq. 9.148

$$\begin{aligned}
\underbrace{1}_{\delta P}\Delta &= \frac{wR^2}{EI}\int_0^\pi \underbrace{(R\sin\theta)}_{M}\underbrace{(1-\cos\theta)}_{\delta M}Rd\theta + \frac{wR^2}{GJ}\int_0^\pi \underbrace{(\theta-\sin\theta)}_{\delta T}\underbrace{R(1-\cos\theta)}_{T}Rd\theta \\
&= \frac{wR^4}{EI}\int_0^\pi \left[(\sin\theta - \sin\theta\cos\theta) + \frac{1}{.265}(\theta - \theta\cos\theta - \sin\theta + \sin\theta\cos\theta)\right]d\theta \\
&= \frac{wR^4}{EI}(\underbrace{2.}_{\text{flexure}} + \underbrace{18.56}_{\text{torsion}}) \\
&= 20.56\frac{wR^4}{EI}
\end{aligned} \tag{9.151}$$

■

9.4 Potential Energy

9.4.1 Derivation

98 From section 9.2.6.1, if U_0 is a potential function, we take its differential

$$dU_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}}d\varepsilon_{ij} \tag{9.152-a}$$

$$dU_0^* = \frac{\partial U_0}{\partial \sigma_{ij}}d\sigma_{ij} \tag{9.152-b}$$

99 However, from Eq. 9.33

$$U_0 = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \quad (9.153\text{-a})$$
$$dU_0 = \sigma_{ij} d\varepsilon_{ij} \quad (9.153\text{-b})$$

thus,

$$\boxed{\begin{aligned} \frac{\partial U_0}{\partial \varepsilon_{ij}} &= \sigma_{ij} \quad (9.154) \\ \frac{\partial U_0^*}{\partial \sigma_{ij}} &= \varepsilon_{ij} \quad (9.155) \end{aligned}}$$

100 We now define the variation of the strain energy density at a point[2]

$$\delta U_0 = \frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} = \sigma_{ij}\delta\varepsilon_{ij} \quad (9.156)$$

101 The principle of virtual work, Eq. 9.103, $\int_\Omega \varepsilon_{ij}\sigma_{ij} d\Omega - \int_\Omega \delta u_i b_i d\Omega - \int_{\Gamma_t} \delta u_i \hat{t}_i d\Gamma = 0$ can be rewritten as

$$\int_\Omega \delta U_0 d\Omega - \int_\Omega \delta u_i b_i d\Omega - \int_{\Gamma_t} \delta u_i \hat{t}_i d\Gamma = 0 \quad (9.157)$$

102 If nor the surface tractions, nor the body forces alter their magnitudes or directions during deformation, the previous equation can be rewritten as

$$\delta \left[\int_\Omega U_0 d\Omega - \int_\Omega u_i b_i d\Omega - \int_{\Gamma_t} u_i \hat{t}_i d\Gamma \right] = 0 \quad (9.158)$$

103 Hence, comparing this last equation, with Eq. 9.85 we obtain

$$\boxed{\begin{aligned} \delta\Pi &= 0 && (9.159) \\ \Pi &\stackrel{\text{def}}{=} U - \mathcal{W}_e && (9.160) \\ &= \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathbf{ub} d\Omega + \int_{\Gamma_t} \mathbf{u\hat{t}} d\Gamma + \mathbf{uP} \right) && (9.161) \end{aligned}}$$

104 We have thus derived the principle of stationary value of the potential energy:

> Of all kinematically admissible deformations (displacements satisfying the essential boundary conditions), the actual deformations (those which correspond to stresses which satisfy equilibrium) are the ones for which the total potential energy assumes a stationalry value.

[2] Note that the variation of strain energy density is, $\delta U_0 = \sigma_{ij}\delta\varepsilon_{ij}$, and the variation of the strain energy itself is $\delta U = \int_\Omega \delta U_0 d\Omega$.

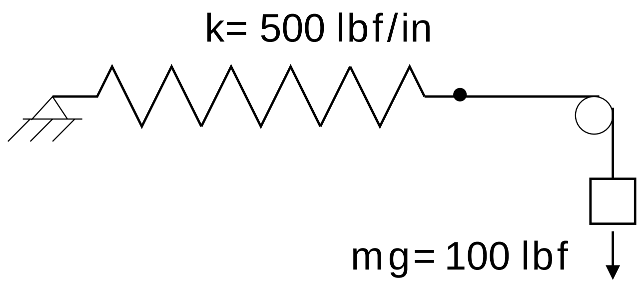

Figure 9.10: Single DOF Example for Potential Energy

105 For problems involving multiple degrees of freedom, it results from calculus that

$$\boxed{\delta\Pi = \frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1 + \frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2 + \ldots + \frac{\partial\Pi}{\partial\Delta_n}\delta\Delta_n} \tag{9.162}$$

106 It can be shown that the minimum potential energy yields a *lower bound* prediction of displacements.

107 As an illustrative example (adapted from Willam, 1987), let us consider the single dof system shown in Fig. 9.10. The strain energy U and potential of the external work $\mathcal{W}$ are given by

$$U = \frac{1}{2}u(Ku) = 250u^2 \tag{9.163-a}$$

$$\mathcal{W}_e = mgu = 100u \tag{9.163-b}$$

Thus the total potential energy is given by

$$\Pi = 250u^2 - 100u \tag{9.164}$$

and will be stationary for

$$\partial\Pi = \frac{d\Pi}{du} = 0 \Rightarrow 500u - 100 = 0 \Rightarrow \boxed{u = 0.2 \text{ in}} \tag{9.165}$$

Substituting, this would yield

$$\begin{array}{rclcl} U & = & 250(0.2)^2 & = & 10 \text{ lbf-in} \\ \mathcal{W} & = & 100(0.2) & = & 20 \text{ lbf-in} \\ \Pi & = & 10 - 20 & = & -10 \text{ lbf-in} \end{array} \tag{9.166}$$

Fig. 9.11 illustrates the two components of the potential energy.

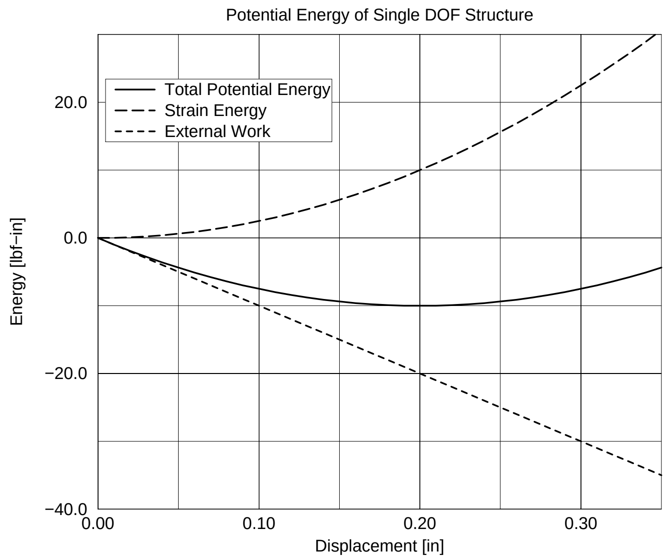

Figure 9.11: Graphical Representation of the Potential Energy

9.4.2 ‡Euler Equations of the Potential Energy

108 A variational statement is obtained by taking the first variation of the variational principle and setting this scalar quantity equal to zero.

109 The variational statement for the general form of the potential energy functional (i.e. Equation 9.174) is

$$\delta\Pi = \int_\Omega \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}\mathrm{d}\Omega - \int_\Omega \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}_0\mathrm{d}\Omega - \int_\Omega \delta\mathbf{u}^T\mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}\mathrm{d}\Gamma = 0 \quad (9.167)$$

which is the Principle of Virtual Work.

110 Since the differential operator $\mathbf{L}$ is linear, the variation of the strains $\delta\boldsymbol{\epsilon}$ can be expressed in terms of the variation of the displacements $\delta\mathbf{u}$

$$\delta\boldsymbol{\epsilon} = \delta(\mathbf{Lu}) = \mathbf{L}\delta\mathbf{u} \quad (9.168)$$

This relationship is exploited to obtain a form of the variational statement in which only variations of the displacements $\delta\mathbf{u}$ are present

$$\delta\Pi = \int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}\boldsymbol{\epsilon}\mathrm{d}\Omega - \int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega + \int_\Omega \delta(\mathbf{Lu})^T\boldsymbol{\sigma}_0\mathrm{d}\Omega - \int_\Omega \delta\mathbf{u}^T\mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T\hat{\mathbf{t}}\mathrm{d}\Gamma = 0 \quad (9.169)$$

which is best suited for obtaining the corresponding Euler equations.

111 A form of the variational statement in which strain-displacement relationship (i.e. Equation ??) is substituted into Equation 9.169

$$\boxed{\begin{aligned}\delta\Pi &= \int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}(\mathbf{Lu})\mathrm{d}\Omega - \int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega + \int_\Omega \delta(\mathbf{Lu})^T\boldsymbol{\sigma}_0\mathrm{d}\Omega \\ &\quad - \int_\Omega \delta\mathbf{u}^T\mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t}\delta\mathbf{u}^T\hat{\mathbf{t}}\mathrm{d}\Gamma = 0 \qquad (9.170)\end{aligned}}$$

is better suited for obtaining the discrete system of equations.

112 To obtain the Euler equations for the general form of the potential energy variational principle the volume integrals defining the virtual strain energy δU in Equation 9.169 must be integrated by parts in order to convert the variation of the strains $\delta(\mathbf{Lu})$ into a variation of the displacements $\delta\mathbf{u}$.

113 Integration by parts of these integrals using Green's theorem (Kreyszig 1988) yields

$$\begin{aligned}\int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}\boldsymbol{\epsilon}\mathrm{d}\Omega &= \oint_{\partial\Omega}\delta\mathbf{u}^T\mathbf{G}(\mathbf{D}\boldsymbol{\epsilon})\mathrm{d}\Gamma - \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T(\mathbf{D}\boldsymbol{\epsilon})\mathrm{d}\Omega \\ \int_\Omega \delta(\mathbf{Lu})^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega &= \oint_{\partial\Omega}\delta\mathbf{u}^T\mathbf{G}(\mathbf{D}\boldsymbol{\epsilon}_0)\mathrm{d}\Gamma - \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T(\mathbf{D}\boldsymbol{\epsilon}_0)\mathrm{d}\Omega \\ \int_\Omega \delta(\mathbf{Lu})^T\boldsymbol{\sigma}_0\mathrm{d}\Omega &= \oint_{\partial\Omega}\delta\mathbf{u}^T\mathbf{G}\boldsymbol{\sigma}_0\mathrm{d}\Gamma - \int_\Omega \delta\mathbf{u}^T\mathbf{L}^T\boldsymbol{\sigma}_0\mathrm{d}\Omega\end{aligned} \qquad (9.171)$$

where $\mathbf{G}$ is a transformation matrix containing the direction cosines for a unit normal vector such that the surface tractions $\mathbf{t}$ are defined as $\mathbf{t} = \mathbf{G}\boldsymbol{\sigma}$ and the surface integrals are over the entire surface of the body $\partial\Omega$.

114 Substituting Equation 9.171 into Equation 9.169, the variational statement becomes

$$\begin{aligned}\delta\Pi = &-\int_\Omega \delta\mathbf{u}^T\{\mathbf{L}^T[\mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) + \boldsymbol{\sigma}_0] + \mathbf{b}\}\mathrm{d}\Omega \\ &+ \int_{\partial\Omega}\delta\mathbf{u}^T\{\mathbf{G}[\mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) + \boldsymbol{\sigma}_0] + \hat{\mathbf{t}}\}\mathrm{d}\Gamma = 0\end{aligned} \qquad (9.172)$$

115 Since $\delta\mathbf{u}$ is arbitrary the expressions in the integrands within the braces must both be equal to zero for $\delta\Pi$ to be equal to zero. Recognizing that the stress-strain relationship (i.e. Equation 9.37) appears in both the volume and surface integrals, the Euler equations are

$$\begin{array}{ccccccc}\mathbf{L}^T\boldsymbol{\sigma} & + & \mathbf{b} & = & \mathbf{0} & \text{on} & \Omega \\ \mathbf{G}\boldsymbol{\sigma} & - & \hat{\mathbf{t}} & = & \mathbf{0} & \text{on} & \Gamma_t\end{array} \qquad (9.173)$$

where the first Euler equation is the equilibrium equation and the second Euler equation defines the natural boundary conditions. The natural boundary conditions are defined on Γ_t rather than $\partial\Omega$ because both the applied surface tractions $\hat{\mathbf{t}}$ and the matrix-vector product $\mathbf{G}\boldsymbol{\sigma}$ are identically zero outside Γ_t.

116 Starting from the Euler equations, it is possible to derive the total potential energy functional by performing the operations just presented in reverse order.

117 Substituting Equations 9.39 and 9.83 into the expression for the total potential energy the functional for the general form of the potential energy variational principle is obtained

$$\boxed{\Pi = \underbrace{\frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}\mathrm{d}\Omega}_{} - \underbrace{\int_\Omega \boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}_0\mathrm{d}\Omega + \int_\Omega \boldsymbol{\epsilon}^T\boldsymbol{\sigma}_0\mathrm{d}\Omega - \int_\Omega \mathbf{u}^T\mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t}\mathbf{u}^T\hat{\mathbf{t}}\mathrm{d}\Gamma}_{}} \tag{9.174}$$

9.4.3 Castigliano's First Theorem

118 A global version of Eq. 9.155 $\frac{\partial U_0}{\partial \varepsilon_{ij}} = \sigma_{ij}$, is Castigliano's theorem.

119 Since we are now considering a general structure, we consider an arbitrary three dimensional structure subjected to a set of external forces (or moments) $\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_n$ with corresponding unknown displacements $\Delta_1, \Delta_2, \cdots, \Delta_n$. The total potential energy is given by

$$\Pi = W_i + W_e = -U + \sum_{i=1}^{n}\hat{P}_i\Delta_i \tag{9.175}$$

120 The strain energy can also be expressed in terms of the displacements Δ_i thus the potential energy will be defined in terms of *generalized coordinates* or *generalized displacements*.

121 For the solid to be in equilibrium, $\delta\Pi = 0$ or

$$\begin{aligned}\delta\Pi &= -\frac{\partial U}{\partial \Delta_1}\delta\Delta_1 - \frac{\partial U}{\partial \Delta_2}\delta\Delta_2 - \cdots - \frac{\partial U}{\partial \Delta_n}\delta\Delta_n \\ &\quad + \hat{P}_1\delta\Delta_1 + \hat{P}_2\delta\Delta_2 \cdots + \hat{P}_n\delta\Delta_n = 0 \end{aligned} \tag{9.176-a}$$

or

$$\left(-\frac{\partial U}{\partial \Delta_1} + \hat{P}_1\right)\delta\Delta_1 + \left(-\frac{\partial U}{\partial \Delta_2} + \hat{P}_2\right)\delta\Delta_2 + \cdots + \left(-\frac{\partial U}{\partial \Delta_n} + \hat{P}_n\right)\delta\Delta_n = 0 \tag{9.177}$$

but since the variation $\delta\Delta_i$ is arbitrary, then each factor within the parenthesis must be equal to zero. Thus

$$\boxed{\frac{\partial U}{\partial \Delta_k} = \hat{P}_k} \tag{9.178}$$

which is Castigliano's first theorem:

> If the strain energy of a body is expressed in terms of displacement components in the direction of the prescribed forces, then the first partial derivative of the strain energy with respect to a displacement, is equal to the corresponding force.

■ **Example 9-7: Fixed End Beam, Variable I**

Considering the beam shown in Fig. 9.12, we can assume the following solution:

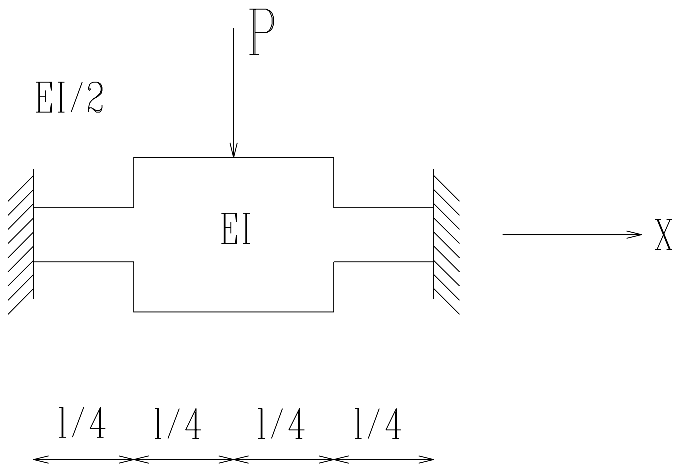

Figure 9.12: Variable Cross Section Fixed Beam

$$v = a_1x^3 + a_2x^2 + a_3x + a_4 \tag{9.179}$$

1. First, this solution *must* satisfy the essential B.C.: $v = v' = 0$ at $x = 0$; and $v = v_{\max}$ and $v' = 0$ at $x = \frac{L}{2}$. This will be enforced by determining the four parameters in terms of a single unknown quantity (4 equations and 4 B.C.'s):

$$\begin{array}{llll} @x = 0 & v = 0 & \Rightarrow a_4 = 0 & \\ @x = 0 & \frac{dv}{dx} = 0 & \Rightarrow a_3 = 0 & \\ @x = \frac{L}{2} & v = v_{\max} & \Rightarrow v_{\max} = a_1\frac{L^3}{8} + a_2\frac{L^2}{4} & \\ @x = \frac{L}{2} & \frac{dv}{dx} = 0 & \Rightarrow \frac{3}{4}a_1L^2 + a_2L = 0 & \Rightarrow a_2 = -\frac{3}{4}a_1L \end{array} \tag{9.180}$$

   upon substitution, we obtain:

$$v = \left(-\frac{16x^3}{L^3} + \frac{12x^2}{L^2}\right)v_{\max} \tag{9.181}$$

   Hence, in this problem the solution is in terms of only one unknown variable $v_{\max}$.

2. In order to apply the principle of Minimum Potential Energy we should evaluate:

   Internal Strain Energy U: for flexural members is given by $U = \int \frac{M^2}{2EI_z}dx$ (Eq. 9.43); recalling that $\frac{M}{EI_z} = \frac{d^2v}{dx^2}$, thus we must evaluate $\frac{d^2v}{dx^2}$ from above:

$$\frac{dv}{dx} = \left(-\frac{48x^2}{L^3} + \frac{24x}{L^2}\right)v_{\max} \tag{9.182}$$

$$\frac{d^2v}{dx^2} = -\frac{24}{L^2}\left(1-\frac{4x}{L}\right)v_{\max} \tag{9.183}$$

which yields

$$U = 2\left[\frac{1}{2}\int E\left(\frac{d^2v}{dx^2}\right)^2 I_z dx\right] \tag{9.184}$$

or:

$$\begin{aligned}\frac{U}{2} &= \frac{E}{2}\int_0^{\frac{L}{4}}\frac{24^2}{L^4}\left(1-\frac{4x}{L}\right)^2 v_{\max}^2\frac{I_z}{2}dx \\ &+\frac{E}{2}\int_{\frac{L}{4}}^{\frac{L}{2}}\frac{24^2}{L^4}\left(1-\frac{4x}{L}\right)^2 v_{\max}^2 I_z dx \\ U &= \frac{72EI_z}{L^3}v_{\max}^2 \qquad (9.185\text{-a})\end{aligned}$$

Potential of the External Work $\mathcal{W}$: For a point load, $\mathcal{W} = Pv_{\max}$

3. Finally,

$$\begin{aligned}\frac{\partial\Pi}{\partial v_{\max}} &= 0 && (9.186\text{-a})\\ \frac{\partial U}{\partial v_{\max}} - \frac{\partial\mathcal{W}}{\partial v_{\max}} &= 0 && (9.186\text{-b})\\ \frac{144EI_z}{L^3}v_{\max} &= P && (9.186\text{-c})\\ v_{\max} &= \boxed{\frac{PL^3}{144EI_z}} && (9.186\text{-d})\end{aligned}$$

4. Note, that had we applied Castigliano's theorem, then

$$\begin{aligned}\frac{\partial U}{\partial v_{\max}} &\stackrel{\text{def}}{=} P && (9.187\text{-a})\\ \frac{144EI_z}{L^3}v_{\max} &= P && (9.187\text{-b})\\ v_{\max} &= \boxed{\frac{PL^3}{144EI_z}} && (9.187\text{-c})\end{aligned}$$

which is identical to the solution obrained through the principle of minimum potential energy.

■

9.4.4 Rayleigh-Ritz Method

122 Continous systems have infinite number of degrees of freedom, those are the displacements at every point within the structure. Their behavior can be described by the Euler Equation, or the partial differential equation of equilibrium. However, only the simplest problems have an exact solution which (satisfies equilibrium, and the boundary conditions).

123 An *approximate* method of solution is the Rayleigh-Ritz method which is based on the principle of virtual displacements. In this method we *approximate* the displacement field by a function

$$u_1 \approx \sum_{i=1}^{n} c_i^1 \phi_i^1 + \phi_0^1 \tag{9.188-a}$$

$$u_2 \approx \sum_{i=1}^{n} c_i^2 \phi_i^2 + \phi_0^2 \tag{9.188-b}$$

$$u_3 \approx \sum_{i=1}^{n} c_i^3 \phi_i^3 + \phi_0^3 \tag{9.188-c}$$

where c_i^j denote undetermined parameters, and ϕ are appropriate functions of positions.

124 ϕ should satisfy three conditions

1. Be continous.
2. Must be *admissible*, i.e. satisfy the essential boundary conditions (the natural boundary conditions are included already in the variational statement. However, if ϕ also satisfy them, then better results are achieved).
3. Must be independent and complete (which means that the exact displacement and their derivatives that appear in Π can be arbitrary matched if enough terms are used. Furthermore, lowest order terms must also beincluded).

In general ϕ is a polynomial or trigonometric function.

125 We determine the parameters c_i^j by requiring that the principle of virtual work for arbitrary variations δc_i^j. or

$$\delta\Pi(u_1, u_2, u_3) = \sum_{i=1}^{n} \left(\frac{\partial \Pi}{\partial c_i^1}\delta c_i^1 + \frac{\partial \Pi}{\partial c_i^2}\delta c_i^2 + \frac{\partial \Pi}{\partial c_i^3}\delta c_i^3 \right) = 0 \tag{9.189}$$

for arbitrary and independent variations of δc_i^1, δc_i^2, and δc_i^3, thus it follows that

$$\boxed{\frac{\partial \Pi}{\partial c_i^j} = 0 \qquad i = 1, 2, \cdots, n; j = 1, 2, 3} \tag{9.190}$$

Thus we obtain a total of $3n$ linearly independent simultaneous equations. From these displacements, we can then determine strains and stresses (or internal forces). Hence we have replaced a problem with an infinite number of d.o.f by one with a finite number.

126 Some general observations

1. c_i^j can either be a set of coefficients with no physical meanings, or variables associated with nodal generalized displacements (such as deflection or displacement).
2. If the coordinate functions ϕ satisfy the above requirements, then the solution converges to the exact one if n increases.
3. For increasing values of n, the previously computed coefficients remain unchanged.
4. Since the strains are computed from the approximate displacements, strains and stresses are generally less accurate than the displacements.
5. The equilibrium equations of the problem are satisfied only in the energy sense $\delta\Pi = 0$ and not in the differential equation sense (i.e. in the weak form but not in the strong one). Therefore the displacements obtained from the approximation generaly do not satisfy the equations of equilibrium.
6. Since the continuous system is approximated by a finite number of coordinates (or d.o.f.), then the approximate system is stiffer than the actual one, and the displacements obtained from the Ritz method converge to the exact ones from below.

■ Example 9-8: Uniformly Loaded Simply Supported Beam; Polynomial Approximation

For the uniformly loaded beam shown in Fig. 9.13
let us assume a solution given by the following infinite series:

$$v = a_1 x(L-x) + a_2 x^2 (L-x)^2 + \dots \tag{9.191}$$

for this particular solution, let us retain only the first term:

$$v = a_1 x(L-x) \tag{9.192}$$

We observe that:

1. Contrarily to the previous example problem the geometric B.C. are immediately satisfied at both $x = 0$ and $x = L$.
2. We can keep v in terms of a_1 and take $\frac{\partial\Pi}{\partial a_1} = 0$ (If we had left v in terms of a_1 and a_2 we should then take both $\frac{\partial\Pi}{\partial a_1} = 0$, and $\frac{\partial\Pi}{\partial a_2} = 0$).
3. Or we can solve for a_1 in terms of $v_{\max}(@x = \frac{L}{2})$ and take $\frac{\partial\Pi}{\partial v_{\max}} = 0$.

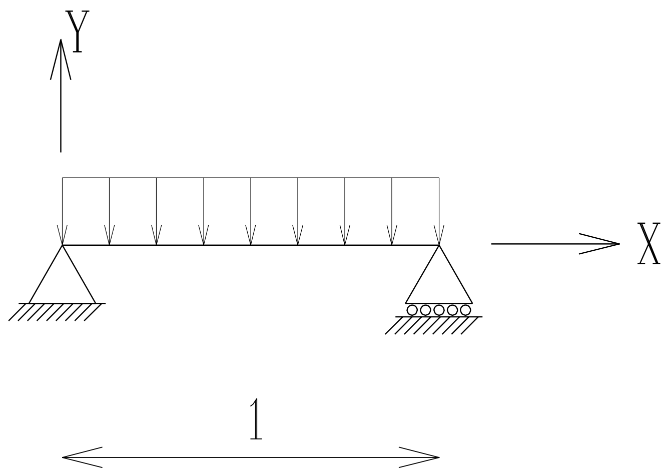

Figure 9.13: Uniformly Loaded Simply Supported Beam Analysed by the Rayleigh-Ritz Method

$$\Pi = U - \mathcal{W} = \int_o^L \frac{M^2}{2EI_z}dx - \int_0^L wv(x)dx \tag{9.193}$$

Recalling that: $\frac{M}{EI_z} = \frac{d^2v}{dx^2}$, the above simplifies to:

$$\Pi = \int_0^L \left[\frac{EI_z}{2} \left(\frac{d^2v}{dx^2} \right)^2 - wv(x) \right] dx \tag{9.194}$$

$$\begin{aligned} &= \int_0^L \left[\frac{EI_z}{2}(-2a_1)^2 - a_1 wx(L-x) \right] dx \\ &= \frac{EI_z}{2} 4a_1^2 L - a_1 w \frac{L^3}{2} + a_1 w \frac{L^3}{3} \\ &= 2a_1^2 EI_z L - \frac{a_1 w L^3}{6} \end{aligned} \tag{9.195}$$

If we now take $\frac{\partial \Pi}{\partial a_1} = 0$, we would obtain:

$$\begin{aligned} 4a_1 EI_z l - \frac{wL^3}{6} &= 0 \\ a_1 &= \frac{wL^2}{24EI_z} \end{aligned} \tag{9.196}$$

Having solved the displacement field in terms of a_1, we now determine $v_{\max}$ at $\frac{L}{2}$:

$$v = \underbrace{\frac{wL^4}{24EI_z}}_{a_1} \left(\frac{x}{L} - \frac{x^2}{L^2} \right)$$

$$= \frac{wL^4}{96EI_z} \tag{9.197}$$

This is to be compared with the exact value of $v_{\max}^{exact} = \frac{5}{384}\frac{wL^4}{EI_z} = \frac{wL^4}{76.8EI_z}$ which constitutes $\approx 17\%$ error.

Note: If two terms were retained, then we would have obtained: $a_1 = \frac{wL^2}{24EI_z}$ & $a_2 = \frac{w}{24EI_z}$ and $v_{\max}$ would be equal to $v_{\max}^{exact}$. (Why?) ■

■ Example 9-9: Uniformly Loaded Simply Supported Beam; Fourrier Series

Let us consider again the problem of Fi. 9.13 but with a trigonometric series for the continuous displacement field:

$$v = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{9.198}$$

we note that the B.C. are satisfied ($v = 0$ at $x = 0$ and $x = L$). The potential energy is given by:

$$\begin{aligned}
\Pi &= U - \mathcal{W} \\
&= \int_0^L \left[\frac{EI_z}{2}\left(\frac{d^2v}{dx^2}\right)^2 - wv(x)\right] dx \\
&= \sum\int_0^L \left[\frac{EI_z}{2}\left(-\frac{n^2\pi^2 a_n}{L^2}\sin\frac{n\pi x}{L}\right)^2 - wa_n \sin\frac{n\pi x}{L}\right] dx \\
&= \dots \\
&= \frac{EI_z}{2}\left(\frac{L}{2}\right)^2 \sum_{n=1}^{\infty} a_n^2\left(\frac{n\pi}{L}\right)^4 + w\left(\frac{L}{\pi}\right)\sum_{n=1}^{\infty}\frac{1}{n}a_n\cos\frac{n\pi x}{L}\,\Big|_0^L \\
&= \frac{\pi^4 EI_z}{4L^3}\sum_{n=1}^{\infty} a_n^2 n^4 - \frac{2wL}{\pi}\sum_{n=1,3,5}^{\infty}\frac{a_n}{n}
\end{aligned} \tag{9.199}$$

Note that for n even, the second term vanishes.

We now take:

$$\frac{\partial\Pi}{\partial a_1} = 0 \quad \frac{\partial\Pi}{\partial a_2} = 0 \quad \dots \quad \frac{\partial\Pi}{\partial a_n} = 0 \tag{9.200}$$

which would yield:

$$a_n = \frac{4wL^4}{EI_z(n\pi)^5} \qquad \text{n} = 1,3,5 \tag{9.201}$$

or:

$$v = \frac{4wL^4}{EI_z\pi^5}\sum_{n=1,3,5}^{\infty}\left(\frac{1}{n}\right)^5 \sin\frac{n\pi x}{L} \tag{9.202}$$

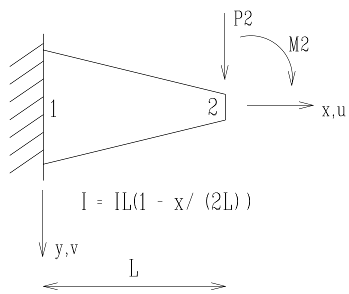

Figure 9.14: Example xx: External Virtual Work

and for $x = \frac{L}{2}$ we would get:

$$v = v_{\max} = \frac{4wL^4}{EI_z\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \dots\right) \tag{9.203}$$

Note that should we consider only the 1st term, then:

$$v_{\max} = \frac{wL^4}{76.5EI_z} \approx v_{\max}^{exact} \tag{9.204}$$

■

■ Example 9-10: Tapered Beam; Fourrier Series

Revisiting the previous problem of a tapered beam subjected to a point load, Fig. 9.14 and using the following approximation

$$v = \sum\nolimits_{n=1,3,\dots} a_n \left(1 - \cos\frac{n\pi x}{2l}\right) \tag{9.205}$$

we seek to solve for v_2 and θ_2, for $n = 1$ and $n = 3$.
Solution:

$$v'' = \sum a_n \left(\frac{n\pi}{2l}\right)^2 \cos\frac{n\pi x}{2l} \tag{9.206}$$

$$U = \frac{1}{2}\int_0^L (v'')^2 EI_z dx \tag{9.207}$$

$$= \frac{EI_1}{2}\int_0^L \left[\sum\nolimits_{n=1,3,} a_n \left(\frac{n\pi}{2l}\right)^2 \cos\frac{n\pi x}{2l}\right]^2 \left(1 - \frac{x}{L}\right) dx \tag{9.208}$$

However, we recall that:

$$\int_0^L \cos\frac{m\pi x}{2l}\cos\frac{n\pi x}{2l}dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} \tag{9.209}$$

$$\int_0^L x\cos\frac{m\pi x}{2l}\cos\frac{n\pi x}{2l}dx = \begin{cases} 0 & m \neq n \\ \frac{L^2}{8} - \frac{L^2}{2n^2\pi^2} & m = n \end{cases} \tag{9.210}$$

Thus combining Eqns. 9.208, 9.209, and 9.210, we obtain:

$$U = \frac{\pi^4 EI_1}{64L^3}\sum_{1,3,5}\left(\frac{3}{4}+\frac{1}{n^2\pi^2}\right)n^4 a_n^2 \tag{9.211}$$

The potential of the external work $\mathcal{W}$ in turn is given by:

$$W = P_2\underbrace{\sum a_n}_{v@x=l} + \frac{M_2\pi}{L}\underbrace{\sum(-1)^{\frac{n-1}{2}}na_n}_{\theta@x=l} \tag{9.212}$$

Finally, taking

$$\frac{\partial\Pi}{\partial a_n} = \frac{\partial U}{\partial a_n} - \frac{\partial\mathcal{W}}{\partial a_n} = 0 \tag{9.213}$$

Combining Eqns. 9.211, 9.212, and 9.213 we solve for a_n:

$$a_n = \frac{32L^3}{\pi^4 EI_1}\frac{\left(P + \frac{n\pi}{2L}(-1)^{\frac{n-1}{2}}M\right)}{\left(\frac{3}{4}+\frac{1}{n^2\pi^2}\right)n^4} \tag{9.214}$$

Solving for $v_2 = \sum a_n$ we obtain:

$$v_2 = \begin{cases} \frac{PL^3}{2.59EI_1} + \frac{ML^2}{1.65EI_1} & \text{n=1} \\ \frac{PL^3}{2.55EI_1} + \frac{ML^2}{1.65EI_1} & \text{n=3} \end{cases} \tag{9.215}$$

Similarly we solve for $\theta_2 = \frac{\pi}{2L}\sum n(-1)^{\frac{n-1}{2}}a_n$

$$\theta_2 \begin{cases} \frac{PL^2}{1.65EI} + \frac{Ml}{1.05EI_1} & \text{n=1} \\ \frac{PL^2}{1.65EI_z} + \frac{ML}{1.04EI_1} & \text{n=3} \end{cases} \tag{9.216}$$

■

9.5 † Complementary Potential Energy

9.5.1 Derivation

127 Eq. 9.129, $\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}d\Omega - \int_{\Gamma_u}\hat{u}_i\delta t_i d\Gamma = 0$ can be rewritten as

$$\delta\left[\int_\Omega U_0 d\Omega - \int_{\Gamma_u}\hat{u}_i\delta t_i d\Gamma\right] = 0 \tag{9.217}$$

or

$$\boxed{\begin{array}{cclr} \delta\Pi^* & = & 0 & (9.218) \\ \Pi^* & \stackrel{\text{def}}{=} & U^* - \mathcal{W}_e^* & (9.219) \end{array}}$$

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which is the principle of stationary complementary energy which states that

> Of all statically admissible states of stress (stresses satisfying the equation of equilibrium), the actual state of stress (the one which satisfy the kinematic conditions) are the ones for which the total complementary potential energy assumes a stationalry value.

9.5.2 Castigliano's Second Theorem

128 Considering again a three dimensional structure subjected to external displacements (or rotations) (or moments) $\hat{\Delta}_1, \hat{\Delta}_2, \cdots, \hat{\Delta}_n$ with corresponding unknown forces (or moments) $P_1, P_2, \cdots, P_n$. The total complementary potential energy is given by

$$\Pi^* = W_i^* + \mathcal{W}_e^* = U_i^* - \sum_{i=1}^{n}\hat{\Delta}_i P_i \tag{9.220}$$

129 The complementary strain energy can also be expressed in terms of the forces P_i thus the complementary potential energy will be defined in terms of *generalized coordinates* or *generalized forces*.

130 For the solid to be in equilibrium, $\delta\Pi^* = 0$ or

$$\begin{aligned}\delta\Pi^* &= \frac{\partial W_i^*}{\partial P_1}\delta P_1 + \frac{\partial W_i^*}{\partial P_2}\delta P_2 + \cdots + \frac{\partial W_i^*}{\partial P_n}\delta P_n \\ &-\hat{\Delta}_1\delta P_1 - \hat{\Delta}_2\delta P_2 \cdots - \hat{\Delta}_n\delta P_n = 0\end{aligned} \tag{9.221-a}$$

or

$$\left(\frac{\partial W_i^*}{\partial P_1} - \hat{\Delta}_1\right)\delta P_1 + \left(\frac{\partial W_i^*}{\partial P_2} - \hat{\Delta}_2\right)\delta P_2 + \cdots + \left(\frac{\partial W_i^*}{\partial P_n} - \hat{\Delta}_n\right)\delta P_n \tag{9.222}$$

but since the variation δP_i is arbitrary, then each factor within the parenthesis must be equal to zero. Thus

$$\boxed{\frac{\partial W_i^*}{\partial P_k} = \hat{\Delta_k}} \tag{9.223}$$

which is Castigliano's second theorem:

> If the complementary strain energy of a body is expressed in terms of forces then the first partial derivative of the strain energy with respect to any one of the forces, is equal to the corresponding displacement at the point where the force is located.

■ Example 9-11: Cantilivered beam

Solve for the displacement of the tip of a cantilever loaded by a point load.

Solution:

From Eq. 9.43, the strain energy is $U = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx$, and for a point load, the external work is $\mathcal{W}_e = P\Delta$ thus the potential energy of the system is

$$\begin{aligned} \Pi &= \mathcal{W}_e - U \\ &= P\Delta - \tfrac{1}{2}\int_0^L \frac{M^2}{EI_z}dx \end{aligned}$$

However, for the point load, the moment is $M = Px$, substituting above

$$\begin{aligned} \Pi &= P\Delta - \tfrac{1}{2}\int_0^L \frac{P^2x^2}{EI}dx \\ &= P\Delta - \tfrac{1}{6}\tfrac{P^2L^3}{EI} \\ \tfrac{d\Pi}{dP} &= \Delta - \tfrac{1}{3}\tfrac{P^2L^3}{EI} = 0 \\ \Rightarrow \Delta &= \tfrac{1}{3}\tfrac{P^2L^3}{EI} \end{aligned}$$

■

9.5.2.1 Distributed Loads

131 Castigliano's theorem can easily be applied to problems in which the structure is subjected to point load or moments, and we seek the deflection under these loads.

132 However when a structure is subjected to say a uniform load, and we wish to determine the deflection at a point where no point load is applied, then we must introduce a fictitious corresponding force R and then write the complementary strain energy interms of R and the applied load.

■ **Example 9-12: Deflection of a Uniformly loaded Beam using Castigliano's Theorem**

Considering a simply supported uniformly beam, we seek the midspan deflection.
Solution:

We introduce a fictitious force R at midspan, and the moment is thus $M(x) = \frac{wL}{2}x + \frac{R}{2}x - w\frac{x^2}{2}$. The complementary strain energy is $U^* = 2\int_0^{\frac{L}{2}} \frac{M(x)}{2EI}dx$ and the displacement

$$\begin{aligned}
\Delta &= \left.\frac{\partial U^*}{\partial R}\right|_{R=0} \\
&= \left.\left(\frac{2}{EI}\int_0^{\frac{L}{2}} M\frac{\partial M}{\partial R}dx\right)\right|_{R=0} \quad = \quad \left.\left\{\frac{2}{EI}\int_0^{\frac{L}{2}}\left[\left(\frac{wL}{2}+\frac{R}{2}\right)x - w\frac{x^2}{2}\right]\frac{x}{2}dx\right\}\right|_{R=0} \\
&= \frac{5wL^4}{384EI}
\end{aligned}$$

■

9.6 Comparison of Alternate Approximate Solutions

133 While we were able to assess the accuracy of our approximate solutions with respect to the exact one, (already known), in general this is not possible. (i.e., If an exact solution is known, there is no need for an approximate one). Thus the question is, given two or more alternate approximate solutions which one is the best?

134 This can be determined by evaluating the potential energy of each approximate solution and identify the lowest one.

■ **Example 9-13: Comparison of MPE Solutions**

With reference to examples (simply supported uniformly loaded beams) we can determine for each one its Potential Energy $\Pi = U - W_e$:

Polynomial Solution: From Eq. 9.196 and 9.195 respectively, we had:

$$\begin{aligned}
a_1 &= \frac{wL^2}{24EI_z} \qquad (9.224)\\
\Pi &= 2a_1^2 EI_z L - \frac{a_1 wL^3}{6} \\
&= 2\left(\frac{wL^2}{24EI_z}\right)^2 EI_z l - \left(\frac{wL^2}{24EI_z}\right)\frac{wL^3}{6}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{w^2L^5}{EI_z}\left(\frac{2}{24^2} - \frac{1}{(6)(24)}\right) \\
&= -\frac{1}{288}\frac{w^2L^5}{EI_z} \qquad (9.225)
\end{aligned}
$$

Trigonometric Solution: From Eq. 9.199 and 9.201 respectively we had:

$$
\Pi = \frac{\pi^4 EI_z}{4L^3}\sum_{n=1}^{\infty} a_n^2 n^4 - \frac{2wL}{\pi}\sum_{n=1,2,3}^{\infty}\frac{a_n}{n} \qquad (9.226)
$$

$$
a_n = \frac{4wL^4}{EI_z(n\pi)^5}\text{n} = 1,\ 3,\ 5 \qquad (9.227)
$$

For $n = 1$:

$$
a_1 = \frac{4wL^4}{EI_z\pi^5} \qquad (9.228)
$$

$$
\begin{aligned}
\Pi &= \frac{\pi^4 EI_z}{4L^3}\left[\frac{4wL^4}{EI_z\pi^5}\right]^2 - \frac{2wL}{\pi}\left[\frac{4wL^4}{EI_z\pi^5}\right] \\
&= \frac{w^2L^5}{EI_z}\left[\frac{16}{4\pi^6} - \frac{8}{\pi^6}\right] \\
&= -\frac{4}{\pi^6}\frac{w^2L^5}{EI_z} \\
&= -\frac{1}{240}\frac{w^2L^5}{EI_z} \qquad (9.229)
\end{aligned}
$$

We note that the Trigonometric solution has a lower potential energy than the polynomial approximation and is thus more accurate (the exact displacement is $v_{\max} = \frac{wL^4}{76.8EI_z}$) as shown in Table 9.3. ■

	$\Pi\frac{EI_z}{w^2L^5}$	$v_{\max}\frac{EI_z}{wL^4}$	% error
Polynomial	$-\frac{1}{288}$	$\frac{1}{96}$	17%
Trigonometric	$-\frac{1}{240}$	$\frac{1}{76.6}$	1%

Table 9.3: Comparison of 2 Alternative Approximate Solutions

9.7 Summary

135 A summary of the various methods introduced in this chapter is shown in Fig. 9.15.

136 The duality between the two variational principles is highlighted by Fig. 9.16, where beginning with kinematically admissible displacements, the principle of virtual work provides

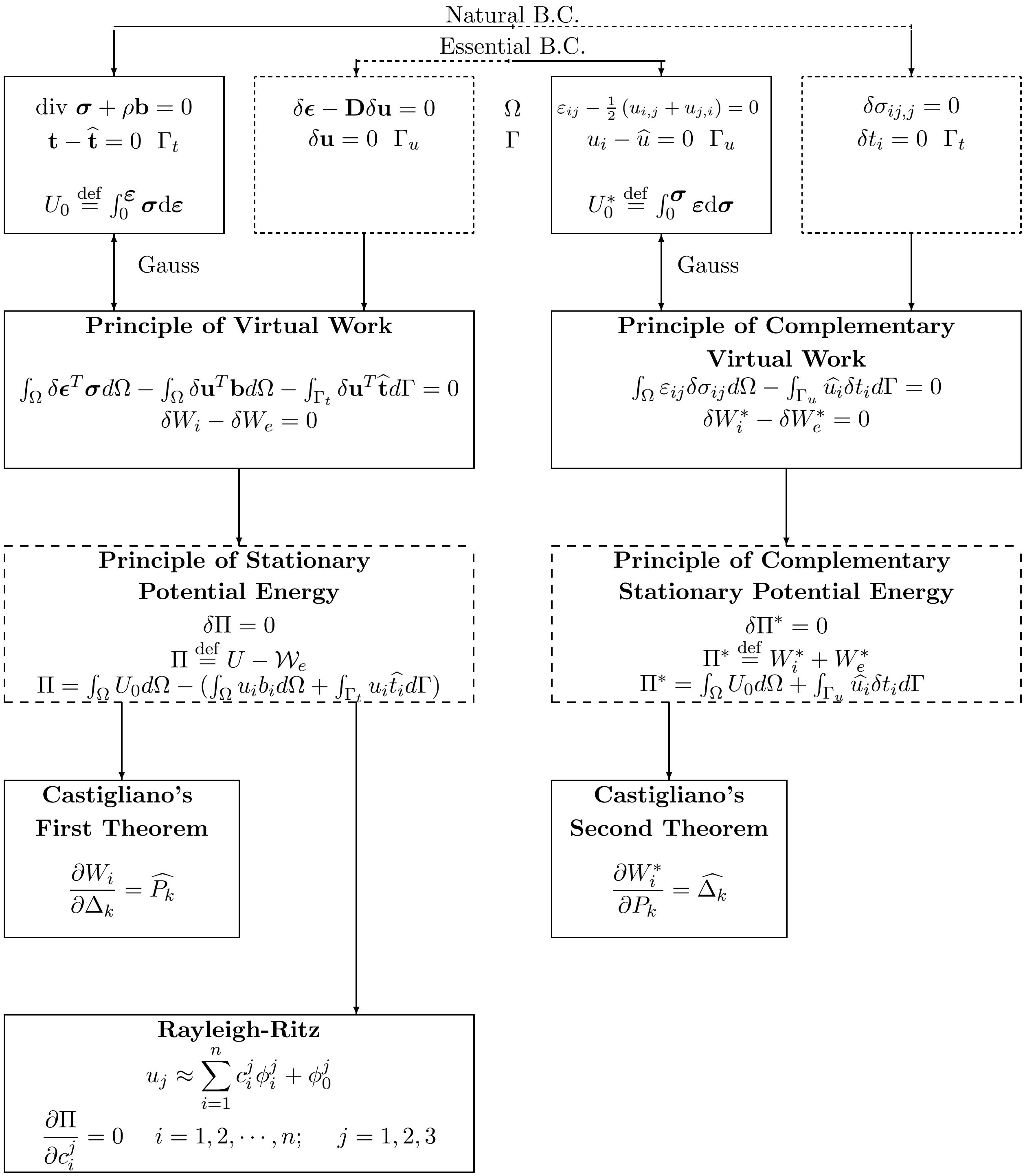


Figure 9.15: Summary of Variational Methods

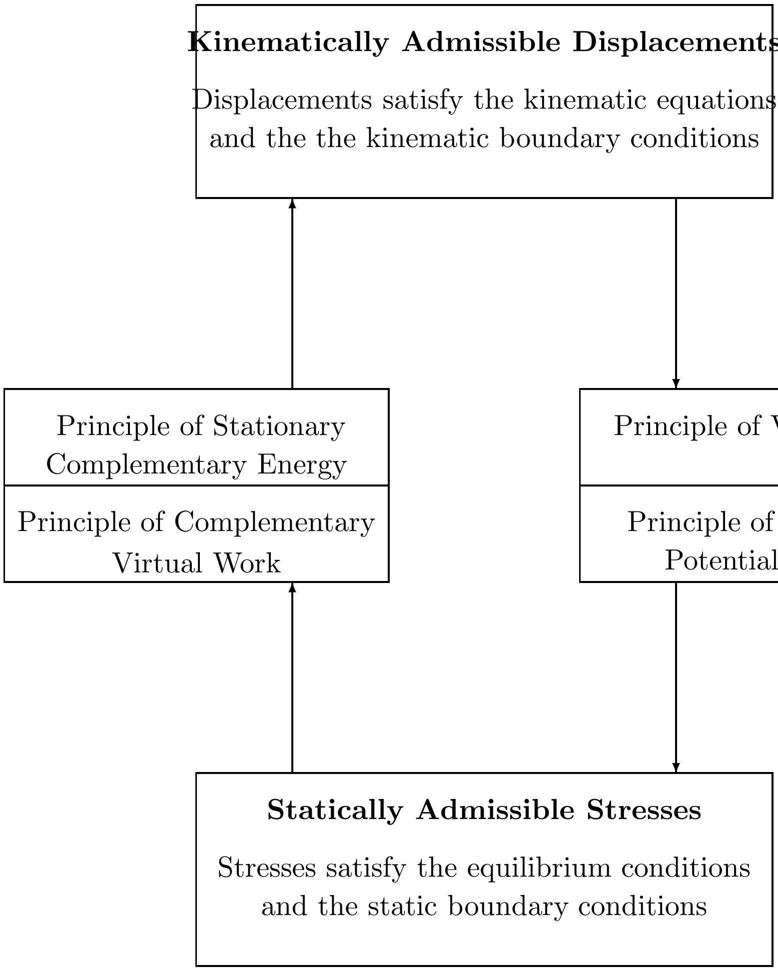


Figure 9.16: Duality of Variational Principles

statically admissible solutions. Similarly, for statically admissible stresses, the principle of complementary virtual work leads to kinematically admissible solutions.

137 Finally, Table 9.4 summarizes some of the major equations associated with one dimensional rod elements.

	U	Virtual Displacement δU		Virtual Force δU^*	
		General	Linear	General	Linear
Axial	$\frac{1}{2}\int_0^L \frac{P^2}{AE} dx$	$\int_0^L \sigma\delta\varepsilon dx$	$\int_0^L \underbrace{E\frac{du}{dx}}_{\sigma}\underbrace{\frac{d(\delta u)}{dx}}_{\delta\varepsilon}\underbrace{Adx}_{d\Omega}$	$\int_0^L \delta\sigma\varepsilon dx$	$\int_0^L \underbrace{\delta P}_{\delta\sigma}\underbrace{\frac{P}{AE}}_{\varepsilon} dx$
Shear	...	$\int_0^L V\delta\gamma_{xy} dx$	...	$\int_0^L \delta V\gamma_{xy} dx$	...
Flexure	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z} dx$	$\int_0^L M\delta\phi dx$	$\int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\sigma}\underbrace{\frac{d^2(\delta v)}{dx^2}}_{\delta\varepsilon} dx$	$\int_0^L \delta M\phi dx$	$\int_0^L \underbrace{\delta M}_{\delta\sigma}\underbrace{\frac{M}{EI_z}}_{\varepsilon} dx$
Torsion	$\frac{1}{2}\int_0^L \frac{T^2}{GJ} dx$	$\int_0^L T\delta\theta dx$	$\int_0^L \underbrace{GJ\frac{d\theta_x}{dx}}_{\sigma}\underbrace{\frac{d(\delta\theta_x)}{dx}}_{\delta\varepsilon} dx$	$\int_0^L \delta T\theta dx$	$\int_0^L \underbrace{\delta T}_{\delta\sigma}\underbrace{\frac{T}{GJ}}_{\varepsilon} dx$

	W	Virtual Displacement δW	Virtual Force δW^*
P	$\Sigma_i\frac{1}{2}P_i\Delta_i$	$\Sigma_i P_i\delta\Delta_i$	$\Sigma_i\delta P_i\Delta_i$
M	$\Sigma_i\frac{1}{2}M_i\theta_i$	$\Sigma_i M_i\delta\theta_i$	$\Sigma_i\delta M_i\theta_i$
w	$\int_0^L w(x)v(x)dx$	$\int_0^L w(x)\delta v(x)dx$	$\int_0^L \delta w(x)v(x)dx$

Table 9.4: Summary of Variational Terms Associated with One Dimensional Elements

Chapter 10

INTERPOLATION FUNCTIONS

10.1 Introduction

27 Application of the Principle of Virtual Displacement requires an *assumed* displacement field. This displacement field can be approximated by *interpolation functions* written in terms of:

1. Unknown polynomial coefficients, most appropriate for continuous systems, and the Rayleigh-Ritz method
$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{10.1}$$
A major drawback of this approach, is that the coefficients have no physical meaning.
2. Unknown nodal deformations, most appropriate for discrete systems and Potential Energy based formulations
$$y = \Delta = N_1 \overline{\Delta}_1 + N_2 \overline{\Delta}_2 + \ldots + N_n \overline{\Delta}_n \tag{10.2}$$

28 For simple problems *both* Eqn. 10.1 and Eqn. 10.2 can readily provide the *exact* solutions of the governing differential equation (such as $\frac{d^4 y}{dx^4} = \frac{q}{EI}$ for flexure), but for more complex ones, one must use an *approximate* one.

10.2 Shape Functions

29 For an element (finite or otherwise), we can write an expression for the generalized displacement (translation/rotation), Δ at any point in terms of all its nodal ones, $\overline{\boldsymbol{\Delta}}$.

$$\Delta = \sum_{i=1}^{n} N_i(X) \overline{\Delta}_i = \lfloor \mathbf{N}(x) \rfloor \{\overline{\boldsymbol{\Delta}}\} \tag{10.3}$$

where:

1. Δ_i is the (generalized) nodal displacement corresponding to d.o.f i

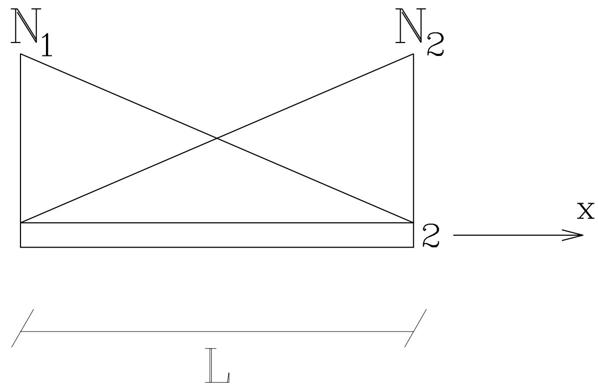


Figure 10.1: Axial Finite Element

2. N_i is an interpolation function, or *shape function* which has the following characteristics:
   (a) $N_i = 1$ at Δ_i
   (b) $N_i = 0$ at Δ_j where $i \neq j$.
3. **N** can be derived on the bases of:
   (a) Assumed deformation state defined in terms of polynomial series.
   (b) Interpolation function (Lagrangian or Hermitian).

30 We shall distinguish between two classes of problems, those involving displacements only, and those involving displacement and their derivatives.

31 The first class requires only continuity of displacement, and will be referred to as C^0 problems (truss, torsion), whereas the second one requires continuity of slopes and will be referred to as C^1 problems.

10.2.1 Axial/Torsional

32 With reference to Fig. 10.1 we start with:

$$\Delta = N_1\Delta_1 + N_2\Delta_2 \tag{10.4}$$
$$\theta_x = N_1\theta_{x1} + N_2\theta_{x2} \tag{10.5}$$

33 Since we have 2 d.o.f's, we will assume a linear deformation state

$$u = a_1x + a_2 \tag{10.6}$$

where u can be either Δ or θ, and the B.C.'s are given by: $u = u_1$ at $x = 0$, and $u = u_2$ at $x = L$. Thus we have:

$$u_1 = a_2 \tag{10.7}$$

$$u_2 = a_1 L + a_2 \tag{10.8}$$

34 Solving for a_1 and a_2 in terms of u_1 and u_2 we obtain:

$$a_1 = \frac{u_2}{L} - \frac{u_1}{L} \tag{10.9}$$

$$a_2 = u_1 \tag{10.10}$$

35 Substituting and rearranging those expressions into Eq. 10.6 we obtain

$$u = (\frac{u_2}{L} - \frac{u_1}{L})x + u_1 \tag{10.11}$$

$$= \underbrace{(1 - \frac{x}{L})}_{N_1} u_1 + \underbrace{\frac{x}{L}}_{N_2} u_2 \tag{10.12}$$

or:

$$\boxed{\begin{array}{l} N_1 = 1 - \frac{x}{L} \\ N_2 = \frac{x}{L} \end{array}} \tag{10.13}$$

10.2.2 Generalization

36 The previous derivation can be generalized by writing:

$$u = a_1 x + a_2 = \underbrace{\lfloor\ x \quad 1\ \rfloor}_{[\mathbf{p}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}} \tag{10.14}$$

where $[\mathbf{p}]$ corresponds to the polynomial approximation, and $\{\mathbf{a}\}$ is the coefficient vector.

37 We next apply the boundary conditions:

$$\underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}} = \underbrace{\left[\begin{array}{cc} 0 & 1 \\ L & 1 \end{array} \right]}_{[\mathcal{L}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}} \tag{10.15}$$

following inversion of $[\mathcal{L}]$, this leads to

$$\underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L} \left[\begin{array}{cc} -1 & 1 \\ L & 0 \end{array} \right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}} \tag{10.16}$$

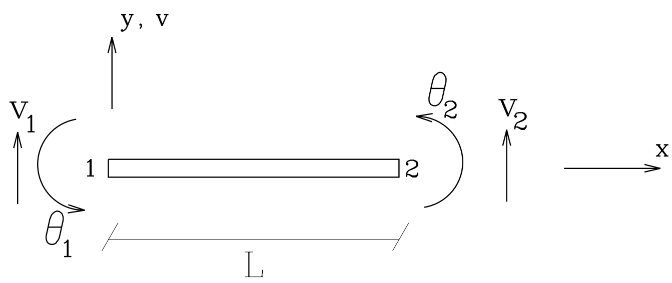


Figure 10.2: Flexural Finite Element

38 Substituting this last equation into Eq. 10.14, we obtain:

$$u = \underbrace{\underbrace{\lfloor\ (1-\frac{x}{L}) \quad \frac{x}{L}\ \rfloor}_{[\mathbf{p}][\mathcal{L}]^{-1}}}_{[\mathbf{N}]} \underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}} \tag{10.17}$$

39 Hence, the shape functions $[\mathbf{N}]$ can be directly obtained from

$$\boxed{[\mathbf{N}] = [\mathbf{p}][\mathcal{L}]^{-1}} \tag{10.18}$$

10.2.3 Flexural

40 With reference to Fig. 10.2. We have 4 d.o.f.'s, $\{\boldsymbol{\Delta}\}_{4\times 1}$: and hence will need 4 shape functions, N_1 to N_4, and those will be obtained through 4 boundary conditions. Therefore we need to assume a polynomial approximation for displacements of degree 3.

$$v = a_1x^3 + a_2x^2 + a_3x + a_4 \tag{10.19}$$

$$\theta = \frac{dv}{dx} = 3a_1x^2 + 2a_2x + a_3 \tag{10.20}$$

41 Note that v can be rewritten as:

$$v = \underbrace{\lfloor\ x^3 \quad x^2 \quad x \quad 1\ \rfloor}_{[\mathbf{p}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\}}_{\{\mathbf{a}\}} \tag{10.21}$$

42 We now apply the boundary conditions:

$$
\begin{array}{lcr}
v = v_1 & \text{a} & \text{t } x = 0 \\
v = v_2 & \text{at} & x = L \\
\theta = \theta_1 = \frac{dv}{dx} & \text{at} & x = 0 \\
\theta = \theta_2 = \frac{dv}{dx} & \text{at} & x = L
\end{array}
$$

or:

$$
\underbrace{\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{bmatrix}}_{[\mathcal{L}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}}_{\{\mathbf{a}\}} \tag{10.22}
$$

43 which when inverted yields:

$$
\underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L^3} \begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}} \tag{10.23}
$$

44 Combining Eq. 10.23 with Eq. 10.21, we obtain:

$$
\Delta = \underbrace{\lfloor\, x^3 \quad x^2 \quad x \quad 1 \,\rfloor}_{[\mathbf{p}]} \underbrace{\frac{1}{L^3} \begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}} \tag{10.24}
$$

$$
= \underbrace{\lfloor\, \underbrace{(1 + 2\xi^3 - 3\xi^2)}_{N_1} \quad \underbrace{x(1-\xi)^2}_{N_2} \quad \underbrace{(3\xi^2 - 2\xi^3)}_{N_3} \quad \underbrace{x(\xi^2 - \xi)}_{N_4} \,\rfloor}_{\underbrace{[\mathbf{p}][\mathcal{L}]^{-1}}_{[\mathbf{N}]}} \underbrace{\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}} \tag{10.25}
$$

where $\xi = \frac{x}{l}$.

45 Hence, the shape functions for the flexural element are given by:

$$
\boxed{
\begin{array}{lcll}
N_1 & = & (1 + 2\xi^3 - 3\xi^2) & (10.26) \\
N_2 & = & x(1-\xi)^2 & (10.27) \\
N_3 & = & (3\xi^2 - 2\xi^3) & (10.28) \\
N_4 & = & x(\xi^2 - \xi) & (10.29)
\end{array}
}
$$

and are shown in Fig 10.3.

46 Table 10.1 illustrates the characteristics of those shape functions

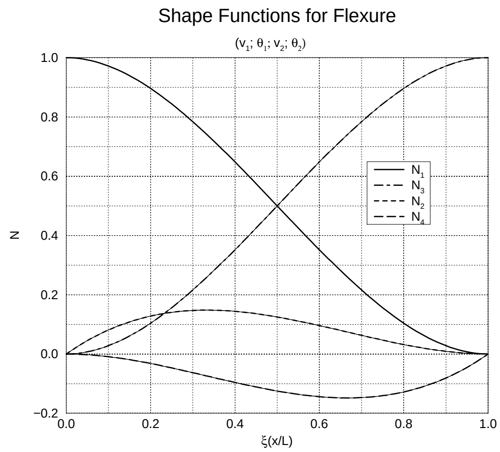


Figure 10.3: Shape Functions for Flexure of Uniform Beam Element.

Function	$\xi = 0$ N_i	$\xi = 0$ $N_{i,x}$	$\xi = 1$ N_i	$\xi = 1$ $N_{i,x}$
$N_1 = (1 + 2\xi^3 - 3\xi^2)$	1	0	0	0
$N_2 = x(1 - \xi)^2$	0	1	0	0
$N_3 = (3\xi^2 - 2\xi^3)$	0	0	1	0
$N_4 = x(\xi^2 - \xi)$	0	0	0	1

Table 10.1: Characteristics of Beam Element Shape Functions

10.2.4 Constant Strain Triangle Element

47 Next we consider a triangular element, Fig. 10.4 with *bi-linear* displacement field (in both x

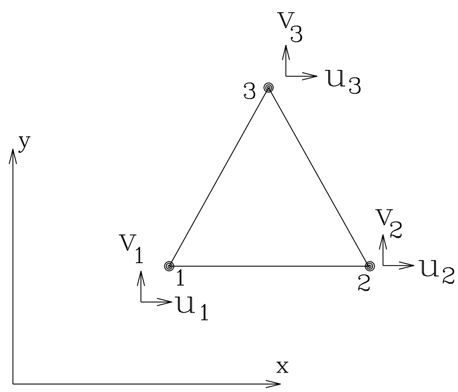


Figure 10.4: *Constant Strain Triangle Element

and y):

$$u = a_1 + a_2 x + a_3 y \tag{10.30}$$

$$v = a_4 + a_5 x + a_6 y \tag{10.31}$$

$$\Delta = \underbrace{\lfloor 1 \quad x \quad y \rfloor}_{[\mathbf{p}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\}}_{\{\mathbf{a}\}} \tag{10.32}$$

48 As before, we first seek the shape functions, and hence we apply the boundary conditions at the nodes for the u displacements first:

$$\underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}} = \underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & y_3 \end{array} \right]}_{[\mathcal{L}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\}}_{\{\mathbf{a}\}} \tag{10.33}$$

49 We then multiply the inverse of $[\mathcal{L}]$ in Eq. 10.33 by $[\mathbf{p}]$ and obtain:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 \tag{10.34}$$

where

$$N_1 = \frac{1}{x_2 y_3}(x_2 y_3 - x y_3 - x_2 y + x_3 y)$$

$$\begin{aligned} N_2 &= \frac{1}{x_2 y_3}(xy_3 - x_3 y) \\ N_3 &= \frac{y}{y_3} \end{aligned} \tag{10.35}$$

We observe that each of the three shape functions is equal to 1 at the corrsponding node, and equal to 0 at the other two.

50 The same shape functions can be derived for v:

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 \tag{10.36}$$

51 Hence, the displacement field will be given by:

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left[\begin{array}{cccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \right\} \tag{10.37}$$

52 The element is refereed to as Constant Strain Triangle (CST) because it has a linear displacement field, and hence a constant strain.

10.3 Interpolation Functions

53 Based on the preceding examples, we now seek to derive a "general formula" for shape functions of polynomials of various orders.

10.3.1 C^0: Lagrangian Interpolation Functions

54 In our earlier approach, the shape functions were obtained by:

1. Assumption of a polynomial function: $\Delta = \lfloor \mathbf{p} \rfloor \{\mathbf{a}\}$
2. Application of the boundary conditions $\{\overline{\boldsymbol{\Delta}}\} = [\mathcal{L}]\{\mathbf{a}\}$
3. Inversion of $[\mathcal{L}]$
4. And finally $[\mathbf{N}] = [\mathbf{p}][\mathcal{L}]^{-1}$

55 By following these operations, we have in effect defined the *Lagrangian Interpolation Functions* for problems with C^0 interelement continuity (i.e continuity of displacement only).

56 The Lagrangian interpolation defines the coefficients ([**N**] in our case) of a polynomial series representation of a function in terms of values defined at discrete points (nodes in our case). For points along a line this would yield:

$$\boxed{N_i = \frac{\prod_{j=1,j\neq i}^{m+1}(x-x_j)}{\prod_{j=1,j\neq i}^{m+1}(x_i-x_j)}} \tag{10.38}$$

57 If expanded, the preceding equation would yield:

$$\begin{aligned} N_1 &= \frac{(x-x_2)(x-x_3)\cdots(x-x_{m+1})}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_{m+1})} \\ N_2 &= \frac{(x-x_1)(x-x_3)\cdots(x-x_{m+1})}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_{m+1})} \\ N_{m+1} &= \frac{(x-x_1)(x-x_2)\cdots(x-x_m)}{(x_{m+1}-x_1)(x_{m+1}-x_2)\cdots(x_{m+1}-x_m)} \end{aligned} \tag{10.39}$$

58 For the axial member, $m = 1$, $x_1 = 0$, and $x_2 = L$, the above equations will result in:

$$\Delta = \frac{(x-L)}{-L}\Delta_1 + \frac{x}{L}\Delta_2 = \underbrace{(1-\frac{x}{L})}_{N_1}\Delta_1 + \underbrace{\frac{x}{L}}_{N_2}\Delta_2 \tag{10.40}$$

which is identical to Eq. 10.12.

10.3.1.1 Constant Strain Quadrilateral Element

59 Next we consider a quadrilateral element, Fig. 10.5 with *bi-linear* displacement field (in both x and y).

60 Using the Lagrangian interpolation function of Eq. 10.38, and starting with the u displacement, we perform two interpolations: the first one along the bottom edge (1-2) and along the top one (4-3).

61 From Eq. 10.38 with $m = 1$ we obtain:

$$\begin{aligned} u_{12} &= \frac{x_2-x}{x_2-x_1}u_1 + \frac{x_1-x}{x_1-x_2}u_2 \\ &= \frac{a-x}{2a}u_1 + \frac{x+a}{2a}u_2 \end{aligned} \tag{10.41}$$

62 Similarly

$$\begin{aligned} u_{43} &= \frac{x_2-x}{x_2-x_1}u_4 + \frac{x_1-x}{x_1-x_2}u_3 \\ &= \frac{a-x}{2a}u_4 + \frac{x+a}{2a}u_3 \end{aligned} \tag{10.42}$$

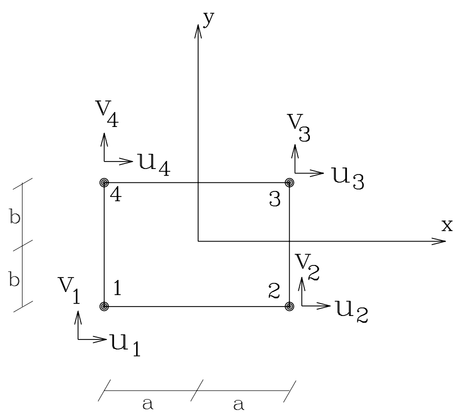


Figure 10.5: Constant Strain Quadrilateral Element

63 Next, we interpolate in the y direction along 1-4 and 2-3 between u_{12} and u_{43}. Again, we use Eq. 10.38 however this time we replace x by y:

$$
\begin{aligned}
u &= \frac{y_2 - y}{y_2 - y_1} u_{12} + \frac{y_1 - y}{y_1 - y_2} u_{43} \\
&= \frac{b-y}{2b}\frac{a-x}{2a} u_1 + \frac{b-y}{2b}\frac{x+a}{2a} u_2 + \frac{y+b}{2b}\frac{a-x}{2a} u_4 + \frac{y+b}{2b}\frac{x+a}{2a} u_3 \\
&= \underbrace{\frac{(a-x)(b-y)}{4ab}}_{N_1} u_1 + \underbrace{\frac{(a+x)(b-y)}{4ab}}_{N_2} u_2 + \underbrace{\frac{(a+x)(b+y)}{4ab}}_{N_3} u_3 + \underbrace{\frac{(a-x)(b+y)}{4ab}}_{N_1} u_4
\end{aligned}
\tag{10.43}
$$

64 One can easily check that at each node i the corresponding N_i is equal to 1, and all others to zero, and that at any point $N_1 + N_2 + N_3 + N_4 = 1$. Hence, the displacement field will be given by:

$$
\left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left[\begin{array}{cccccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \right\}
\tag{10.44}
$$

10.3.1.2 Solid Rectangular Trilinear Element

65 By extension to the previous derivation, the shape functions of a solid rectangular trilinear solid element, Fig. 10.6 will be given by:

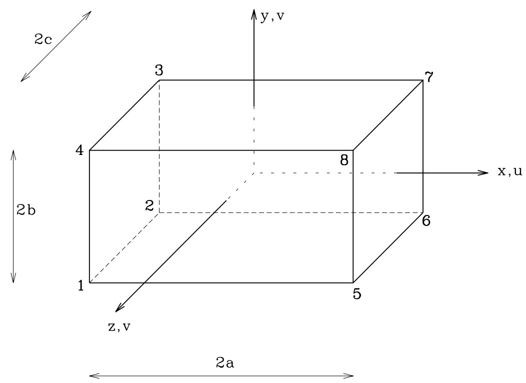


Figure 10.6: Solid Trilinear Rectangular Element

$$\left\{ \begin{array}{c} u \\ v \\ w \end{array} \right\} = \left[\begin{array}{cccccccccccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{array} \right\} \tag{10.45}$$

where

$$N_i = \frac{(a \pm x)(b \pm y)(c \pm z)}{8abc} \tag{10.46}$$

10.3.2 C^1: Hermitian Interpolation Functions

66 For problems involving the first derivative of the shape function, that is with C^1 interelement continuity (i.e continuity of first derivative or slope) such as for flexure, *Hermitian interpolation functions* rather than Lagrangian ones should be used.

Constant					a_1				
Linear				a_2x		a_3y			
Quadratic			a_4x^2		a_5xy		a_6y^2		
Cubic		a_7x^3		a_8x^2y		a_9xy^2		$a_{10}y^3$	
Quartic	$a_{11}x^4$		$a_{12}x^3y$		$a_{13}x^2y^2$		$a_{14}xy^3$		$a_{15}x^4$

(10.47)

Table 10.2: Interpretation of Shape Functions in Terms of Polynomial Series (1D & 2D)

Element	Terms	# of Nodes (terms)
Linear	a_1, a_2	2
Quadratic	a_1, a_2, a_4	3
Bi-Linear (triangle)	a_1, a_2, a_3,	3
Bi-Linear (quadrilateral)	a_1, a_2, a_3, a_5	4
Bi-Quadratic (Serendipity)	a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_9	8
Bi-Quadratic (Lagrangian)	a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{13}	9

Table 10.3: Polynomial Terms in Various Element Formulations (1D & 2D)

67 Hermitian interpolation functions are piecewise cubic functions which satisfy the conditions of displacement and slope (C^0, C^1) continuities. They are exensively used in CAD as Bezier curves.

10.4 Interpretation of Shape Functions in Terms of Polynomial Series

68 A schematic interpretation of shape functions in terms of polynomial series terms is given in Table 10.2.

69 Polynomial terms present in various element formulations is shown in Table 10.3

10.5 Characteristics of Shape Functions

1. The basis of derivation of shape functions could be:
   (a) A polynomial relation
      i. Exact
      ii. Approximation
   (b) Or other
      i. Logarithmic

ii. Trigonometric

2. Shape functions should
   (a) be continuous, of the type required by the variational principle.
   (b) exhibit rigid body motion (i.e. $v = a_1 + \ldots$)
   (c) exhibit constant strain.
3. Shape functions should be complete, and meet the same requirements as the coefficients of the Rayleigh Ritz method.
4. Shape functions can often be written in non-dimensional coordinates (i.e. $\xi = \frac{x}{l}$). This will be exploited later by Isoparametric elements.

Chapter 11

FINITE ELEMENT FORMULATION

27 Having introduced the virtual displacement method in chapter 9, the shape functions in chapter 10, and finally having reviewed the basic equations of elasticity in chapter 8, we shall present a general energy based formulation of the element stiffness matrix in this chapter.

28 Whereas chapter 2 derived the stiffness matrices of one dimensional rod elements, the approach used could not be generalized to general finite element. Alternatively, the derivation of this chapter will be applicable to both one dimensional rod elements or contnuum (2D or 3D) elements.

11.1 Strain Displacement Relations

29 The displacement Δ at any point inside an element can be written in terms of the shape functions $\lfloor \mathbf{N} \rfloor$ and the nodal displacements $\{\overline{\boldsymbol{\Delta}}\}$

$$\Delta(x) = \lfloor \mathbf{N}(x) \rfloor \{\overline{\boldsymbol{\Delta}}\} \tag{11.1}$$

The strain is then defined as:

$$\varepsilon(x) = [\mathbf{B}(x)]\{\overline{\boldsymbol{\Delta}}\} \tag{11.2}$$

where $[\mathbf{B}]$ is the matrix which relates joint displacements to strain field and is clearly expressed in terms of derivatives of $\mathbf{N}$.

11.1.1 Axial Members

$$u(x) = \underbrace{\lfloor \underbrace{(1-\frac{x}{L})}_{N_1} \quad \underbrace{\frac{x}{L}}_{N_2} \rfloor}_{\lfloor \mathbf{N} \rfloor} \underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}} \tag{11.3}$$

$$\varepsilon(x) = \varepsilon_x = \frac{du}{dx} = \underbrace{\left\lfloor \underbrace{-\frac{1}{L}}_{\frac{\partial N_1}{\partial x}} \quad \underbrace{\frac{1}{L}}_{\frac{\partial N_2}{\partial x}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}} \tag{11.4}$$

11.1.2 Flexural Members

30 Using the shape functions for flexural elements previously derived in Eq. 10.29 we have:

$$\varepsilon = \frac{y}{\rho} = y\frac{d^2v}{dx^2} \tag{11.5}$$

$$\frac{1}{\rho} = \frac{M}{EI} \tag{11.6}$$

$$= y\frac{d^2v}{dx^2} \tag{11.7}$$

$$= y \underbrace{\left\lfloor \underbrace{\frac{6}{L^2}(2\xi-1)}_{\frac{\partial^2 N_1}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi-2)}_{\frac{\partial^2 N_2}{\partial x^2}} \quad \underbrace{\frac{6}{L^2}(-2\xi+1)}_{\frac{\partial^2 N_3}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi-1)}_{\frac{\partial^2 N_4}{\partial x^2}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}} \tag{11.8}$$

11.2 Virtual Displacement and Strains

31 In anticipation of the application of the principle of virtual displacement, we define the vectors of virtual displacements and strain in terms of nodal displacements and shape functions:

$$\delta\Delta(x) = [\mathbf{N}(x)]\{\delta\overline{\mathbf{\Delta}}\} \tag{11.9}$$

$$\delta\varepsilon(x) = [\mathbf{B}(x)]\{\delta\overline{\mathbf{\Delta}}\} \tag{11.10}$$

11.3 Element Stiffness Matrix Formulation

32 In one dimensional elements with initial strain (temperature effect, support settlement, or other) such that:

$$\varepsilon_x = \underbrace{\frac{\sigma_x}{E}}_{\text{due to load}} + \underbrace{\varepsilon_x^0}_{\text{initial strain}} \tag{11.11}$$

thus:

$$\sigma_x = E\varepsilon_x - E\varepsilon_x^0 \tag{11.12}$$

33 Generalizing, and in matrix form:

$$\boxed{\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\epsilon}\} - [\mathbf{D}]\{\boldsymbol{\epsilon}^0\}} \tag{11.13}$$

where $[\mathbf{D}]$ is the constitutive matrix which relates stress and strain vectors.

34 The element will be subjected to a load $q(x)$ acting on its surface

35 Let us now apply the principle of virtual displacement and restate some known relaations:

$$\delta U = \delta W \tag{11.14}$$

$$\delta U = \int_\Omega \lfloor \delta\boldsymbol{\epsilon} \rfloor \{\boldsymbol{\sigma}\} d\Omega \tag{11.15}$$

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\epsilon}\} - [\mathbf{D}]\{\boldsymbol{\epsilon}^0\} \tag{11.16}$$

$$\{\boldsymbol{\epsilon}\} = [\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\} \tag{11.17}$$

$$\{\delta\boldsymbol{\epsilon}\} = [\mathbf{B}]\{\delta\overline{\boldsymbol{\Delta}}\} \tag{11.18}$$

$$\lfloor \delta\boldsymbol{\epsilon} \rfloor = \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T \tag{11.19}$$

36 Combining Eqns. 11.14, 11.15, 11.16, 11.19, and 11.17, the internal virtual strain energy is given by:

$$\begin{aligned}\delta U &= \int_\Omega \underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T}_{\{\delta\boldsymbol{\epsilon}\}} \underbrace{[\mathbf{D}][\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\}}_{\{\boldsymbol{\sigma}\}} d\Omega - \int_\Omega \underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T}_{\{\delta\boldsymbol{\epsilon}\}} \underbrace{[\mathbf{D}]\{\boldsymbol{\epsilon}^0\}}_{\{\boldsymbol{\sigma}^0\}} d\Omega \\ &= \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, d\Omega \{\overline{\boldsymbol{\Delta}}\} - \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\varepsilon^i\} d\Omega\end{aligned} \tag{11.20}$$

37 the virtual external work in turn is given by:

$$\delta W = \underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor}_{\text{Virt. Nodal Displ.}} \underbrace{\{\mathbf{F}\}}_{\text{Nodal Force}} + \int_l \lfloor \delta\Delta \rfloor q(x) dx \tag{11.21}$$

38 combining this equation with:

$$\{\delta\Delta\} = [\mathbf{N}]\{\delta\overline{\boldsymbol{\Delta}}\} \tag{11.22}$$

yields:

$$\delta W = \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \{\mathbf{F}\} + \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \int_0^l [\mathbf{N}]^T q(x)\, dx \tag{11.23}$$

39 Equating the internal strain energy Eqn. 11.20 with the external work Eqn. 11.23, we obtain:

$$\underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, d\Omega}_{[\mathbf{k}]} \{\overline{\boldsymbol{\Delta}}\} - \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\varepsilon^i\} d\Omega}_{\{\mathbf{F}^0\}}}_{\delta U} =$$

$$\underbrace{\lfloor\delta\overline{\boldsymbol{\Delta}}\rfloor\{\mathbf{F}\} + \lfloor\delta\overline{\boldsymbol{\Delta}}\rfloor \underbrace{\int_0^l [\mathbf{N}]^T q(x)\,dx}_{\{\mathbf{F}^e\}}}_{\delta W} \tag{11.24}$$

40 Cancelling out the $\lfloor\delta\overline{\boldsymbol{\Delta}}\rfloor$ term, this is the same equation of equilibrium as the one written earlier on. It relates the (unknown) nodal displacement $\left\{\overline{\boldsymbol{\Delta}}\right\}$, the structure stiffness matrix $[\mathbf{k}]$, the external nodal force vector $\{\mathbf{F}\}$, the distributed element force $\{\mathbf{F}^e\}$, and the vector of initial displacement.

41 From this relation we define:

The element stiffness matrix:

$$\boxed{[\mathbf{k}] = \int_\Omega [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]d\Omega} \tag{11.25}$$

Element initial force vector:

$$\boxed{\{\mathbf{F}^i\} = \int_\Omega [\mathbf{B}]^T[\mathbf{D}]\{\varepsilon^i\}d\Omega} \tag{11.26}$$

Element equivalent load vector:

$$\boxed{\{\mathbf{F}^e\} = \int_0^l [\mathbf{N}]\,q(x)\,dx} \tag{11.27}$$

and the general equation of equilibrium can be written as:

$$\boxed{[\mathbf{k}]\{\boldsymbol{\Delta}\} - \{\mathbf{F}^0\} = \{\mathbf{F}\} + \{\mathbf{F}^e\}} \tag{11.28}$$

11.3.1 Stress Recovery

42 Whereas from the preceding section, we derived a general relationship in which the nodal displacements are the primary unknowns, we next seek to determine the internal (generalized) stresses which are most often needed for design.

43 Recalling that we have:

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\epsilon}\} \tag{11.29}$$

$$\{\boldsymbol{\epsilon}\} = [\mathbf{B}]\{\boldsymbol{\Delta}\} \tag{11.30}$$

With the vector of nodal displacement $\{\boldsymbol{\Delta}\}$ known, those two equations would yield:

$$\boxed{\{\boldsymbol{\sigma}\} = [\mathbf{D}]\cdot[\mathbf{B}]\{\boldsymbol{\Delta}\}} \tag{11.31}$$

Chapter 12

SOME FINITE ELEMENTS

12.1 Introduction

27 Having first introduced the method of virtual displacements in Chapter 9, than the shape functions [**N**] (Chapter 10) which relate internal to external nodal displacements, than the basic equations of elasticity (Chapter 8) which defined the [**D**] matrix, and finally having applied the virtual displacement method to finite element in chapter 11, we now revisit some one dimensional element whose stiffness matrix was earlier derived, and derive the stiffness matrices of additional two dimensional finite elements.

12.2 Truss Element

28 The shape functions of the truss element were derived in Eq. 10.13:

$$\begin{aligned} N_1 &= 1 - \frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned}$$

29 The corresponding strain displacement relation [**B**] is given by:

$$\begin{aligned} \varepsilon_x &= \frac{d\mathbf{u}}{dx} \\ &= \left[\ \frac{dN_1}{dx} \quad \frac{dN_2}{dx}\ \right] \\ &= \underbrace{\left[\ -\frac{1}{L} \quad \frac{1}{L}\ \right]}_{[\mathbf{B}]} \end{aligned} \tag{12.1}$$

30 For the truss element, the constitutive matrix [**D**] reduces to the scalar E; Hence, substituting into Eq. 11.25 $[\mathbf{k}] = \int_\Omega [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]d\Omega$ and with $d\Omega = Adx$ for element with constant cross

sectional area we obtain:

$$[\mathbf{k}] = A\int_0^L \left\{ \begin{array}{c} -\frac{1}{L} \\ \frac{1}{L} \end{array} \right\} \cdot E \cdot \lfloor\ -\frac{1}{L} \quad \frac{1}{L}\ \rfloor dx$$

$$\boxed{\begin{array}{l} [\mathbf{k}] = \dfrac{AE}{L^2}\displaystyle\int_0^L \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] dx \\ = \frac{AE}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \end{array}} \tag{12.2}$$

31 We observe that this stiffness matrix is identical to the one earlier derived in Eq. 2.45.

12.3 Flexural Element

32 For a beam element, for which we have previously derived the shape functions in Eq. 10.29 and the [**B**] matrix in Eq. 11.8, substituting in Eq. 11.25:

$$[\mathbf{k}] = \int_0^l \int_A [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, y^2\, dA\, dx \tag{12.3}$$

and noting that $\displaystyle\int_A y^2\, dA = I_z$ Eq. 11.25 reduces to

$$[\mathbf{k}] = \int_0^l [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}] I_z\, dx \tag{12.4}$$

33 For this simple case, we have: $[\mathbf{D}] = E$, thus:

$$[\mathbf{k}] = EI_z \int_0^l [\mathbf{B}]^T [\mathbf{B}]\, dx \tag{12.5}$$

34 Using the shape function for the beam element from Eq. 10.29, and noting the change of integration variable from dx to $d\xi$, we obtain

$$[\mathbf{k}] = EI_z \int_0^1 \left\{ \begin{array}{c} \frac{6}{L^2}(2\xi - 1) \\ -\frac{2}{L}(3\xi - 2) \\ \frac{6}{L^2}(-2\xi + 1) \\ -\frac{2}{L}(3\xi - 1) \end{array} \right\} \lfloor\ \frac{6}{L^2}(2\xi - 1) \quad -\frac{2}{L}(3\xi - 2) \quad \frac{6}{L^2}(-2\xi + 1) \quad -\frac{2}{L}(3\xi - 1)\ \rfloor \underbrace{L d\xi}_{dx} \tag{12.6}$$

or

$$\boxed{[\mathbf{k}] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad \theta_1 \quad & \quad v_2 \quad & \quad \theta_2 \quad \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right] \end{array}} \tag{12.7}$$

Which is identical to the beam stiffness matrix derived in Eq. 2.45 from equilibrium relations.

12.4 Triangular Element

35 Having retrieved the stiffness matrices of simple one dimensional elements using the principle of virtual displacement, we next consider two dimensional continuum elements starting with the triangular element of constant thickness t made out of isotropic linear elastic material. The element will have two d.o.f's at each node:

$$\left\{\overline{\mathbf{\Delta}}\right\} = \lfloor\; u_1 \quad u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \;\rfloor^t \tag{12.8}$$

12.4.1 Strain-Displacement Relations

36 The strain displacement relations is required to determine $[\mathbf{B}]$

37 For the 2D plane elasticity problem, the strain vector $\{\boldsymbol{\epsilon}\}$ is given by:

$$\{\boldsymbol{\epsilon}\} = \lfloor\; \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \;\rfloor^t \tag{12.9}$$

hence we can rewrite the strains in terms of the derivatives of the shape functions through the matrix $[\mathbf{B}]$:

$$\left\{\begin{array}{c}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{array}\right\} = \underbrace{\left[\begin{array}{cc}\frac{\partial\mathbf{N}}{\partial x} & 0\\ 0 & \frac{\partial\mathbf{N}}{\partial y}\\ \frac{\partial\mathbf{N}}{\partial y} & \frac{\partial\mathbf{N}}{\partial x}\end{array}\right]}_{[\mathbf{B}]} \underbrace{\left\{\begin{array}{c}\mathbf{u}\\ \mathbf{v}\end{array}\right\}}_{\left\{\overline{\mathbf{\Delta}}\right\}} \tag{12.10}$$

38 We note that because we have 3 u and 3 v displacements, the size of $[\mathbf{B}]$ and $[\overline{\mathbf{\Delta}}]$ are 3×6 and 6×1 respectively.

39 Differentiating the shape functions from Eq. 10.35 we obtain:

$$\underbrace{\left\{\begin{array}{c}\varepsilon_x\\ \hline \varepsilon_y\\ \hline \gamma_{xy}\end{array}\right\}}_{\{\boldsymbol{\epsilon}\}} = \underbrace{\left[\begin{array}{ccc|ccc}
\underbrace{-\frac{1}{x_2}}_{\frac{\partial N_1}{\partial x}} & \underbrace{\frac{1}{x_2}}_{\frac{\partial N_2}{\partial x}} & \underbrace{0}_{\frac{\partial N_3}{\partial x}} & 0 & 0 & 0\\ \hline
0 & 0 & 0 & \underbrace{\frac{x_3-x_2}{x_2y_3}}_{\frac{\partial N_1}{\partial y}} & \underbrace{-\frac{-x_3}{x_2y_3}}_{\frac{\partial N_2}{\partial y}} & \underbrace{\frac{1}{y_3}}_{\frac{\partial N_3}{\partial y}}\\ \hline
\underbrace{\frac{x_3-x_2}{x_2y_3}}_{\frac{\partial N_1}{\partial y}} & \underbrace{-\frac{-x_3}{x_2y_3}}_{\frac{\partial N_1}{\partial y}} & \underbrace{\frac{1}{y_3}}_{\frac{\partial N_3}{\partial x}} & \underbrace{-\frac{1}{x_2}}_{\frac{\partial N_1}{\partial x}} & \underbrace{\frac{1}{x_2}}_{\frac{\partial N_2}{\partial x}} & \underbrace{0}_{\frac{\partial N_3}{\partial x}}
\end{array}\right]}_{[\mathbf{B}]} \underbrace{\left\{\begin{array}{c}u_1\\ u_2\\ u_3\\ \hline v_1\\ v_2\\ v_3\end{array}\right\}}_{\left\{\overline{\mathbf{\Delta}}\right\}} \tag{12.11}$$

12.4.2 Stiffness Matrix

40 With the constitutive matrix [**D**] given by Eq. **??**, the strain-displacement relation [**B**] by Eq. 12.11, we can substitute those two quantities into the general equation for stiffness matrix, Eq. 11.25:

$$
\begin{aligned}
[\mathbf{k}] &= \int_\Omega [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]d\Omega \\
&= \int_\Omega \underbrace{\left[\begin{array}{c|c|c} -\frac{1}{x_2} & 0 & \frac{x_3-x_2}{x_2y_3} \\ \frac{1}{x_2} & 0 & -\frac{-x_3}{x_2y_3} \\ 0 & 0 & \frac{1}{y^3} \\ \hline 0 & \frac{x_3-x_2}{x_2y_3} & -\frac{1}{x_2} \\ 0 & -\frac{-x_3}{x_2y_3} & \frac{1}{x_2} \\ 0 & \frac{1}{y_3} & 0 \end{array}\right]}_{[\mathbf{B}]^T} \underbrace{\frac{E}{1-\nu^2}\left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2}) \end{array}\right]}_{[\mathbf{D}]} \\
& \quad \underbrace{\left[\begin{array}{ccc|ccc} -\frac{1}{x_2} & \frac{1}{x_2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{x_3-x_2}{x_2y_3} & -\frac{-x_3}{x_2y_3} & \frac{1}{y_3} \\ \hline \frac{x_3-x_2}{x_2y_3} & -\frac{-x_3}{x_2y_3} & \frac{1}{y_3} & -\frac{1}{x_2} & \frac{1}{x_2} & 0 \end{array}\right]}_{[\mathbf{B}]} \underbrace{tdxdy}_{dvol} \qquad (12.12) \\
&= \gamma \left[\begin{array}{cccccc} y_3^2+\alpha x_{3-2}^2 & -y_3^2-\alpha x_3x_{3-2} & \alpha x_2x_{3-2} & -\beta y_3x_{3-2} & \nu x_3y_3+\alpha y_3x_{3-2} & -\nu x_2y_3 \\ -y_3^2-\alpha x_3x_{3-2} & y_3^2+\alpha x_3^2 & -\alpha x_2x_3 & \nu y_3x_{3-2}+\alpha x_3y_3 & -\beta x_3y_3 & \nu x_2y_3 \\ \alpha x_2x_{3-2} & -\alpha x_2x_3 & \alpha x_2^2 & -\alpha x_2y_3 & \alpha x_2y_3 & 0 \\ -\beta y_3x_{3-2} & \nu y_3x_{3-2}+\alpha x_3y_3 & -\alpha x_2y_3 & \alpha y_3^2+x_{3-2}^2 & -\alpha y_3^2-x_3x_{3-2} & x_2x_{3-2} \\ \nu x_3y_3+\alpha y_3x_{3-2} & -\beta x_3y_3 & \alpha x_2y_3 & -\alpha y_3^2-x_3x_{3-2} & \alpha y_3^2+x_3^2 & -x_2x_3 \\ -\nu x_2y_3 & \nu x_2y_3 & 0 & x_2x_{3-2} & -x_2x_3 & x_2^2 \end{array}\right]
\end{aligned}
$$

where $\alpha = \frac{1-\nu}{2}$, $\beta = \frac{1+\nu}{2}$, $\gamma = \frac{ET}{2(1-\nu^2)x_2y_3}$, $x_{3-2} = x_3 - x_2$, and $y_{3-2} = y_3 - y_2$.

12.4.3 Internal Stresses

41 Recall from Eq. 11.31 that $\{\boldsymbol{\sigma}\} = [\mathbf{D}]\cdot[\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\}$ hence for this particular element we will have:

$$
\underbrace{\left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right\}}_{\{\boldsymbol{\sigma}\}} = \underbrace{\frac{E}{1-\nu^2}\left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2}) \end{array}\right]}_{[\mathbf{D}]} \underbrace{\left[\begin{array}{cccccc} -\frac{1}{x_2} & \frac{1}{x_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_3-x_2}{x_2y_3} & -\frac{-x_3}{x_2y_3} & \frac{1}{y_3} \\ \frac{x_3-x_2}{x_2y_3} & -\frac{-x_3}{x_2y_3} & \frac{1}{y_3} & -\frac{1}{x_2} & \frac{1}{x_2} & 0 \end{array}\right]}_{[\mathbf{B}]} \underbrace{\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline v_1 \\ v_2 \\ v_3 \end{array}\right\}}_{\{\overline{\boldsymbol{\Delta}}\}}
$$

$$
= \kappa \begin{bmatrix} -y_3 & y_3 & 0 & \nu x_{3-2} & -\nu x_3 & \nu x_2 \\ -\nu y_3 & \nu y_3 & 0 & x_{3-2} & -x_3 & x_2 \\ \alpha x_{3-2} & -\alpha x_3 & \alpha x_2 & -\alpha y_3 & \alpha y_3 & 0 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline v_1 \\ v_2 \\ v_3 \end{array} \right\} \tag{12.13}
$$

where $\kappa = \frac{E}{(1-\nu^2)x_2y_3}$

42 We should note that for this element the stress is independent of x and y because a linear displacement relation was assumed resulting in a constant strain and stress (for linear elastic material).

12.4.4 Observations

43 For this element we should note that:

1. Both σ and ε are constants
2. Interelement equilibrium conditions are not satisfied
3. Interelement continuity of displacement is satisfied

12.5 Quadrilateral Element

Chapter 13

GEOMETRIC NONLINEARITY

13.1 Strong Form

27 Column buckling theory originated with Leonhard Euler in 1744.

28 An initially straight member is concentrically loaded, and all fibers remain elastic until buckling occur.

29 For buckling to occur, it must be assumed that the column is slightly bent as shown in Fig. 13.1. Note, in reality no column is either perfectly straight, and in all cases a minor imperfection

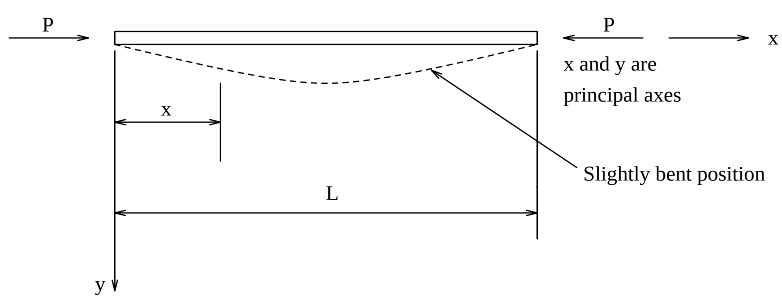


Figure 13.1: Euler Column

is present.

13.1.1 Lower Order Differential Equation

30 At any location x along the column, the imperfection in the column compounded by the concentric load P, gives rise to a moment

$$M_z = -Py \tag{13.1}$$

Note that the value of y is irrelevant.

31 Recalling that

$$\frac{d^2y}{dx^2} = \frac{M_z}{EI} \tag{13.2}$$

upon substitution, we obtain the following differential equation

$$\frac{d^2y}{dx^2} - \frac{P}{EI}y = 0 \tag{13.3}$$

32 Letting $k^2 = \frac{P}{EI}$, the solution to this second-order linear differential equation is

$$y = -A \sin kx - B \cos kx \tag{13.4}$$

33 The two constants are determined by applying the boundary conditions

1. $y = 0$ at $x = 0$, thus $B = 0$
2. $y = 0$ at $x = L$, thus

$$A \sin kL = 0 \tag{13.5}$$

34 This last equation can be satisfied if: 1) $A = 0$, that is there is no deflection; 2) $kL = 0$, that is no applied load; or 3)

$$kL = n\pi \tag{13.6}$$

Thus buckling will occur if $\frac{P}{EI} = \left(\frac{n\pi}{L}\right)^2$ or

$$P = \frac{n^2\pi^2 EI}{L^2}$$

35 The fundamental buckling mode, i.e. a single curvature deflection, will occur for $n = 1$; Thus Euler critical load for a pinned column is

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{13.7}$$

36 The corresponding critical stress is

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}} \tag{13.8}$$

where $I = Ar^2$.

37 Note that buckling will take place with respect to the weakest of the two axis.

13.1.2 Higher Order Differential Equation

38 In the preceding approach, the buckling loads were obtained for a column with specified boundary conditons. A second order differential equation, valid specifically for the member being analyzed was used.

39 In the next approach, we derive a single fourth order equation which will be applicable to any column regardelss of the boundary conditions.

40 Considering a beam-column subjected to axial and shear forces as well as a moment, Fig. 13.2, taking the moment about i for the beam segment and assuming the angle $\frac{dv}{dx}$ between the axis of the beam and the horizontal axis is small, leads to

$$M - \left(M + \frac{dM}{dx}dx\right) + w\frac{(dx)^2}{2} + \left(V + \frac{dV}{dx}\right)dx - P\left(\frac{dv}{dx}\right)dx = 0 \tag{13.9}$$

41 Neglecting the terms in dx^2 which are small, and then differentiating each term with respect to x, we obtain

$$\frac{d^2M}{dx^2} - \frac{dV}{dx} - P\frac{d^2v}{dx^2} = 0 \tag{13.10}$$

42 However, considering equilibrium in the y direction gives

$$\frac{dV}{dx} = -w \tag{13.11}$$

43 From beam theory, neglecting axial and shear deformations, we have

$$M = -EI\frac{d^2v}{dx^2} \tag{13.12}$$

44 Substituting Eq. 13.11 and 13.12 into 13.10, and assuming a beam of uniform cross section, we obtain

$$\boxed{EI\frac{d^4v}{dx^4} - P\frac{d^2v}{dx^2} = w} \tag{13.13}$$

45 Introdcing $k^2 = \frac{P}{EI}$, the general solution of this fourth order differential equation to any set of boundary conditions is

$$v = C_1\sin kx + C_2\cos kx + C_3x + C_4 \tag{13.14}$$

46 If we consider again the stability of a hinged-hinged column, the boundary conditions are

$$\begin{array}{ccccccccc} v & = & 0, & v_{,xx} & = & 0 & \text{at} & x = & 0 \\ v & = & 0, & v_{,xx} & = & 0 & \text{at} & x = & L \end{array} \tag{13.15}$$

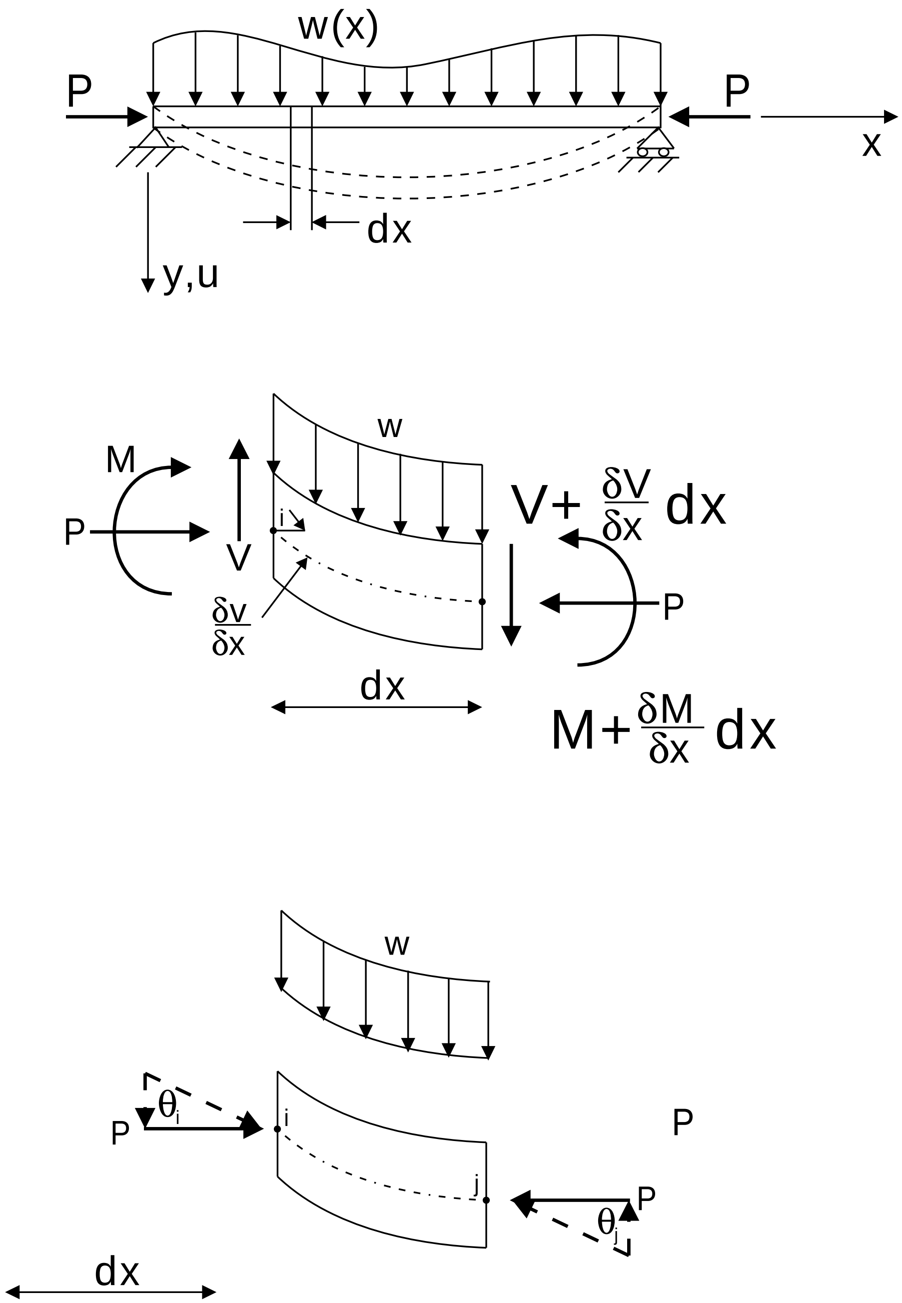


Figure 13.2: Simply Supported Beam Column; Differential Segment; Effect of Axial Force P

substitution of the two conditions at $x = 0$ leads to $C_2 = C_4 = 0$. From the remaining conditions, we obtain

$$\begin{aligned} C_1 \sin kL + C_3 L &= 0 && \text{(13.16-a)} \\ -C_1 k^2 \sin kl &= 0 && \text{(13.16-b)} \end{aligned}$$

these relations are satisfied either if $C_1 = C_3 = 0$ or if $\sin kl = C_3 = 0$. The first alternative leads to the trivial solution of equilibrium at all loads, and the second to $kL = n\pi$ for $n = 1, 2, 3 \cdots$. For $n = 1$, the critical load is

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{13.17}$$

which was derived earlier using the lower order differential equation.

47 Next we consider a column with one end fixed (at $x = 0$), and one end hinged (at $x = L$). The boundary conditions are

$$\begin{array}{rclrclrcl} v &=& 0, & v_{,xx} &=& 0 & \text{at } x &=& 0 \\ v &=& 0, & v_{,x} &=& 0 & \text{at } x &=& L \end{array} \tag{13.18}$$

These boundary conditions will yield $C_2 = C_4 = 0$, and

$$\sin kL - kL \cos kL = 0 \tag{13.19}$$

But since $\cos kL$ can not possibly be equal to zero, the preceding equation can be reduced to

$$\tan kL = kL \tag{13.20}$$

which is a transcendental algebraic equation and can only be solved numerically. We are essentially looking at the intersection of $y = x$ and $y = \tan x$, Fig. 13.3 and the smallest positive root is $kL = 4.4934$, since $k^2 = \frac{P}{EI}$, the smallest critical load is

$$\boxed{P_{cr} = \frac{(4.4934)^2}{L^2} EI = \frac{\pi^2}{(0.699L)^2} EI} \tag{13.21}$$

Note that if we were to solve for x such that $v_{,xx} = 0$ (i.e. an inflection point), then $x = 0.699L$.

48 We observe that in using the higher order differential equation, we can account for both natural and essential boundary conditions.

13.1.3 Slenderness Ratio

49 For different boundary conditions, we define the slenderness ratio

$$\lambda = \frac{l_e}{r}$$

where l_e is the effective length and is equal to $l_e = kl$ and r the radius of gyration ($r = \sqrt{\frac{I}{A}}$).

50 l_e is the distance between two adjacent (fictitious or actual) inflection points, Fig. 13.4

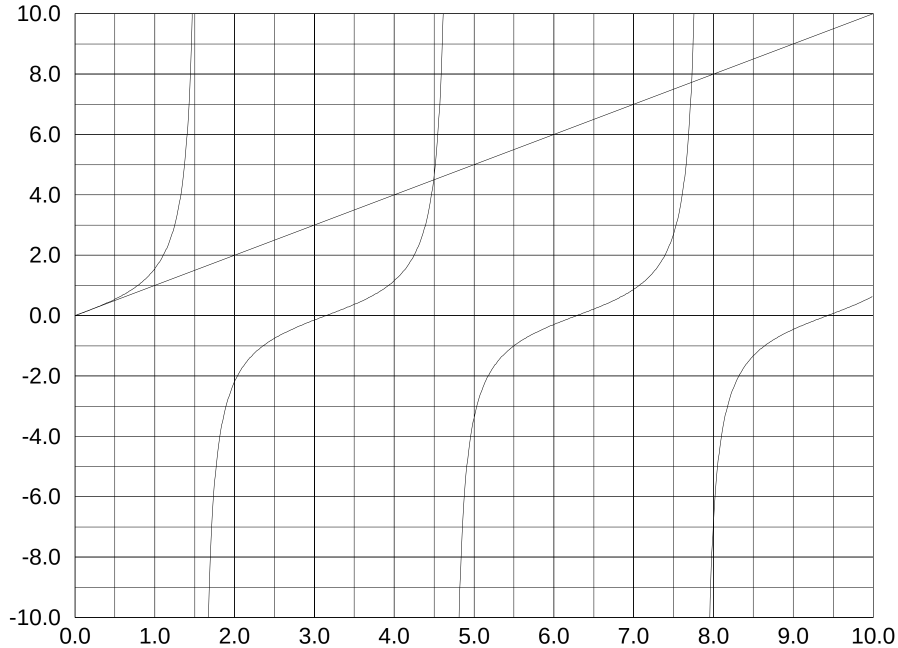


Figure 13.3: Solution of the Tanscendental Equation for the Buckling Load of a Fixed-Hinged Column

13.2 Weak Form

13.2.1 Strain Energy

51 Considering a uniform section prismatic element, Fig. **??**, subjected to axial and flexural deformation (no shear), the Lagrangian finite strain-displacement relation is given by **??**

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \tag{13.22}$$

thus, the total strain would be

$$\varepsilon_{xx} = \underbrace{\frac{du}{dx}}_{\text{Axial}} \underbrace{-y\left(\frac{d^2v}{dx^2}\right)}_{\text{Flexure}} + \underbrace{\frac{1}{2}\left(\frac{dv}{dx}\right)^2}_{\text{Large Deformation}} \tag{13.23}$$

52 we note that the first and second terms are the familiar components of axial and flexural strains respectively, and the third one (which is nonlinear) is obtained from large-deflection strain-displacement.

53 The Strain energy of the element is given by

$$U^e = \frac{1}{2}\int_\Omega E\varepsilon_{xx}^2 d\Omega \tag{13.24}$$

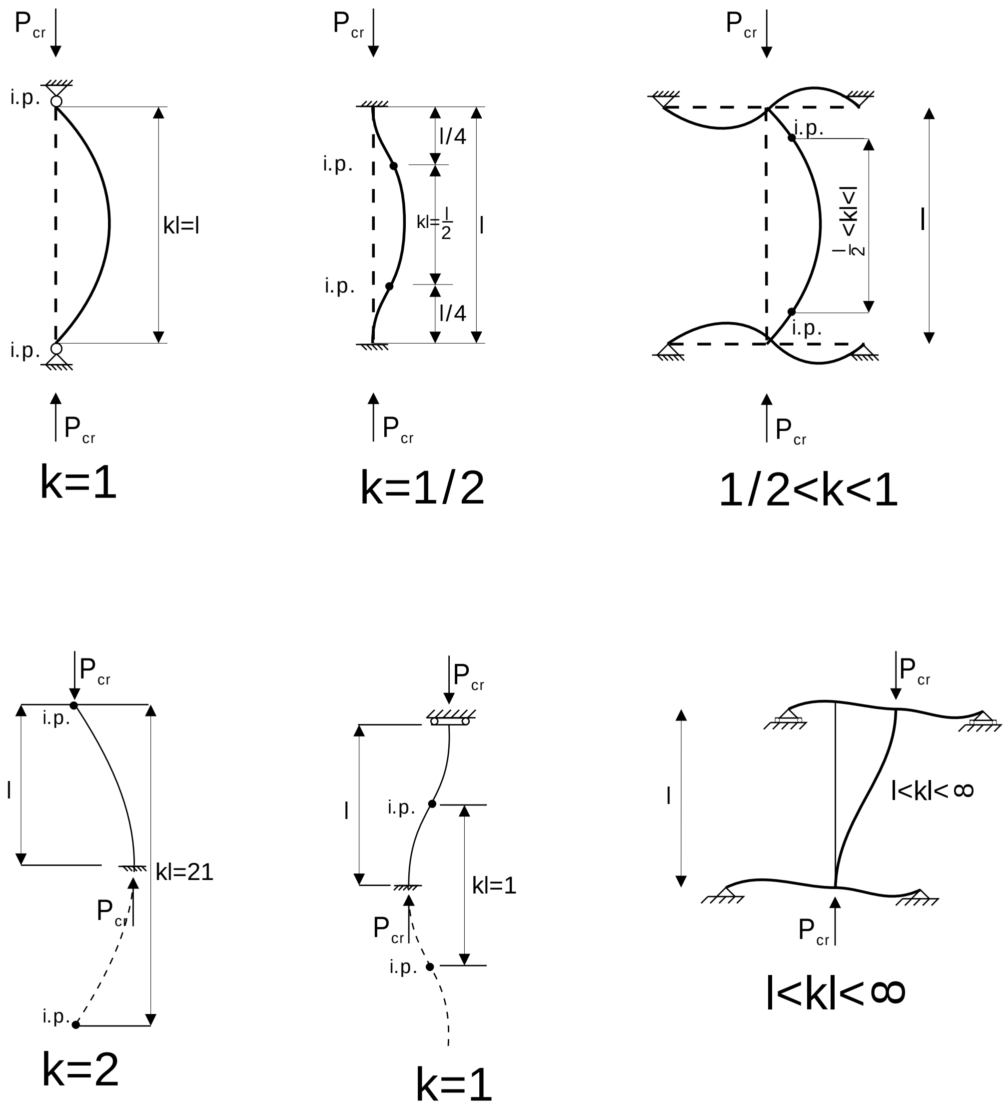


Figure 13.4: Critical lengths of columns

54 Substituting Eq. 13.23 into U^e we obtain

$$\begin{aligned} U^e &= \frac{1}{2}\int_L\int_A \left[\left(\frac{du}{dx}\right)^2 + y^2\left(\frac{d^2v}{dx^2}\right)^2 + \frac{1}{4}\left(\frac{dv}{dx}\right)^4 - 2y\left(\frac{du}{dx}\right)\left(\frac{d^2v}{dx^2}\right)\right. \\ &\left. -y\left(\frac{d^2v}{dx^2}\right)\left(\frac{dv}{dx}\right)^2 + \left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right)^2\right] EdAdx \end{aligned} \tag{13.25-a}$$

55 Noting that

$$\int_A dA = A; \quad \int_A ydA = 0; \quad \int_A y^2 dA = I \tag{13.26}$$

for y measured from the centroid, U^e reduces to

$$U^e = \frac{1}{2}\int_L E\left[A\left(\frac{du}{dx}\right)^2 + I\left(\frac{d^2v}{dx^2}\right)^2 + \frac{A}{4}\left(\frac{dv}{dx}\right)^4 + A\left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right)^2\right]dx \tag{13.27}$$

We discard the highest order term $\frac{A}{4}\left(\frac{dv}{dx}\right)^4$ in order to transform the above equation into a linear instability formulation.

56 Under the assumption of an independent prebuckling analysis for axial loading, the axial load P_x is

$$P_x = EA\frac{du}{dx} \tag{13.28}$$

Thus Eq. 13.27 reduces to

$$U^e = \frac{1}{2}\int_L \left[EA\left(\frac{du}{dx}\right)^2 + EI\left(\frac{d^2v}{dx^2}\right)^2 + P_x\left(\frac{dv}{dx}\right)^2\right]dx \tag{13.29}$$

57 We can thus decouple the strain energy into two components, one associated with axial and the other with flexural deformations

$$U^e = U_a^e + U_f^e \tag{13.30-a}$$

$$U_a^e = \frac{1}{2}\int_L EA\left(\frac{du}{dx}\right)^2 dx \tag{13.30-b}$$

$$U_f^e = \frac{1}{2}\int_L \left[EI\left(\frac{d^2v}{dx^2}\right)^2 + P_x\left(\frac{dv}{dx}\right)^2\right]dx \tag{13.30-c}$$

13.2.2 Euler Equation

58 Recall, from Eq. 9.15 that a functional in terms of two field variables (u and v) with higher order derivatives of the form

$$\Pi = \int\int F(x, y, u, v, u_{,x}, u_{,y}, v_{,x}, v_{,y}, \cdots, v_{,yy})dxdy \tag{13.31}$$

There would be as many Euler equations as dependent field variables, Eq. 9.16

$$\left\{ \begin{array}{lcl} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial u_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial u_{,yy}} & = & 0 \\ \frac{\partial F}{\partial v} \boxed{- \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{,y}} \boxed{+ \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial v_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{,yy}} & = & 0 \end{array} \right. \tag{13.32}$$

59 For the problem at hand, those two equations reduce to

$$U_f^e = \int_L \underbrace{\frac{1}{2}\left[EIv_{,xx}^2 + P_x v_{,x}^2\right]}_{F} dx \tag{13.33}$$

and the corresponding Euler equation will be

$$-\frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}} = 0 \tag{13.34}$$

The terms of the Euler Equation are given by

$$\frac{\partial F}{\partial v_{,x}} = P_x v_{,x} \tag{13.35-a}$$

$$\frac{\partial F}{\partial v_{,xx}} = EIv_{,xx} \tag{13.35-b}$$

Substituting into the Euler equation, and assuming constant P_x, and EI, we obtain

$$\boxed{EI\frac{d^4v}{dx^4} - P_x\frac{d^2v}{dx^2} = 0} \tag{13.36}$$

which is identical to Eq. 13.13

13.2.3 Discretization

60 Assuming a functional representation of the transverse displacements in terms of the four joint displacements

$$v = \mathbf{N}\overline{\mathbf{v}} \tag{13.37-a}$$

$$\frac{dv}{dx} = \mathbf{N}_{,x}\overline{\mathbf{v}} \tag{13.37-b}$$

$$\frac{d^2v}{dx^2} = \mathbf{N}_{,xx}\overline{\mathbf{v}} \tag{13.37-c}$$

61 Substituting this last equation into Eq. 13.30-c, the element potential energy is given by

$$\begin{aligned}\Pi^e &= U_f^e + W^e && (13.38\text{-a})\\ &= \frac{1}{2}\lfloor \overline{\mathbf{v}}_e \rfloor [\mathbf{k}_e] \{\overline{\mathbf{v}}_e\} + \frac{1}{2}\lfloor \overline{\mathbf{v}}_e \rfloor [\mathbf{k}_g] \{\overline{\mathbf{v}}_e\} - \lfloor \overline{\mathbf{v}} \rfloor \{\mathbf{P}\} && (13.38\text{-b})\end{aligned}$$

where

$$[\mathbf{k}_e] = \left[\int_L EI \{\mathbf{N}_{,xx}\} \lfloor \mathbf{N}_{,xx} \rfloor dx\right] \tag{13.39}$$

and

$$\boxed{[\mathbf{k}_g] = \left[P \int_L \{\mathbf{N}_{,x}\} \lfloor \mathbf{N}_{,x} \rfloor dx\right]} \tag{13.40}$$

where $[\mathbf{k}_e]$ is the conventional element flexural stiffness matrix.

62 $[\mathbf{k}_g]$ introduces the considerations related to elastic instability. We note that its terms solely depend on geometric parameters (length), therefore this matrix is often referred to as the *geometric* stiffness matrix.

63 Using the shape functions for flexural elements, Eq. 10.29, and substituting into Eq. 13.39 and Eq. 13.40 we obtain

$$\mathbf{k}_e = \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right] \end{array} \tag{13.41}$$

which is the same element stiffness matrix derived earlier in Eq. 12.7.

64 The geometric stiffness matrix is given by

$$\boxed{\mathbf{k}_g = \frac{P}{L} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{array}\right] \end{array}} \tag{13.42}$$

65 The equilibrium relation is thus

$$\mathbf{k}\overline{\mathbf{v}} = \overline{\mathbf{P}} \tag{13.43}$$

where the element stiffness matrix is expressed in terms of both the elastic and geometric components)

$$\boxed{\mathbf{k} = \mathbf{k}_e + \mathbf{k}_g} \tag{13.44}$$

66 In a global formulation,we would have

$$\boxed{\mathbf{K} = \mathbf{K}_e + \mathbf{K}_g} \tag{13.45}$$

67 We assume that *conservative loading* is applied, that is the direction of the load does not "follow" the deflected direction of the member upon which it acts.

13.3 Elastic Instability

68 In elastic instability, the intensity of the axial load system to cause buckling is yet unknown, the incremental stiffness matrix must first be numerically evaluated using an arbitrary chosen load intensity (since $\mathbf{K}_g$ is itself a function of P).

69 For buckling to occur, the intensity of the axial load system must be λ times the initially arbitrarily chosen intensity of the force. Note that for a structure, the initial distribution of $\overline{\mathbf{P}}^*$ must be obtained from a linear elastic analysis. Hence, the buckling load, $\overline{\mathbf{P}}$ is given by

$$\overline{\mathbf{P}} = \lambda \overline{\mathbf{P}}^* \tag{13.46}$$

70 Since the geometric stiffness matrix is proportional to the internal forces at the start, it follows that

$$\mathbf{K}_g = \lambda \mathbf{K}_g^* \tag{13.47}$$

where $\mathbf{K}_g^*$ corresponds to the geometric stiffness matrix for unit values of the applied loading ($\lambda = 1$).

71 The elastic stiffness matrix $\mathbf{K}_e$ remains a constant, hence we can write

$$(\mathbf{K}_e + \lambda \mathbf{K}_g^*)\overline{\mathbf{v}} - \lambda \overline{\mathbf{P}}^* = 0$$

72 The displacements are in turn given by

$$\overline{\mathbf{v}} = (\mathbf{K}_e + \lambda \mathbf{K}_g^*)^{-1} \lambda \overline{\mathbf{P}}^*$$

and for the displacements to tend toward infinity, then

$$\boxed{|\mathbf{K}_e + \lambda \mathbf{K}_g^*| = 0} \tag{13.48}$$

which can also be expressed as

$$\boxed{|\mathbf{K}_g^{-1}\mathbf{K}_e + \lambda\mathbf{I}| = 0} \tag{13.49}$$

73 Alternatively, it can simply be argued that there is no unique solution (bifurcation condition) to $\overline{\mathbf{v}}$.

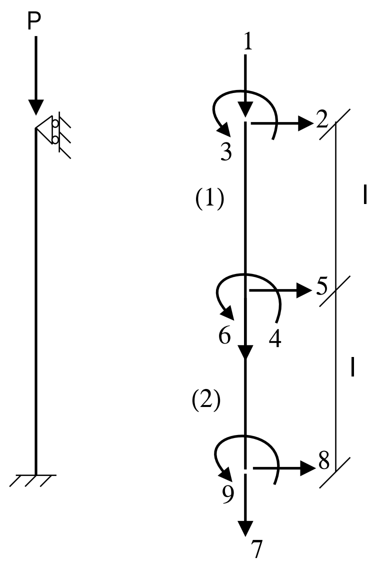
74 The lowest value of λ, λ_{crit} will give the buckling load for the structure and the buckling loads will be given by

$$\boxed{\overline{\mathbf{P}}_{crit} = \lambda_{crit}\overline{\mathbf{P}}^*} \tag{13.50}$$

75 The corresponding deformed shape is directly obtained from the corresponding eigenvector.

■ Example 13-1: Column Stability

Determine the buckling load of the following column.



Solution:

The following elastic stiffness matrices are obtained

$$\mathbf{k}_e^1 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \tag{13.51-a}$$

$$\mathbf{k}_e^2 = \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \tag{13.51-b}$$

Similarly, the geometric stiffness matrices are given by

$$\mathbf{k}_g^1 = \frac{-P}{L} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix} \end{array} \tag{13.52-a}$$

$$\mathbf{k}_g^2 = \frac{-P}{L} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix} \end{array} \tag{13.52-b}$$

The structure's stiffness matrices $\mathbf{K}_e$ and $\mathbf{K}_g$ can now be assembled from the element stiffnesses. Eliminating rows and columns 2, 7, 8, 9 corresponding to zero displacements in the column, we obtain

$$\mathbf{K}_e = \frac{EI}{L^3} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 \\ 0 & 0 & 4L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix} \end{array} \tag{13.53}$$

and

$$\mathbf{K}_g = \frac{-P}{L} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{15}L^2 & \frac{-L}{10} & \frac{-L^2}{30} \\ 0 & 0 & \frac{-L}{10} & \frac{12}{5} & 0 \\ 0 & 0 & \frac{-L^2}{30} & 0 & \frac{4}{15}L^2 \end{bmatrix} \end{array} \tag{13.54}$$

noting that in this case $\mathbf{K}_g^* = \mathbf{K}_g$ for $P = 1$, the determinant $|\mathbf{K}_e + \lambda\mathbf{K}_g^*| = 0$ leads to

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} & \begin{vmatrix} \frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 \\ 0 & 0 & 4L^2 - \frac{2}{15}\frac{\lambda L^4}{EI} & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} \\ 0 & 0 & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 24 - \frac{12}{5}\frac{\lambda L^2}{EI} & 0 \\ 0 & 0 & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} & 0 & 8L^2 - \frac{4}{15}\frac{\lambda L^4}{EI} \end{vmatrix} \end{array} = 0 \tag{13.55}$$

introducing $\phi = \frac{AL^2}{I}$ and $\mu = \frac{\lambda L^2}{EI}$, the determinant becomes

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} & \begin{vmatrix} \phi & -\phi & 0 & 0 & 0 \\ -\phi & 2\phi & 0 & 0 & 0 \\ 0 & 0 & 2\left(2 - \frac{\mu}{15}\right) & -6L + \frac{\mu}{10} & 2 + \frac{\mu}{30} \\ 0 & 0 & -6L + \frac{\mu}{10} & 12\left(2 - \frac{\mu}{5}\right) & 0 \\ 0 & 0 & 2 + \frac{\mu}{30} & 0 & 4\left(2 - \frac{\mu}{15}\right) \end{vmatrix} \end{array} = 0 \tag{13.56}$$

Expanding the determinant, we obtain the cubic equation in μ

$$3\mu^3 - 220\mu^2 + 3,840\mu - 14,400 = 0 \tag{13.57}$$

and the lowest root of this equation is $\boxed{\mu = 5.1772}$.

We note that from Eq. 13.21, the exact solution for a column of length L was

$$P_{cr} = \frac{(4.4934)^2}{l^2}EI = \frac{(4.4934)^2}{(2L)^2}EI = \boxed{5.0477\frac{EI}{L^2}} \tag{13.58}$$

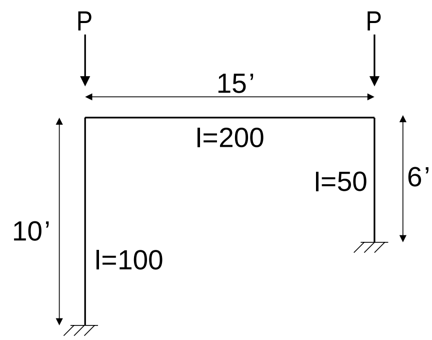
and thus, the numerical value is about 2.6 percent higher than the exact one. The *mathematica* code for this operation is:

```
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:={
{e a/l  , 0          , 0          , -e a/l , 0           , 0          },
{0      , 12 e i/l^3 , 6 e i/l^2  , 0      , -12 e i/l^3 , 6 e i/l^2  },
{0      , 6 e i/l^2  , 4 e i/l    , 0      , -6 e i/l^2  , 2 e i/l    },
{-e a/l , 0          , 0          , e a/l  , 0           , 0          },
{ 0     , -12 e i/l^3 , -6 e i/l^2 , 0     , 12 e i/l^3  , -6 e i/l^2 },
```

```
{ 0     , 6e i/l^2    , 2 e i/l    , 0      , -6 e i/l^2  , 4 e i/l    }
}
ke1=N[ke[e,a,l,i]]
ke2=N[ke[e,a,l,i]]

(* Assemble structure elastic stiffness matrices *)
ke={
{ke1[[3,3]], ke1[[3,5]]            , ke1[[3,6]]           },
{ ke1[[5,3]], ke1[[5,5]]+ke2[[2,2]], ke1[[5,6]]+ke2[[2,3]]},
{ ke1[[6,3]], ke1[[6,5]]+ke2[[3,2]], ke1[[6,6]]+ke2[[3,3]]}
}
WriteString["mat.out",MatrixForm[ke1]]
WriteString["mat.out",MatrixForm[ke2]]
WriteString["mat.out",MatrixForm[ke]]
(* Define geometric stiffness matrices *)
kg[p_,l_]:=p/l{
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , 6/5  ,  1/10    , 0 , - 6/5  , 1/10     },
{0 , 1/10 , 2 l^2/15 , 0 , - 1/10 , - l^2/30 },
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , -6/5 , - 1/10   , 0 ,  6/5   , - 1/10   },
{0 , 1/10 , - l^2/30 , 0 , - 1/10 , 2 l^2/15 }
}
kg1=kg[p,l]
kg2=kg[p,l]
(* Assemble structure geometric stiffness matrices *)
kg={
{kg1[[3,3]], kg1[[3,5]]            , kg1[[3,6]]           },
{ kg1[[5,3]], kg1[[5,5]]+kg2[[2,2]], kg1[[5,6]]+kg2[[2,3]]},
{ kg1[[6,3]], kg1[[6,5]]+kg2[[3,2]], kg1[[6,6]]+kg2[[3,3]]}
}
(* Determine critical loads in terms of p (note p=1) *)
p=1
keigen= l^2 (Inverse[kg] . ke)/( e i)
pcrit=N[Eigenvalues[keigen]]
(* Alternatively*)
knew =ke - x kg
pcrit2=NSolve[Det[knew]==0,x]
```

■ Example 13-2: Frame Stability

Determine the buckling load for the following frame. Neglect axial deformation.



Solution:
The element stiffness matrices are given by

$$\mathbf{k}_e^1 = \begin{array}{c} \begin{array}{cccc} u_1 & \theta_2 & 0 & 0 \end{array} \\ \begin{bmatrix} 20 & 1,208 & \cdots & \cdots \\ 1,208 & 96,667 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array} \tag{13.59-a}$$

$$\mathbf{k}_g^1 = -P \begin{array}{c} \begin{array}{cccc} u_1 & \theta_2 & 0 & 0 \end{array} \\ \begin{bmatrix} 0.01 & 0.10 & \cdots & \cdots \\ 0.10 & 16.00 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array} \tag{13.59-b}$$

$$\mathbf{k}_e^2 = \begin{array}{c} \begin{array}{cccc} 0 & \theta_2 & 0 & \theta_3 \end{array} \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 128,890 & \cdots & 64,440 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 64,440 & \cdots & 128,890 \end{bmatrix} \end{array} \tag{13.59-c}$$

$$\mathbf{k}_e^3 = \begin{array}{c} \begin{array}{cccc} u_1 & \theta_3 & 0 & 0 \end{array} \\ \begin{bmatrix} 47 & 1,678 & \cdots & \cdots \\ 1,678 & 80,556 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array} \tag{13.59-d}$$

$$\begin{array}{ccc} & & \begin{array}{cccc} u_1 & \theta_3 & 0 & 0 \end{array} \\ \mathbf{k}_g^3 & = & -P\left[\begin{array}{cccc} 0.01667 & 0.1 & \cdots & \cdots \\ 0.1 & 9.6 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right] \end{array} \tag{13.59-e}$$

The global equilibrium relation can now be written as

$$(\mathbf{K}_e - P\mathbf{K}_g)\,\boldsymbol{\delta} = 0 \tag{13.60}$$

$$\begin{array}{c} \begin{array}{ccc} u_1 & \theta_2 & \theta_3 \end{array} \\ \left|\begin{array}{ccc} (66.75) - P(0.026666) & (1,208.33) - P(0.1) & (1,678.24) - P(0.1) \\ (1,208.33) - P(0.1) & (225,556.) - P(16.) & (64,444.4) - P(0) \\ (1,678.24) - P(0.1) & (64,444.) - P(0) & (209,444.) - P(9.6) \end{array}\right| = 0 \end{array} \tag{13.61}$$

The smallest buckling load amplification factor λ is thus equal to $2,017$ kips.

```
(* Initialize constants *)
a1=0
a2=0
a3=0
i1=100
i2=200
i3=50
l1=10 12
l2=15 12
l3=6 12
e1=29000
e2=e1
e3=e1
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:={
{e a/l  , 0            , 0           , -e a/l , 0            , 0           },
{0      , 12 e i/l^3   , 6 e i/l^2   , 0      , -12 e i/l^3  , 6 e i/l^2   },
{0      , 6 e i/l^2    , 4 e i/l     , 0      , -6 e i/l^2   , 2 e i/l     },
{-e a/l , 0            , 0           , e a/l  , 0            , 0           },
{ 0     , -12 e i/l^3  , -6 e i/l^2  , 0      , 12 e i/l^3   , -6 e i/l^2  },
{ 0     , 6e i/l^2     , 2 e i/l     , 0      , -6 e i/l^2   , 4 e i/l     }
}
ke1=ke[e1,a1,l1,i1]
ke2=ke[e2,a2,l2,i2]
ke3=ke[e3,a3,l3,i3]
(* Define geometric stiffness matrices *)
kg[l_,p_]:=p/l{
{0 , 0    , 0         , 0 , 0      , 0         },
{0 , 6/5  ,  1/10     , 0 , - 6/5  , 1/10      },
{0 , 1/10 , 2 l^2/15  , 0 , - 1/10 , - l^2/30  },
{0 , 0    , 0         , 0 , 0      , 0         },
{0 , -6/5 , - 1/10    , 0 ,  6/5   , - 1/10    },
{0 , 1/10 , - l^2/30  , 0 , - 1/10 , 2 l^2/15  }
```

```
}
kg1=kg[l1,1]
kg3=kg[l3,1]
(* Assemble structure elastic and geometric stiffness matrices *)
ke={
{ ke1[[2,2]]+ke3[[2,2]] , ke1[[2,3]]            , ke3[[2,3]]             },
{ ke1[[3,2]]            , ke1[[3,3]]+ke2[[3,3]] , ke2[[3,6]]             },
{ ke3[[3,2]]            , ke2[[6,3]]            , ke2[[6,6]]+ke3[[3,3]]  }
}
kg={
{ kg1[[2,2]]+kg3[[2,2]] , kg1[[2,3]] , kg3[[2,3]]  },
{ kg1[[3,2]]            , kg1[[3,3]] , 0           },
{ kg3[[3,2]]            , 0          , kg3[[3,3]]  }
}
(* Determine critical loads in terms of p (note p=1) *)
p=1
keigen=Inverse[kg] . ke
pcrit=N[Eigenvalues[keigen]]
modshap=N[Eigensystems[keigen]]
```

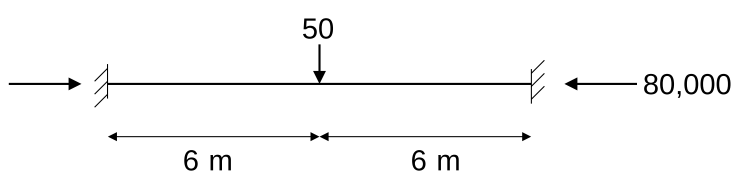
■

13.4 Geometric Non-Linearity

76 From Eq. 13.44 it is evident that since k_g depends on the magnitude of P_x, which itself may be an unknown in a framework, then we do have a *geometrically non-linear* problem.

■ **Example 13-3: Effect of Axial Load on Flexural Deformation**

Determine the midspan displacement and member end forces for the beam-column shown below in terms of P_x; The concentrated force is 50kN applied at midspan, E=2×10^9 kN/m^2 and I=2×10^{-3} m^4.



Solution:

Using two elements for the beam column, the only degrees of freedom are the deflection and rotation at midspan (we neglect the axial deformation).

The element stiffness and geometric matrices are given by

$$[\mathbf{K}_e^1] = \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 0 & v_1 & \theta_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 222,222. & 666,666. & 0 & -222,222. & 666,666. \\ 0 & 666,666. & 2,666,666 & 0. & -666,666. & 1,333,333 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -222,222. & -666,666. & 0 & 222,222. & -666,666. \\ 0 & 666,666. & 1,333,333 & 0. & -666,666. & 2,666,666 \end{bmatrix} \end{array} \tag{13.62}$$

$$[\mathbf{K}_e^2] = \begin{array}{c} \begin{array}{cccccc} 0 & v_1 & \theta_2 & 0 & 0 & 0 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 222,222. & 666,666. & 0 & -222,222. & 666,666. \\ 0 & 666,666. & 2,666,666 & 0. & -666,666. & 1,333,333 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -222,222. & -666,666. & 0 & 222,222. & -666,666. \\ 0 & 666,666. & 1,333,333 & 0. & -666,666. & 2,666,666 \end{bmatrix} \end{array} \tag{13.63}$$

$$[\mathbf{K}_g^1] = \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 0 & v_1 & \theta_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16,000 & -8,000 & 0 & 16,000 & -8,000 \\ 0 & -8,000 & -64,000 & 0 & 8,000 & 16,000 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16,000 & 8,000 & 0 & -16,000 & 8,000 \\ 0 & -8,000 & 16,000 & 0. & 8,000 & -64,000 \end{bmatrix} \end{array} \tag{13.64}$$

$$[\mathbf{K}_g^2] = \begin{array}{c} \begin{array}{cccccc} 0 & v_1 & \theta_2 & 0 & 0 & 0 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16,000 & -8,000 & 0 & 16,000 & -8,000 \\ 0 & -8,000 & -64,000 & 0 & 8,000 & 16,000 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16,000 & 8,000 & 0 & -16,000 & 8,000 \\ 0 & -8,000 & 16,000 & 0. & 8,000 & -64,000 \end{bmatrix} \end{array} \tag{13.65}$$

Assembling the stiffness and geometric matrices we get

$$[\mathbf{K}] = \begin{array}{c} \begin{array}{cc} v_1 & \theta_2 \end{array} \\ \begin{bmatrix} 412,444. & 0. \\ 0. & 5,205,330 \end{bmatrix} \end{array} \tag{13.66}$$

and the displacements would be

$$\left\{ \begin{array}{c} v_1 \\ \theta_2 \end{array} \right\} = \left\{ \begin{array}{c} -0.00012123 \\ 0 \end{array} \right\} \tag{13.67}$$

and the member end forces for element 1 are given by

$$
\begin{aligned}
\left\{\begin{array}{c} P_{lft} \\ V_{lft} \\ M_{lft} \\ P_{rgt} \\ V_{rgt} \\ M_{rgt} \end{array}\right\}
&= \left[[\mathbf{K}_e^1] + [\mathbf{K}_g^1]\right] \left\{\begin{array}{c} u_{lft} \\ v_{lft} \\ \theta_{lft} \\ u_{rgt} \\ v_{rgt} \\ \theta_{rgt} \end{array}\right\} \\
&= \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 0 & v_1 & \theta_2 \end{array} \\ \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & ,0 \\
0 & 206,222. & 658,667. & 0 & -206,222. & 658,667. \\
0 & 658,667. & 260,2670 & 0 & -658,667. & 1,349,330 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -206,222 & -658,667. & 0 & 206,222. & -658,667. \\
0 & 658,667. & 1,349,330 & 0 & -658,667. & 2,602,670
\end{array}\right] \end{array}
\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -0.00012123 \\ 0 \end{array}\right\} \\
&= \left\{\begin{array}{c} 0 \\ 25. \\ 79.8491 \\ 0. \\ -25. \\ 79.8491 \end{array}\right\}
\end{aligned}
\tag{13.68-a}
$$

Note that had we not accounted for the axial forces, then

$$
\left\{\begin{array}{c} v_1 \\ \theta_2 \end{array}\right\} = \left\{\begin{array}{c} -0.0001125 \\ 0 \end{array}\right\} \tag{13.69-a}
$$

$$
\left\{\begin{array}{c} P_{lft} \\ V_{lft} \\ M_{lft} \\ P_{rgt} \\ V_{rgt} \\ M_{rgt} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 25. \\ 75. \\ 0. \\ -25. \\ 75. \end{array}\right\} \tag{13.69-b}
$$

Alternatively, if instead of having a compressive force, we had a tensile force, then

$$
\left\{\begin{array}{c} v_1 \\ \theta_2 \end{array}\right\} = \left\{\begin{array}{c} -0.000104944 \\ 0 \end{array}\right\} \tag{13.70-a}
$$

$$
\left\{\begin{array}{c} P_{lft} \\ V_{lft} \\ M_{lft} \\ P_{rgt} \\ V_{rgt} \\ M_{rgt} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 25. \\ 70.8022 \\ 0. \\ -25. \\ 70.8022 \end{array}\right\} \tag{13.70-b}
$$

We observe that the compressive force increased the displacements and the end moments, whereas a tensile one stiffens the structure by reducing them.

The *Mathematica* to solve this problem follows

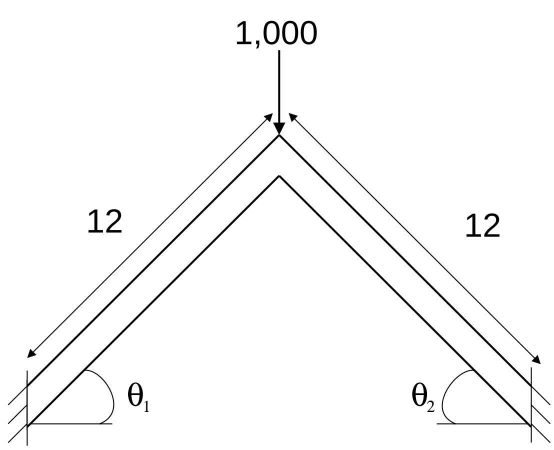
```
(* Initialize constants *)
OpenWrite["mat.out"]
a1=0
a2=0
e=2 10^9
i=2 10^(-3)
i1=i
i2=i1
l=6
l1=l
l2=6
p=-80000 (* negative compression *)
load={-50,0}
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:={
{e a/l  , 0           , 0          , -e a/l , 0            , 0          },
{0      , 12 e i/l^3  , 6 e i/l^2  , 0      , -12 e i/l^3  , 6 e i/l^2  },
{0      , 6 e i/l^2   , 4 e i/l    , 0      , -6 e i/l^2   , 2 e i/l    },
{-e a/l , 0           , 0          , e a/l  , 0            , 0          },
{ 0     , -12 e i/l^3 , -6 e i/l^2 , 0      , 12 e i/l^3   , -6 e i/l^2 },
{ 0     , 6e i/l^2    , 2 e i/l    , 0      , -6 e i/l^2   , 4 e i/l    }
}
ke1=N[ke[e,a1,l1,i1]]
ke2=N[ke[e,a2,l2,i2]]
(* Assemble structure elastic stiffness matrices *)
ke=N[{
{ ke1[[5,5]]+ke2[[2,2]], ke1[[5,6]]+ke2[[2,3]]},
{ ke1[[6,5]]+ke2[[3,2]], ke1[[6,6]]+ke2[[3,3]]}
}]
WriteString["mat.out",MatrixForm[ke1]]
WriteString["mat.out",MatrixForm[ke2]]
WriteString["mat.out",MatrixForm[ke]]
(* Define geometric stiffness matrices *)
kg[p_,l_]:=p/l {
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , 6/5  ,  1/10    , 0 , - 6/5  , 1/10     },
{0 , 1/10 , 2 l^2/15 , 0 , - 1/10 , - l^2/30 },
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , -6/5 , - 1/10   , 0 ,  6/5   , - 1/10   },
{0 , 1/10 , - l^2/30 , 0 , - 1/10 , 2 l^2/15 }
}
kg1=N[kg[p,l1]]
kg2=N[kg[p,l2]]
(* Assemble structure geometric stiffness matrices *)
kg=N[{
{ kg1[[5,5]]+kg2[[2,2]], kg1[[5,6]]+kg2[[2,3]]},
{ kg1[[6,5]]+kg2[[3,2]], kg1[[6,6]]+kg2[[3,3]]}
}]
```

```
(* Determine critical loads and normalize wrt p  *)
keigen=Inverse[kg] . ke
pcrit=N[Eigenvalues[keigen] p]
(* Note that this gives lowest pcrit=1.11  10^6, exact value is 1.095 10^6 *)

(* Add elastic to geometric structure stiffness matrices *)
k=ke+kg
(* Invert stiffness matrix and solve for displacements *)
km1=Inverse[k]
dis=N[km1 . load]
(* Displacements of element 1*)
dis1={0, 0, 0, 0, dis[[1]], dis[[2]]}
k1=ke1+kg1
(* Member end forces for element 1 with axial forces  *)
endfrc1=N[k1 . dis1]

(* Member end forces for element 1 without axial forces  *)
knopm1=Inverse[ke]
disnop=N[knopm1 . load]
disnop1={0, 0, 0, 0, disnop[[1]], disnop[[2]]}
(* Displacements of element 1*)
endfrcnop1=N[ke1 . disnop1]
```

■

■ Example 13-4: Bifurcation

Analyse the stability of the following structure. Compare the axial force caused by the coupled membrane/flexural effects with the case where there is no interaction.



Solution:

In the following solution, we will first determine the axial forces based on the elastic stiffness matrix only. Then, on the basis of those axial forces, we shall determine the geometric stiffness matrix, and solve for the displacements. Because of the non-linearity of the problem, we may have to iterate in order to reach convergence. Following each analysis, we shall recompute the geometric stiffness matrix on the basis of the axial loads detemined from the previous iteration.

Note that convergence will be reached only for stable problems. If the method fails to converge, it implies possible biffurcation which could be caused by elastic displacements approaching $L\sin\theta$, due to either θ being too small, or E being too small (i.e not stiff enough).

```
NEEDS SOME CORRECTION
(* Initialize constants *)
a1 = 1
a2 = 1
i1 = 1 1^3/12
i2 = i1
l1 = 12
l2 = 12
e1 = 200000
e2 = e1
e3 = e1
theta1  =N[Pi/8]
theta2 = Pi-theta1
load ={0, -1000, 0}
normold = 0
epsilon = 0.01
puncpl = load[[2]] / (Sin[theta1] 2)
(*
   Define elastic stiffness matrices
*)
ke[e_,a_,l_,i_] := {
{e a/l  , 0           , 0          , -e a/l , 0           , 0          },
{0      , 12 e i/l^3  , 6 e i/l^2  , 0      , -12 e i/l^3 , 6 e i/l^2  },
{0      , 6 e i/l^2   , 4 e i/l    , 0      , -6 e i/l^2  , 2 e i/l    },
{-e a/l , 0           , 0          , e a/l  , 0           , 0          },
{ 0     , -12 e i/l^3 , -6 e i/l^2 , 0      , 12 e i/l^3  , -6 e i/l^2 },
{ 0     , 6e i/l^2    , 2 e i/l    , 0      , -6 e i/l^2  , 4 e i/l    }
}
(*
  Define geometric stiffness matrix
*)
kg[l_,p_] := p/l {
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , 6/5  ,  1/10    , 0 , - 6/5  , 1/10     },
{0 , 1/10 , 2 l^2/15 , 0 , - 1/10 , - l^2/30 },
{0 , 0    , 0        , 0 , 0      , 0        },
{0 , -6/5 , - 1/10   , 0 ,  6/5   , - 1/10   },
{0 , 1/10 , - l^2/30 , 0 , - 1/10 , 2 l^2/15 }
```

```
}
(*
   Define Transformation matrix and its transpose
*)
gam[theta_] := {
{ Cos[theta] , Sin[theta], 0 , 0          , 0          , 0 },
{ -Sin[theta], Cos[theta], 0 , 0          , 0          , 0 },
{ 0          , 0         , 1 , 0          , 0          , 0 },
{ 0          , 0         , 0 , Cos[theta]  , Sin[theta] , 0 },
{ 0          , 0         , 0 , -Sin[theta] , Cos[theta] , 0 },
{ 0          , 0         , 0 , 0          , 0          , 1 }
}
gamt[theta_] := {
{ Cos[theta] , -Sin[theta], 0 , 0          , 0           , 0 },
{ Sin[theta] , Cos[theta] , 0 , 0          , 0           , 0 },
{ 0          , 0          , 1 , 0          , 0           , 0 },
{ 0          , 0          , 0 , Cos[theta]  , -Sin[theta] , 0 },
{ 0          , 0          , 0 , Sin[theta]  , Cos[theta]  , 0 },
{ 0          , 0          , 0 , 0          , 0           , 1 }
}
(*
  Define functions for local displacments and loads
*)
u[theta_,v1_,v2_] := Cos[theta] v1 + Sin[theta] v2
(*
  Transformation and transpose matrices
*)
gam1 = gam[theta1]
gam2 = gam[theta2]
gam1t = gamt[theta1]
gam2t = gamt[theta2]
(*
  Element elastic stiffness matrices
*)
ke1 = ke[e1, a1, l1, i1]
ke2 = ke[e2, a2, l2, i2]
Ke1 = gam1t . ke1 . gam1
Ke2 = gam2t . ke2 . gam2
(*
  Structure's global stiffness matrix
*)
Ke={
{ Ke1[[4,4]] + Ke2[[1,1]] , Ke1[[4,5]] + Ke2[[1,2]] , Ke1[[4,6]] + Ke2[[1,3]] },
{ Ke1[[5,4]] + Ke2[[2,1]] , Ke1[[5,5]] + Ke2[[2,2]] , Ke1[[5,6]] + Ke2[[2,3]] },
{ Ke1[[6,4]] + Ke2[[3,1]] , Ke1[[6,5]] + Ke2[[3,2]] , Ke1[[6,6]] + Ke2[[3,3]] }
}
(*
  ======= uncoupled analysis ==========
*)
dise=Inverse[Ke].load
u[theta_,diseg1_,diseg2_] := Cos[theta] diseg1 + Sin[theta] diseg2
uu1 = u[ theta1, dise[[1]], dise[[2]] ]
```

```
uu2 = u[ theta2, dise[[1]], dise[[2]] ]
up1 = a1 e1 uu1/l1
up2 = a2 e2 uu2/l2
(*
  ========== Coupled Nonlinear Analysis ==============
  Start Iteration
*)
diseg = N[dise]
For[ iter = 1 , iter <= 100, ++iter,
(* displacements in local coordinates *)
disloc={ 0,0,0,
         u[ theta1, diseg[[1]], diseg[[2]] ],
         u[ theta2, diseg[[1]], diseg[[2]] ],
         0};
(* local force *)
ploc = ke1 . disloc;
p1 = ploc[[4]];
p2 = p1;
kg1 = kg[ l1 , p1 ];
kg2 = kg[ l2 , p2 ];
Kg1 = gam1t . kg1 . gam1;
Kg2 = gam2t . kg2 . gam2;
Kg={
{ Kg1[[4,4]] + Kg2[[1,1]] , Kg1[[4,5]] + Kg2[[1,2]] , Kg1[[4,6]] + Kg2[[1,3]] },
{ Kg1[[5,4]] + Kg2[[2,1]] , Kg1[[5,5]] + Kg2[[2,2]] , Kg1[[5,6]] + Kg2[[2,3]] },
{ Kg1[[6,4]] + Kg2[[3,1]] , Kg1[[6,5]] + Kg2[[3,2]] , Kg1[[6,6]] + Kg2[[3,3]] }
};
(*
  Solve
*)
Ks = Ke + Kg;
diseg = Inverse[Ks] . load;
normnew = Sqrt[ diseg . diseg ];
ratio = ( normnew-normold ) / normnew;
Print["Iteration ",N[iter],"; u1 ",N[u1],"; p1 ",N[p1]," ratio ",N[ratio]];
normold = normnew;
If[ Abs[ ratio ] < epsilon, Break[] ]
]
Print[" p1 ",N[p1]," up1 ",N[up1]," p1/up1 ",N[p1/up1]," ratio ",N[ratio]]
```

■

13.5 Summary

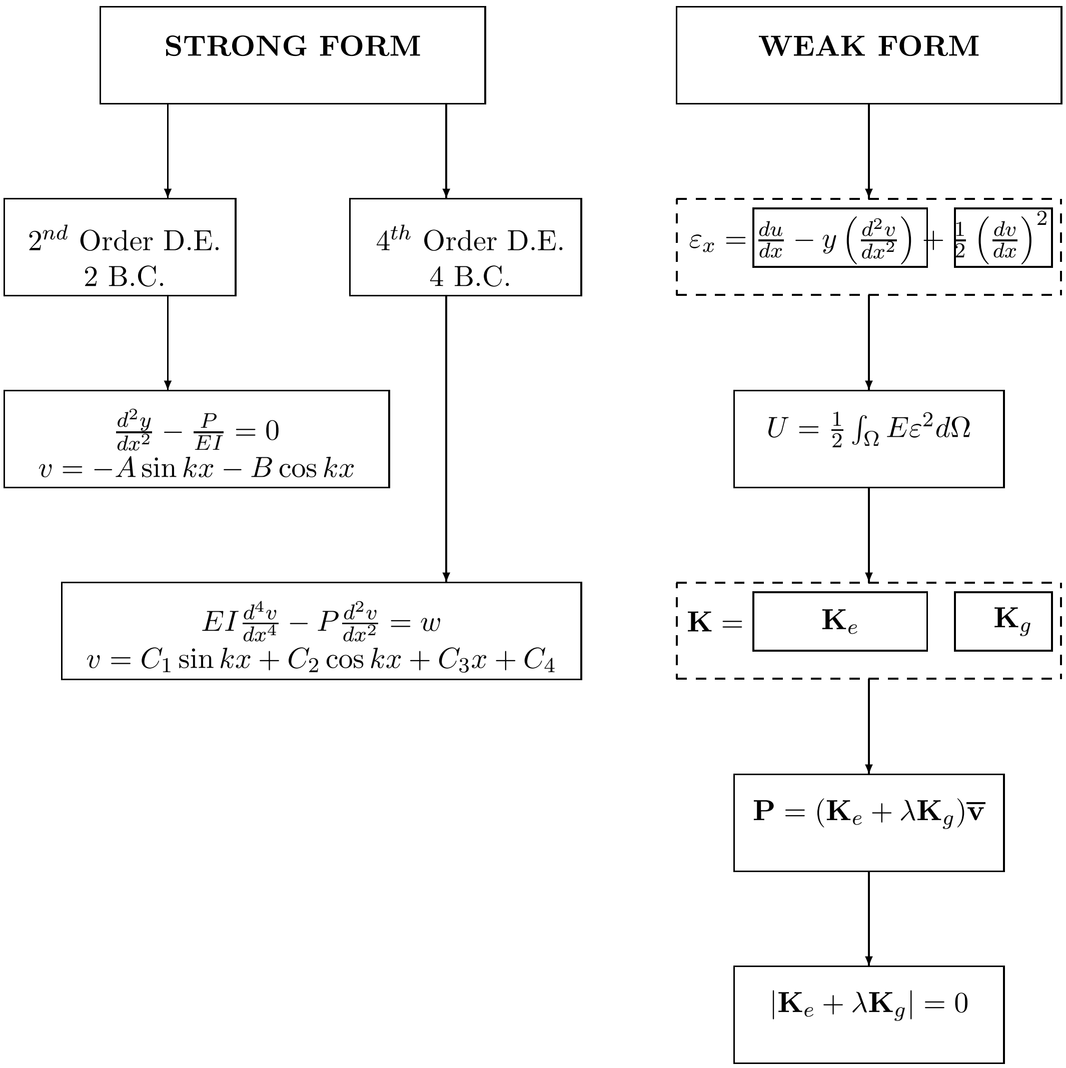


Figure 13.5: Summary of Stability Solutions

Appendix A

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Appendix B

REVIEW of MATRIX ALGEBRA

Because of the discretization of the structure into a finite number of nodes, its solution will always lead to a matrix formulation. This matrix representation will be exploited by the computer ability to operate on vectors and matrices. Hence, it is essential that we do get a thorough understanding of basic concepts of matrix algebra.

B.1 Definitions

Matrix:

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1j} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2j} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{ij} & \dots & A_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mj} & \dots & A_{mn} \end{bmatrix} \tag{2.1}$$

We would indicate the size of the matrix as $[\mathbf{A}]_{m\times n}$, and refer to an individual term of the matrix as A_{ij}. Note that matrices, and vectors are usually boldfaced when typeset, or with a tilde when handwritten $\tilde{A}$.

Vectors: are one column matrices:

$$\{\mathbf{X}\} = \begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_i \\ \vdots \\ B_m \end{Bmatrix} \tag{2.2}$$

A row vector would be

$$\lfloor \mathbf{C} \rfloor = \lfloor\ B_1 \quad B_2 \quad \dots \quad B_i \quad \dots \quad B_m\ \rfloor \tag{2.3}$$

Note that scalars, vectors, and matrices are tensors of order 0, 1, and 2 respectively.

Square matrix: are matrices with equal number of rows and columns. $[\mathbf{A}]_{m\times m}$

Symmetry: $A_{ij} = A_{ji}$

Identity matrix: is a square matrix with all its entries equal to zero except the diagonal terms which are equal to one. It is often denoted as $[\mathbf{I}]$, and

$$I_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{2.4}$$

Diagonal matrix: is a square matrix with all its entries equal to zero except the diagonal terms which are different from zero. It is often denoted as $[\mathbf{D}]$, and

$$D_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ \neq 0, & \text{if } i = j \end{cases} \tag{2.5}$$

Upper Triangular matrix: is a square matrix with all its entries equal to zero, except those along and above the diagonal. It is often denoted as $[\mathbf{U}]$, and

$$U_{ij} = \begin{cases} 0, & \text{if } i > j \\ \neq 0, & \text{if } i \leq j \end{cases} \tag{2.6}$$

Lower Triangular matrix: is a square matrix with all its entries equal to zero except those along and below the diagonal. It is often denoted as $[\mathbf{L}]$, and

$$L_{ij} = \begin{cases} 0, & \text{if } i < j \\ \neq 0, & \text{if } i \geq j \end{cases} \tag{2.7}$$

Orthogonal matrices: $[\mathbf{A}]_{m\times n}$ and $[\mathbf{B}]_{m\times n}$ are said to be orthogonal if $[\mathbf{A}]^T[\mathbf{B}] = [\mathbf{B}]^T[\mathbf{A}] = [\mathbf{I}]$

A square matrix $[\mathbf{C}]_{m\times m}$ is orthogonal if $[\mathbf{C}]^T[\mathbf{C}] = [\mathbf{C}][\mathbf{C}]^T = [\mathbf{I}]$

Trace of a matrix: $tr(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$

Submatrices: are matrices within a matrix, for example

$$[\mathbf{A}] = \left[\begin{array}{cc|c} 5 & 3 & 1 \\ 4 & 6 & 2 \\ \hline 10 & 3 & 4 \end{array}\right] = \begin{bmatrix} [\mathbf{A}_{11}] & [\mathbf{A}_{12}] \\ [\mathbf{A}_{21}] & [\mathbf{A}_{22}] \end{bmatrix} \tag{2.8}$$

$$[\mathbf{B}] = \left[\begin{array}{cc} 1 & 5 \\ 2 & 4 \\ \hline 3 & 2 \end{array}\right] = \begin{bmatrix} [\mathbf{B}_1] \\ [\mathbf{B}_2] \end{bmatrix} \tag{2.9}$$

$$[\mathbf{A}][\mathbf{B}] = \begin{bmatrix} [\mathbf{A}_{11}][\mathbf{B}_1] + [\mathbf{A}_{12}][\mathbf{B}_2] \\ [\mathbf{A}_{21}][\mathbf{B}_1] + [\mathbf{A}_{22}][\mathbf{B}_2] \end{bmatrix} \tag{2.10}$$

$$[\mathbf{A}_{11}][\mathbf{B}_1] = \begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 37 \\ 16 & 44 \end{bmatrix} \tag{2.11}$$

$$
\begin{aligned}
[\mathbf{A}_{22}]\,[\mathbf{B}_2] &= [4]\,[3\;\;2] = [12\;\;8] && (2.12)\\
[\mathbf{A}]\,[\mathbf{B}] &= \begin{bmatrix} 14 & 34 \\ 22 & 48 \\ 28 & 70 \end{bmatrix} && (2.13)
\end{aligned}
$$

Operate on submatrices just as if we were operating on individual matrix elements.

B.2 Elementary Matrix Operations

Transpose: of a matrix $[\mathbf{A}]_{m\times n}$ is another matrix $[\mathbf{B}] = [\mathbf{B}]^T_{n\times m}$ such that $B_{ij} = A_{ji}$ Note that

$$([\mathbf{A}]\,[\mathbf{B}])^T = [\mathbf{B}]^T\,[\mathbf{A}]^T \quad (2.14)$$

Addition (subtraction):

$$
\begin{aligned}
[\mathbf{A}]_{m\times n} &= [\mathbf{B}]_{m\times n} + [\mathbf{C}]_{m\times n} && (2.15)\\
A_{ij} &= B_{ij} + C_{ij} && (2.16)\\
& && (2.17)
\end{aligned}
$$

Scalar Multiplication:

$$
\begin{aligned}
[\mathbf{B}] &= k\cdot[\mathbf{A}] && (2.18)\\
B_{ij} &= kA_{ij}
\end{aligned}
$$

Matrix Multiplication: of two matrices is possible if the number of columns of the first one is equal to the number of rows of the second.

$$
\begin{aligned}
[\mathbf{A}]_{m\times n} &= [\mathbf{B}]_{m\times p}\cdot[\mathbf{C}]_{p\times n} && (2.19)\\
A_{ij} &= \underbrace{\lfloor\mathbf{B}_i\rfloor_{1\times p}\cdot\{\mathbf{C}_j\}_{p\times 1}}_{1\times 1}\\
&= \sum_{r=1}^{p} B_{ir}C_{rj} && (2.20)\\
& && (2.21)
\end{aligned}
$$

Some important properties of matrix products include:

Associative: $[\mathbf{A}]([\mathbf{B}][\mathbf{C}]) = ([\mathbf{A}][\mathbf{B}])[\mathbf{C}]$

Distributive: $[\mathbf{A}]([\mathbf{B}]+[\mathbf{C}]) = [\mathbf{A}][\mathbf{B}]+[\mathbf{A}][\mathbf{C}]$

Non-Commutativity: $[\mathbf{A}][\mathbf{B}] \neq [\mathbf{B}][\mathbf{A}]$

B.3 Determinants

The Determinant of a matrix $[\mathbf{A}]_{n\times n}$, denoted as $\det \mathbf{A}$ or $|\mathbf{A}|$, is recursively defined as

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j} \det \mathbf{A}_{1j} \tag{2.22}$$

Where $\mathbf{A}_{1j}$ is the $(n-1)$x$(n-1)$ matrix obtained by eliminating the ith row and the jth column of matrix $\mathbf{A}$. For a 2×2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a21 \tag{2.23}$$

For a 3×3 matrix

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} && (2.24)\\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) && (2.25)\\
&\quad + a_{13}(a_{21}a_{32} - a_{31}a_{22}) && (2.26)\\
&= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} && (2.27)\\
&\quad + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} && (2.28)
\end{aligned}$$

Can you write a computer program to compute the determinant of an $n\times n$ matrix?

We note that an $n\times n$ matrix would have a determinant which contains $n!$ terms each one involving n multiplications. Hence if $n = 10$ there would be $10! = 3,628,800$ terms, each one involving 9 multiplications hence over 30 million floating operations should be performed in order to evaluate the determinant.

This is why it is impractical to use Cramer's rule to solve a system of linear equations.

Some important properties of deteminants:

1. The determinant of the transpose of a matrix is equal to the determinant of the matrix
$$| \mathbf{A} |=| \mathbf{A}^T | \tag{2.29}$$
2. If at least one row or one column is a linear combination of the other rows or columns, then the determinant is zero. The inverse is also true, if the determinant is equal to zero, then at least one row or one column is a linear combination of of other rows or columns.
3. If there is linear dependancy between rows, then there is also one between columns and *vice-versa*.
4. The determinant of an upper or lower triangular matrix is equal to the product of the main diagonal terms.
5. The determinant of the product of two square matrices is equal to the product of the individual determinants
$$| \mathbf{AB} |=| \mathbf{A} || \mathbf{B} | \tag{2.30}$$

B.4 Singularity and Rank

If the deteminant of a matrix $[\mathbf{A}]_{n\times n}$ is zero, than the matrix is said to be singular. As we have seen earlier, this means that there is at least one row or one column which is a linear combinations of the others. Should we remove this row and column, we can repeat the test for singularity until the size of the submatrix is $r \times r$. Then we refer to r as the rank of the matrix or $rank(\mathbf{A}) = r$. We deduce that the rank of a nonsingular $n \times n$ matrix is n. If the rank of a matrix r is less than its size n, we say that it has $n - r$ rank deficiency.

If n is the size of the global stiffness matrix of a structure in which the boundary conditions have not been accounted for ($n =$ is equal to the total number of nodes times the total number of degrees of freedom per node) would have a rank r equal to n minus the number of possible rigid body motions (3 and 6 in two and three dimensional respectively).

B.5 Inversion

The inverse of a square (nonsingular) matrix $[\mathbf{A}]$ is denoted by $[\mathbf{A}]^{-1}$ and is such that

$$[\mathbf{A}]\,[\mathbf{A}]^{-1} = [\mathbf{A}]^{-1}\,[\mathbf{A}] = [\mathbf{I}] \tag{2.31}$$

Some observations

1. The inverse of the transpose of a matrix is equal to the transpose of the inverse

$$\left[\mathbf{A}^T\right]^{-1} = \left[\mathbf{A}^{-1}\right]^T \tag{2.32}$$

2. The inverse of a matrix product is the reverse product of the inverses

$$([\mathbf{A}]\,[\mathbf{B}])^{-1} = [\mathbf{B}]^{-1}\,[\mathbf{A}]^{-1} \tag{2.33}$$

3. The inverse of a symmetric matrix is also symmetric
4. The inverse of a diagonal matrix is another diagonal one with entries equal to the inverse of the entries of the original matrix.
5. The inverse of a triangular matrix is a triangular matrix.
6. It is computationally more efficient to decompose a matrix ($[\mathbf{A}] = [\mathbf{L}]\,[\mathbf{D}]\,[\mathbf{U}]$) using upper and lower decomposition or Gauss elimination) than to invert a matrix.

B.6 Eigenvalues and Eigenvectors

A special form of the system of linear equation

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{nn} \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} = \left\{ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \right\} \tag{2.34}$$

is one in which the right hand side is a multiple of the solution:

$$[\mathbf{A}]\{\mathbf{x}\} = \lambda\{\mathbf{x}\} \tag{2.35}$$

which can be rewritten as

$$[\mathbf{A} - \lambda\mathbf{I}]\{\mathbf{x}\} = \mathbf{0} \tag{2.36}$$

A nontrivial solution to this system of equations is possible if and only if $[\mathbf{A} - \lambda\mathbf{I}]$ is singular or

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{2.37}$$

or

$$[\mathbf{A}] = \begin{vmatrix} A_{11} - \lambda & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} - \lambda & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \dots & A_{nn} - \lambda \end{vmatrix} = 0 \tag{2.38}$$

When the determinant is expanded, we obtain an n^{th} order polynomial in terms of λ which is known as the characteristic equation of $[\mathbf{A}]$. The n solutions (which can be real or complex) are the eigenvalues of $[\mathbf{A}]$, and each one of them λ_i satisfies

$$[\mathbf{A}]\{\mathbf{x}_i\} = \lambda_i\{\mathbf{x}_i\} \tag{2.39}$$

where $\{\mathbf{x}_i\}$ is a corresponding eigenvector.

It can be shown that:

1. The n eigenvalues of real symmetric matrices of rank n are all real.
2. The eigenvectors are orthogonal and form an orthogonal basis in E_n.

Eigenvalues and eigenvectors are used in stability (buckling) analysis, dynamic analysis, and to assess the performance of finite element formulations.

Appendix C

SOLUTIONS OF LINEAR EQUATIONS

Note this chapter is incomplete

C.1 Introduction

76 Given a system of linear equations $[\mathbf{A}]_{n\times n}\{\mathbf{x}\} = \{\mathbf{b}\}$ (which may result from the direct stiffness method), we seek to solve for $\{\mathbf{x}\}$. *Symbolically* this operation is represented by: $\{\mathbf{x}\} = [\mathbf{A}]^{-1}\{\mathbf{b}\}$

77 There are two approaches for this operation:

Direct inversion using Cramer's rule where $[\mathbf{A}]^{-1} = \frac{[adj\mathbf{A}]}{[\mathbf{A}]}$. However, this approach is computationally very inefficient for $n \geq 3$ as it requires evaluation of n high order determinants.

Decomposition: where in the most general case we seek to decompose $[\mathbf{A}]$ into $[\mathbf{A}] = [\mathbf{L}][\mathbf{D}][\mathbf{U}]$ and where:

$[\mathbf{L}]$	lower triangle matrix
$[\mathbf{D}]$	diagonal matrix
$[\mathbf{U}]$	upper triangle matrix

There are two classes of solutions

Direct Method: characterized by known, finite number of operations required to achieve the decomposition yielding exact results.

Indirect methods: or iterative decomposition technique, with no *a-priori* knowledge of the number of operations required yielding an aapproximate solution with user defined level of accuracy.

C.2 Direct Methods

C.2.1 Gauss, and Gaus-Jordan Elimination

78 Given $[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$, we seek to transform this equation into

1. **Gaus Elimination:** $[\mathbf{U}]\{\mathbf{x}\} = \{\mathbf{y}\}$ where $[\mathbf{U}]$is an upper triangle, and then backsubstitute from the bottom up to solve for the unknowns. Note that in this case we operate on both $[\mathbf{A}]$ & $\{\mathbf{b}\}$, yielding $\{\mathbf{x}\}$.
2. **Gauss-Jordan Elimination:** is similar to the Gaus Elimination, however tather than transforming the $[\mathbf{A}]$ matrix into an upper diagonal one, we transform $[\mathbf{A}|\mathbf{I}]$ into $[\mathbf{I}|\mathbf{A}^{-1}]$. Thus no backsubstitution is needed and the matrix inverse can be explicitely obtained.

■ **Example C-1: Gauss Elimination**

In this first example we simply seek to solve for the unknown vector $\{\mathbf{x}\}$ given:

$$\left\{\begin{array}{llll} +10x_1 & +x_2 & -5x_3 & = 1. \\ -20x_1 & +3x_2 & +20x_3 & = 2. \\ +5x_1 & +3x_2 & +5x_3 & = 6. \end{array}\right. \tag{3.1}$$

Solution:

1. Add $\frac{20}{10}$ times the first equation to the second one will elliminate the x_1 coefficient from the second equation.
2. Substract $\frac{5}{10}$ times the first equation from the third one will elliminate the x_1 coefficient from the third equation.

$$\left\{\begin{array}{llll} 10.x_1 & +x_2 & -5.x_3 & = 1. \\ & +5.x_2 & +10.x_3 & = 4. \\ & +2.5x_2 & +7.5x_3 & = 5.5 \end{array}\right.$$

3. Substract $\frac{2.5}{5}$ times the second equation from the third one will elliminate the x_2 coefficient from the last equation

$$\left\{\begin{array}{llll} 10.x_1 & +x_2 & -5.x_3 & = 1. \\ & +5.x_2 & +10x_3 & = 4. \\ & & +2.5x_3 & = 3.5 \end{array}\right.$$

4. Now we can backsubstitute and solve from the bottom up:

$$x_3 = \frac{3.5}{2.5} = 1.4 \tag{3.2}$$

$$x_2 = \frac{4. - 10.x_3}{5.} = -2. \tag{3.3}$$

$$x_1 = \frac{1. - x_2 + 5.x_3}{10.} = 1. \tag{3.4}$$

■

■ **Example C-2: Gauss-Jordan Elimination**

In this second example we will determine both $\{\mathbf{x}\}$ and the matrix inverse $[\mathbf{A}]^{-1}$.

Solution:

The operation is identical to the first, however we augment the matrix $[\mathbf{A}]$ by $[\mathbf{I}]$: $[\mathbf{A}|\mathbf{I}]$, and operate simultaneously on the two submatrices.

1. Initial matrix

$$\left[\begin{array}{lll|lll} 10 & 1 & -5 & 1 & 0 & 0 \\ -20 & 3 & 20 & 0 & 1 & 0 \\ 5 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} 1 \\ 2 \\ 6 \end{array}\right\} \tag{3.5}$$

2. Elimination of the first column:
   (a) row 1=0.1(row 1)
   (b) row 2=(row2)+20(new row 1)
   (c) row 3=(row 3) -5(new row 1)

$$\left[\begin{array}{lll|lll} 1 & 0.1 & -0.5 & 0.1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & 2.5 & 7.5 & -0.5 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} 0.1 \\ 4 \\ 5.5 \end{array}\right\} \tag{3.6}$$

3. Elimination of second column
   (a) row 2=0.2(row 2)
   (b) row 1=(row 1)-0.1(new row 2)
   (c) row 3=(row 3) -2.5(new row 2)

$$\left[\begin{array}{lll|lll} 1 & 0 & -0.7 & 0.06 & -0.02 & 0 \\ 0 & 1 & 2 & 0.4 & 0.2 & 0 \\ 0 & 0 & 2.5 & -1.5 & -0.5 & 1 \end{array}\right] \left\{\begin{array}{c} 0.02 \\ 0.8 \\ 3.5 \end{array}\right\} \tag{3.7}$$

4. Elimination of the third column
   (a) row 3=0.4(row 3)
   (b) row 1=(row 1)+0.7(new row 3)
   (c) row 2=(row 2)-2(new row 3)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.36 & -0.16 & 0.28 \\ 0 & 1 & 0 & 1.6 & 0.6 & -0.8 \\ 0 & 0 & 1 & -0.6 & -0.2 & 0.4 \end{array}\right] \underbrace{\left\{\begin{array}{c} 1 \\ -2 \\ 1.4 \end{array}\right\}}_{\{\mathbf{x}\}} \tag{3.8}$$

This last equation is $[\mathbf{I}|\mathbf{A}^{-1}]$ ■

C.2.1.1 Algorithm

79 Based on the preceding numerical examples, we define a two step algorithm for the Gaussian ellimination.

80 Defining a_{ij}^k to be the coefficient of the i^{th} row & j^{th} column at the k^{th} reduction step with $i \geq k$ & $j \geq k$:

Reduction:

$$\boxed{\begin{array}{lll} a_{ik}^{k+1} & = 0 & k < i \leq n \\ a_{ij}^{k+1} & = a_{ij}^k - \frac{a_{ik}^k a_{kj}^k}{a_{kk}^k} & k < i \leq n; \quad k < j \leq n \\ b_{ij}^{k+1} & = b_{ij}^k - \frac{a_{ik}^k b_{kj}^k}{a_{kk}^k} & k < i \leq n; \quad 1 < j \leq m \end{array}} \tag{3.9}$$

Backsubstitution:

$$\boxed{x_{ij} = \frac{b_{ij}^i - \sum_{k=i+1}^n a_{ik}^i x_{kj}}{a_{ii}^i}} \tag{3.10}$$

Note that Gauss-Jordan produces both the solution of the equations as well as the inverse of the original matrix. However, if the inverse is not desired it requires three times (N^3) more operations than Gauss or LU decomposition ($\frac{N^3}{3}$).

C.2.2 LU Decomposition

81 In the previous decomposition method, the right hand side ($\{\mathbf{b}\}$ must have been known before decomposition (unless we want to detemine the inverse of the matrix which is computationaly more expensive).

82 In some applications it may be desirable to decompose the matrix without having the RHS completed. For instance, in the direct stiffness method we may have multiple load cases yet we would like to invert only once the stiffness matrix.

This will be achieved through the following decomposition:

$$\boxed{[\mathbf{A}] = [\mathbf{L}][\mathbf{U}]} \tag{3.11}$$

It can be shown that:

1. Both decompositions are equivalent.
2. Count on number of operation show that the 2 methods yield the same number of operations. Number of operations in LU decomposition is equal to the one in Gauss elimination.

83 The solution consists in:

Decomposition: of the matrix independently of the right hand side vector

$$[\mathbf{A}] = [\mathbf{L}]\,[\mathbf{U}] \tag{3.12}$$

$$[\mathbf{L}]\underbrace{[\mathbf{U}]\,\{\mathbf{x}\}}_{\{\mathbf{y}\}} = \{\mathbf{b}\} \tag{3.13}$$

Backsubstitution: for each right hand side vector

1. Solve for $\{\mathbf{y}\}$ from $[\mathbf{L}]\{\mathbf{y}\} = \{\mathbf{b}\}$ starting from top
2. Solve for $\{\mathbf{x}\}$ from $[\mathbf{U}]\{\mathbf{x}\} = \{\mathbf{y}\}$ starting from bottom

84 The vector $\{\mathbf{y}\}$ is the same as the one to which $\{\mathbf{b}\}$ was reduced to in the Gauss Elimination.

C.2.2.1 Algorithm

1. Given:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & & \ddots \\ l_{n1} & l_{n2}\cdots & 1 & \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & \ddots & & \\ & & & u_{nn} \end{bmatrix} \tag{3.14}$$

2. solve:

$$\begin{array}{lll} a_{11} = u_{11} & a_{12} = u_{12}\cdots & a_{1n} = u_{1n} \\ a_{21} = l_{21}u_{11} & a_{22} = l_{21}u_{12} + u_{22} & a_{2n} = l_{21}u_{1n} + u_{2n} \\ \vdots & & \\ a_{n1} = l_{n1}u_{11} & a_{n2} = l_{n1}u_{12} + l_{n2}u_{22} & a_{nn} = \sum_{k=1}^{n-1} l_{nk}u_{kn} + u_{nn} \end{array} \tag{3.15}$$

3. let:

$$[\mathbf{A}]^F = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ l_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & & & \vdots \\ l_{n1} & l_{n2} & \cdots & u_{nn} \end{bmatrix} \tag{3.16}$$

4. Take row by row or column by column

$$\boxed{\begin{array}{ll} l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}}{u_{jj}} & i > j \\ u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} & i \leq j \\ l_{ii} = 1 & \end{array}} \tag{3.17}$$

Note:

1. Computed elements l_{ij} or u_{ij} may always overwrite corresponding element a_{ij}
2. If $[\mathbf{A}]$ is symmetric $[\mathbf{L}]^T \neq [\mathbf{U}]$, symmetry is destroyed in $[\mathbf{A}]^F$

For symmetric matrices, LU decomposition reduces to:

$$\boxed{\begin{array}{ll} u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} & i \leq j \\ l_{ii} = 1 & \\ l_{ij} = \frac{u_{ji}}{u_{jj}} & \end{array}} \tag{3.18}$$

■ **Example C-3: Example**

Given:

$$\mathbf{A} = \begin{bmatrix} 7 & 9 & -1 & 2 \\ 4 & -5 & 2 & -7 \\ 1 & 6 & -3 & -4 \\ 3 & -2 & -1 & -5 \end{bmatrix} \tag{3.19}$$

Solution:

Following the above procedure, it can be decomposed into:

Row 1: $u_{11} = a_{11} = 7$; $u_{12} = a_{12} = 9$; $u_{13} = a_{13} = -1$; $u_{14} = a_{14} = 2$

Row 2:

$$\begin{array}{llll} l_{21} & = \frac{a_{21}}{u_{11}} & = \frac{4}{7} & \\ u_{22} & = a_{22} - l_{21}u_{12} & = -5 - 4\frac{9}{7} & = -10.1429 \\ u_{23} & = a_{23} - l_{21}u_{13} & = 2 + 4\frac{1}{7} & = 2.5714 \\ u_{24} & = a_{24} - l_{21}u_{14} & = -7 - 4\frac{2}{7} & = -10.1429 \end{array}$$

Row 3:

$$\begin{array}{llll} l_{31} & = \frac{a_{31}}{u_{11}} & = \frac{1}{7} & \\ l_{32} & = \frac{a_{32} - l_{31}u_{12}}{u_{22}} & = \frac{6-(0.1429)(9)}{-10.1429} & = -0.4647 \\ u_{33} & = a_{33} - l_{31}u_{13} - l_{32}u_{23} & = -3 - (0.1429)(-1) - (-0.4647)(2.5714) & = -1.6622 \\ u_{34} & = a_{34} - l_{31}u_{14} - l_{32}u_{24} & = -4 - (0.1429)(2) - (-0.4647)(-8.1429) & = -8.0698 \end{array}$$

Row 4:

$$
\begin{array}{llll}
l_{41} & = \frac{a_{41}}{u_{11}} & = \frac{3}{7} & \\
l_{42} & = \frac{a_{42}-l_{41}u_{12}}{u_{22}} & = \frac{-2-(0.4286)(9)}{-10.1429} & = -0. \\
l_{43} & = \frac{a_{43}-l_{41}u_{13}-l_{42}u_{23}}{u_{33}} & = \frac{-1-(0.4286)(-1)-(0.5775)(2.5714)}{-1.6622} & = 1.23 \\
u_{44} & = a_{44}-l_{41}u_{14}-l_{42}u_{24}-l_{43}u_{34} & = -5-(0.4286)(2)-(0.5775)(-8.1429)-(1.2371)(-8.0698) & = 8.82
\end{array}
$$

or

$$
\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ .5714 & 1 & 0 & 0 \\ .1429 & -.4647 & 1 & 0 \\ .4286 & .5775 & 1.2371 & 1 \end{bmatrix}}_{[\mathbf{L}]} \underbrace{\begin{bmatrix} 7 & 9 & -1 & 2 \\ 0 & -10.1429 & 2.571 & -8.143 \\ 0 & 0 & -1.662 & -8.069 \\ 0 & 0 & 0 & 8.8285 \end{bmatrix}}_{[\mathbf{U}]}}_{[\mathbf{A}]} \tag{3.20}
$$

■

C.2.3 Cholesky's Decomposition

85 If $[\mathbf{A}]$ is symmetric $[\mathbf{A}]^F$ is not. For example:

$$
\begin{bmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{bmatrix} = \begin{bmatrix} 1 & & \\ .25 & 1 & \\ .5 & -1.5 & 1 \end{bmatrix} \begin{bmatrix} 16 & 4 & 8 \\ & 4 & -6 \\ & & 9 \end{bmatrix} \tag{3.21}
$$

86 In the most general case, we will have:

$$
[\mathbf{A}] = [\mathbf{L}^*][\mathbf{D}][\mathbf{U}^*]^T \tag{3.22}
$$

87 For aa symmetric $[\mathbf{A}]$ matrix, $[\mathbf{U}^*]$ should be the transpose of $[\mathbf{L}^*]$ or

$$
[\mathbf{A}] = [\mathbf{L}^*][\mathbf{D}][\mathbf{L}^*]^T \tag{3.23}
$$

88 Furthermore, the diagonal matrix $[\mathbf{D}]$ can be factored as as the product of two matrices: $[\mathbf{D}] = [\mathbf{D}]^{\frac{1}{2}}[\mathbf{D}]^{\frac{1}{2}}$ Thus:

$$
[\mathbf{A}] = \underbrace{[\mathbf{L}^*][\mathbf{D}]^{\frac{1}{2}}}_{[\mathbf{L}]}\underbrace{[\mathbf{D}]^{\frac{1}{2}}[\mathbf{L}^*]^T}_{[\mathbf{L}]^T} \tag{3.24}
$$

89 This algorithm can be summarized as:

$$
\boxed{\begin{array}{ll} l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2} & \\ l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1}}{l_{jj}} & i > j \end{array}} \tag{3.25}
$$

90 Note:

1. Decomposition takes place by columns
2. l_{ij} will occupy same space as a_{ij}

■ Example C-4: Cholesky's Decomposition

Given:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 10 & 4 \\ 6 & 13 & 13 & 6 \\ 10 & 13 & 27 & 2 \\ 4 & 6 & 2 & 72 \end{bmatrix} \tag{3.26}$$

Solution:

Column 1:

$$\begin{array}{llll} l_{11} & = \sqrt{a_{11}} & = \sqrt{4} & = 2 \\ l_{21} & = \frac{a_{21}}{l_{11}} & = \frac{6}{2} & = 3 \\ l_{31} & = \frac{a_{31}}{l_{11}} & = \frac{10}{2} & = 5 \\ l_{41} & = \frac{a_{41}}{l_{11}} & = \frac{4}{2} & = 2 \end{array}$$

Column 2:

$$\begin{array}{llll} l_{22} & = \sqrt{a_{22} - l_{21}^2} & = \sqrt{13 - 3^2} & = 2 \\ l_{32} & = \frac{a_{32} - l_{31} l_{21}}{l_{22}} & = \frac{13-(5)(3)}{2} & = -1 \\ l_{42} & = \frac{a_{42} - l_{41} l_{21}}{l_{22}} & = \frac{6-(2)(3)}{2} & = 0 \end{array}$$

Column 3:

$$\begin{array}{llll} l_{33} & = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} & = \sqrt{27 - 5^2 - (-1)^2} & = 1 \\ l_{43} & = \frac{a_{43} - l_{41} l_{31} - l_{42} l_{32}}{l_{33}} & = \frac{2-(2)(5)-(0)(-1)}{1} & = -8 \end{array}$$

Column 4:

$$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2} = \sqrt{72 - (2)^2 - (0)^2 - (-8)^2} = 2$$

or

$$\underbrace{\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & -1 & 1 & 0 \\ 2 & 0 & -8 & 2 \end{bmatrix}}_{[\mathbf{L}]} \underbrace{\begin{bmatrix} 2 & 3 & 5 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{[\mathbf{U}]}}_{[\mathbf{A}]} \tag{3.27}$$

■

C.2.4 Pivoting

C.3 Indirect Methods

91 Iterative methods are most suited for

1. Very large systems of equation $n > 10$, or 100,000
2. systems with a known "guess" of the solution

92 The most popular method is the Gauss Seidel.

C.3.1 Gauss Seidel

$$\begin{array}{ccccccc} c_{11}x_1 & + & c_{12}x_2 & + & c_{13}x_3 & = & r_1 \\ c_{21}x_1 & + & c_{22}x_2 & + & c_{23}x_3 & = & r_2 \\ c_{31}x_1 & + & c_{32}x_2 & + & c_{33}x_3 & = & r_3 \end{array} \tag{3.28}$$

solve 1^{st} equation for x_1 using initial "guess" for x_2, x_3.

$$x_1 = \frac{r_1 - c_{12}x_2 - c_{13}x_3}{c_{11}} \tag{3.29}$$

solve 2^{nd} equation for x_2 using the computed value of x_1 & initial guess of x_3

$$x_2 = \frac{r_2 - c_{21}x_1 - c_{23}x_3}{c_{22}} \tag{3.30}$$

so on & so forth $\cdots$
Note:

1. The iterative process can be considered to have converged if:

$$|\frac{\mathbf{x}^k - \mathbf{x}^{k-1}}{\mathbf{x}^{lk}}| \leq \varepsilon \tag{3.31}$$

2. The convergence can be accelerated by relaxation

$$x_i^k = \lambda x_i^k + (1-\lambda)x_i^{k-1} \tag{3.32}$$

where λ is a weight factor between 0. and 2. For values below 1 we have *underrelaxation*, and for values greater than 1 we have *overrelaxation*. The former is used for nonconvergent systems, whereas the later is used to accelerate convergence of converging ones. optimum λ for frame analysis is around 1.8.

C.4 Ill Conditioning

93 An ill condition system of linear equations is one in which a small perturbation of the coefficient a_{ij} results in large variation in the results $\mathbf{x}$. Such a system arises in attempting to solve for the intersection of two lines which are nearly parallel, or the decomposition of a structure stiffness matrix in which very stiff elements are used next to very soft ones.

C.4.1 Condition Number

94 Ill conditioning can be detected by determining the *condition number* κ of the matrix.

$$\kappa = \frac{\lambda_{max}}{\lambda min} \tag{3.33}$$

where λ_{max} and λmin are the maximum and minimum eigenvalues of the coefficient matrix.

95 In the decomposition of a matrix, truncation errors may result in a loss of precision which has been quantified by:

$$s = p - \log \kappa \tag{3.34}$$

where p is the number of decimal places to which the coefficient matrix is represented in the computer, and s is the number of correct decimal places in the solution.

96 Note that because the formula involves $\log \kappa$, the eigenvalues need only be approximately evaluated.

C.4.2 Pre Conditioning

97 If a matrix $[\mathbf{K}]$ has an unacceptably high condition number, it can be *preconditioned*through a congruent operation:

$$[\mathbf{K}'] = [\mathbf{D}_1][\mathbf{K}][\mathbf{D}_2] \tag{3.35}$$

However there are no general rules for selecting $[\mathbf{D}_1]$ and $[\mathbf{D}_2]$.

C.4.3 Residual and Iterative Improvements

Appendix D

TENSOR NOTATION

NEEDS SOME EDITING

76 Equations of elasticity are expressed in terms of tensors, where

- A tensor is a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.
- A tensor is classified by the rank or order
- A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar.
- A tensor of order one has three coordinate components in space, hence it is a vector.
- In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.

77 For example, force and a stress are tensors of order 1 and 2 respectively.

78 To express tensors, there are three distinct notations which can be used: 1) Engineering; 2) indicial; or 3) Dyadic.

79 Whereas the Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, the tensor and the dyadic form will lead to shorter and more compact forms.

D.1 Engineering Notation

In the engineering notation, we carry on the various subscript(s) associated with each coordinate axis, for example σ_{xx}, σ_{xy}.

D.2 Dyadic/Vector Notation

80 Uses bold face characters for tensors of order one and higher, $\boldsymbol{\sigma}, \boldsymbol{\epsilon}$. This notation is independent of coordinate systems.

81 Since scalar operations are in general not applicable to vectors, we define

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} && \text{(4.1-a)}\\
\mathbf{A}\times\mathbf{B} &= -\mathbf{B}\times\mathbf{A} && \text{(4.1-b)}\\
\mathbf{A} &= A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k} && \text{(4.1-c)}\\
\mathbf{A}\cdot\mathbf{B} &= |\mathbf{A}||\mathbf{B}|\cos(\mathbf{A},\mathbf{B}) &&\\
&= A_xB_x+A_yB_y+A_zB_z && \text{(4.1-d)}\\
\mathbf{A}\otimes\mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z \end{vmatrix} && \text{(4.1-e)}\\
\text{grad } A = \boldsymbol{\nabla} A &= \mathbf{i}\frac{\partial A}{\partial x}+\mathbf{j}\frac{\partial A}{\partial y}+\mathbf{k}\frac{\partial A}{\partial z} && \text{(4.1-f)}\\
\text{div } \mathbf{A} = \boldsymbol{\nabla}\cdot\mathbf{A} &= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\mathbf{i}A_x+\mathbf{j}A_y+\mathbf{k}A_z) &&\\
&= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} && \text{(4.1-g)}\\
\text{Laplacian } \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} &= \frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2} && \text{(4.1-h)}\\
& && \text{(4.1-i)}
\end{aligned}
$$

D.3 Indicial/Tensorial Notation

This notation uses letter appended indices (sub or super scripts) to the letter representing the tensor quantity of interest. i.e. $a^i; \tau_{ij}; \varepsilon^{ij}$, where the number of indices is the rank of the tensor (see sect. B.4).

82 The following rules define tensorial notation:

1. If there is one letter index, that index goes from i to n. For instance:

$$
a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} \qquad i = 1,3 \tag{4.2}
$$

   assuming that $n = 3$.

2. A repeated index will take on all the values of its range, and the resulting tensors summed. For instance:

$$a_{1i}x_i = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \tag{4.3}$$

3. Tensor's order:
   - First order tensor (such as force) has only one free index:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor \tag{4.4}$$

   - Second order tensor (such as stress or strain) will have two free indeces.

$$D_{ij} \begin{bmatrix} D_{11} & D_{22} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{4.5}$$

   - A fourth order tensor (such as Elastic constants) will have four free indeces.
4. Derivatives of tensor with respect to x_i is written as $,i$. For example:

$$\frac{\partial \Phi}{\partial x_i} = \Phi_{,i} \quad \frac{\partial v_i}{\partial x_i} = v_{i,i} \quad \frac{\partial v_i}{\partial x_j} = v_{i,j} \quad \frac{\partial T_{i,j}}{\partial x_k} = T_{i,j,k} \tag{4.6}$$

Usefulness of the indicial notation is in presenting systems of equations in compact form. For instance:

$$x_i = c_{ij}z_j \tag{4.7}$$

this simple compacted equation (expressed as $\mathbf{x} = \mathbf{cz}$ in dyadic notation), when expanded would yield:

$$\begin{aligned} x_1 &= c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\ x_2 &= c_{21}z_1 + c_{22}z_2 + c_{23}z_3 \\ x_3 &= c_{31}z_1 + c_{32}z_2 + c_{33}z_3 \end{aligned} \tag{4.8-a}$$

Similarly:

$$A_{ij} = B_{ip}C_{jq}D_{pq} \tag{4.9}$$

$$\begin{aligned} A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{12} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\ A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22} \end{aligned} \tag{4.10-a}$$

Appendix E

INTEGRAL THEOREMS

76 Some useful integral theorems are presented here without proofs. Schey's textbook *div grad curl and all that* provides an excellent informal presentation of related material.

E.1 Integration by Parts

The integration by part formula is

$$\boxed{\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)dx} \tag{5.1}$$

or

$$\boxed{\int_a^b udv = uv|_a^b - \int_a^b vdu} \tag{5.2}$$

E.2 Green-Gradient Theorem

Green's theorem is

$$\boxed{\oint (Rdx + Sdy) = \int_\Gamma \left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right) dxdy} \tag{5.3}$$

E.3 Gauss-Divergence Theorem

77 The general form of the Gauss' integral theorem is

$$\boxed{\int_\Gamma \mathbf{v}.\mathbf{n}d\Gamma = \int_\Omega \mathrm{div}\mathbf{v}d\Omega} \tag{5.4}$$

or

$$\boxed{\int_\Gamma v_i n_i d\Gamma = \int_\Omega v_{i,i} d\Omega} \tag{5.5}$$

78 In 2D-3D Gauss' integral theorem is

$$\boxed{\int\int\int_V \text{div } \mathbf{q} dV = \int\int_S \mathbf{q}^T.\mathbf{n} dS} \tag{5.6}$$

or

$$\boxed{\int\int\int_V v_{i,i} dV = \int\int_S v_i n_i dS} \tag{5.7}$$

79 Alternatively

$$\boxed{\int\int\int_V \phi \text{div } \mathbf{q} dV = \int\int_S \phi \mathbf{q}^T.\mathbf{n} dS - \int\int\int_V (\boldsymbol{\nabla}\phi)^T \mathbf{q} dV} \tag{5.8}$$

80 For 2D-1D transformations, we have

$$\boxed{\int\int_A \text{div } \mathbf{q} dA = \oint_s \mathbf{q}^T \mathbf{n} ds} \tag{5.9}$$

or

$$\boxed{\int\int_A \phi \text{div } \mathbf{q} dA = \oint_s \phi \mathbf{q}^T \mathbf{n} ds - \int\int_A (\boldsymbol{\nabla}\phi)^T \mathbf{q} dA} \tag{5.10}$$

DRAFT

LECTURE NOTES

CVEN 3525/3535

STUCTURAL ANALYSIS

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Spring 1999

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PREFACE

Whereas there are numerous excellent textbooks covering Structural Analysis, or Structural Design, I felt that there was a need for a single reference which

- Provides a **succinct, yet rigorous**, coverage of Structural Engineering.
- **Combines**, as much as possible, Analysis with Design.
- Presents numerous, **carefully selected, example problems**.

in a properly type set document.

As such, and given the reluctance of undergraduate students to go through extensive verbage in order to capture a key concept, I have opted for an unusual format, one in which each key idea is clearly distinguishable. In addition, such a format will hopefully foster group learning among students who can easily reference misunderstood points.

Finally, whereas all problems have been taken from a variety of references, I have been very careful in not only properly selecting them, but also in enhancing their solution through appropriate figures and LaTeX typesetting macros.

Structural Engineering can be characterized as the art of molding materials we don't really understand into shapes we cannot really analyze so as to withstand forces we cannot really assess in such a way that the public does not really suspect.

-Really Unknown Source

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Chapter 1

INTRODUCTION

1.1 Structural Engineering

1 Structural engineers are responsible for the detailed analysis and design of:

Architectural structures: Buildings, houses, factories. They must work in close cooperation with an architect who will ultimately be responsible for the design.

Civil Infrastructures: Bridges, dams, pipelines, offshore structures. They work with transportation, hydraulic, nuclear and other engineers. For those structures they play the leading role.

Aerospace, Mechanical, Naval structures: aeroplanes, spacecrafts, cars, ships, submarines to ensure the structural safety of those important structures.

1.2 Structures and their Surroundings

2 Structural design is affected by various environmental constraints:

1. Major movements: For example, elevator shafts are usually shear walls good at resisting lateral load (wind, earthquake).
2. Sound and structure interact:
   - A **dome** roof will concentrate the sound
   - A **dish** roof will diffuse the sound
3. Natural light:
   - A flat roof in a building may not provide adequate light.
   - A Folded plate will provide adequate lighting (analysis more complex).
   - A bearing and shear wall building may not have enough openings for daylight.
   - A Frame design will allow more light in (analysis more complex).
4. Conduits for cables (electric, telephone, computer), HVAC ducts, may dictate type of floor system.
5. Net clearance between columns (unobstructed surface) will dictate type of framing.

1.3 Architecture & Engineering

3 Architecture must be the product of a **creative** collaboration of architects and engineers.

4 Architect stress the overall, rather than elemental approach to design. In the design process, they conceptualize a space-form scheme as a total system. They are **generalists**.

5 The engineer, partly due to his/her education think in reverse, starting with details and without sufficient regards for the overall picture. (S)he is a **pragmatist** who "knows everything about nothing".

6 Thus there is a conceptual **gap** between architects and engineers at all levels of design.

7 Engineer's education is more specialized and in depth than the architect's. However, engineer must be kept aware of overall architectural objective.

8 In the last resort, it is the architect who is the leader of the construction team, and the engineers are his/her servant.

9 A possible compromise might be an **Architectural Engineer**.

1.4 Architectural Design Process

10 Architectural design is hierarchical:

Schematic: conceptual overall space-form feasibility of basic schematic options. Collaboration is mostly between the owner and the architect.

Preliminary: Establish basic physical properties of major subsystems and key components to prove design feasibility. Some collaboration with engineers is necessary.

Final design: final in-depth design refinements of all subsystems and components and preparation of working documents ("blue-prints"). Engineers play a leading role.

1.5 Architectural Design

11 Architectural design must respect various constraints:

Functionality: Influence of the adopted structure on the purposes for which the structure was erected.

Aesthetics: The architect often imposes his aesthetic concerns on the engineer. This in turn can place severe limitations on the structural system.

Economy: It should be kept in mind that the two largest components of a structure are labors and materials. Design cost is comparatively negligible.

1.6 Structural Analysis

12 Given an **existing** structure subjected to a certain load determine internal forces (axial, shear, flexural, torsional; or stresses), deflections, and verify that no unstable failure can occur.

13 Thus the basic structural requirements are:

Strength: stresses should not exceed critical values: $\sigma < \sigma_f$

Stiffness: deflections should be controlled: $\Delta < \Delta_{max}$

Stability: buckling or cracking should also be prevented

1.7 Structural Design

14 Given a set of forces, **dimension** the structural element.

Steel/wood Structures Select appropriate section.

Reinforced Concrete: Determine dimensions of the element and internal reinforcement (number and sizes of reinforcing bars).

15 For **new structures**, **iterative** process between analysis and design. A preliminary design is made using **rules of thumbs** (best known to Engineers with design experience) and analyzed. Following design, we check for

Serviceability: deflections, crack widths under the applied load. Compare with acceptable values specified in the design code.

Failure: and compare the failure load with the applied load times the appropriate factors of safety.

If the design is found not to be acceptable, then it must be modified and reanalyzed.

16 For **existing structures rehabilitation**, or verification of an old infrastructure, analysis is the most important component.

17 In summary, analysis is always required.

1.8 Load Transfer Elements

18 From Strength of Materials, Fig. 1.1

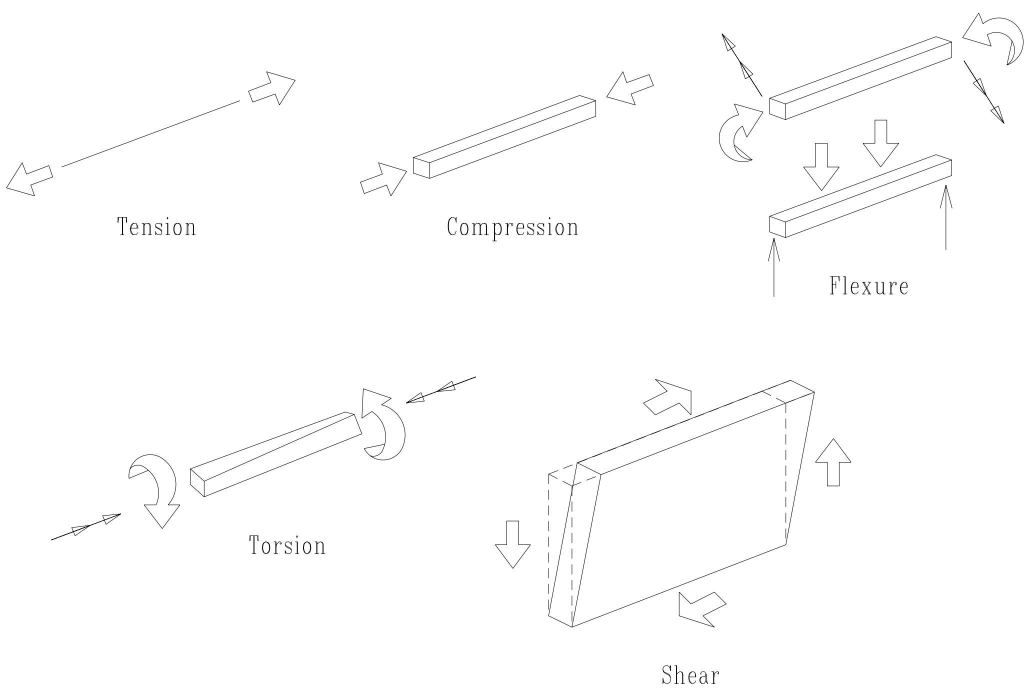

Figure 1.1: Types of Forces in Structural Elements (1D)

Axial: cables, truss elements, arches, membrane, shells

Flexural: Beams, frames, grids, plates

Torsional: Grids, 3D frames

Shear: Frames, grids, shear walls.

1.9 Structure Types

19 Structures can be classified as follows:

Tension & Compression Structures: only, no shear, flexure, or torsion

Cable (tension only): The high strength of steel cables, combined with the efficiency of simple tension, makes cables ideal structural elements to span large distances such as bridges, and dish roofs, Fig. 1.2

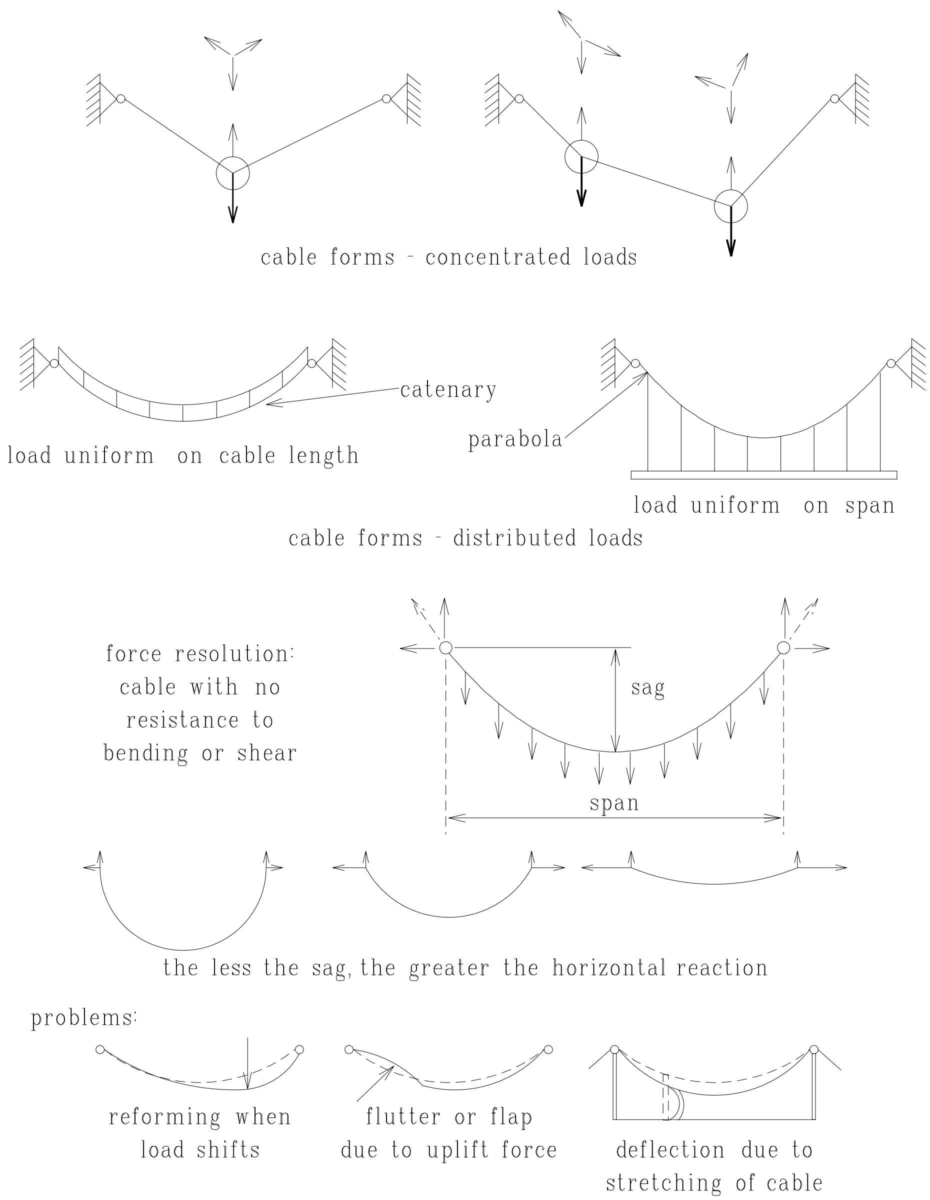

Figure 1.2: Basic Aspects of Cable Systems

Arches (mostly compression) is a “reversed cable structure”. In an arch, we seek to minimize flexure and transfer the load through axial forces only. Arches are used for large span roofs and bridges, Fig. 1.3

Trusses have **pin connected** elements which can transmit axial forces only (tension and compression). Elements are connected by either slotted, screwed, or **gusset plate** connectors. However, due to construction details, there may be **secondary stresses** caused by relatively rigid connections. Trusses are used for joists, roofs, bridges, electric tower, Fig. 1.4

Post and Beams: Essentially a support column on which a “beam” rests, Fig. 1.5, and 1.6.

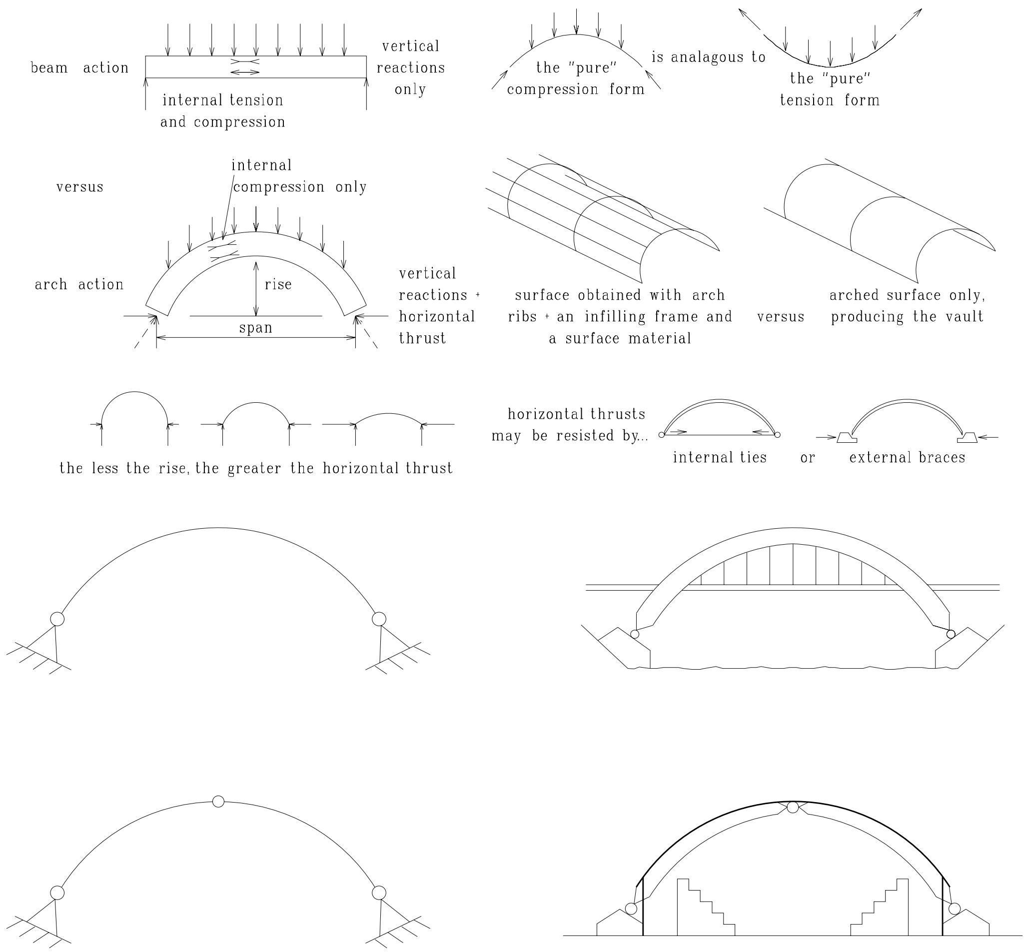

Figure 1.3: Basic Aspects of Arches

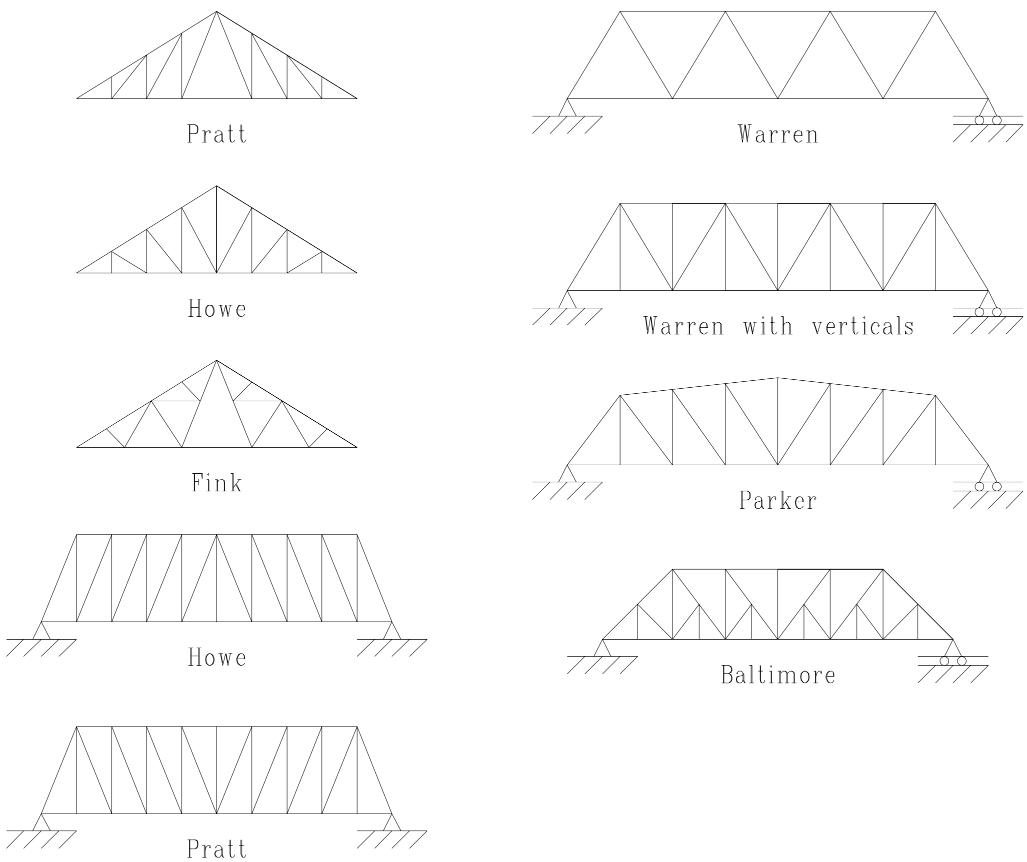

Figure 1.4: Types of Trusses

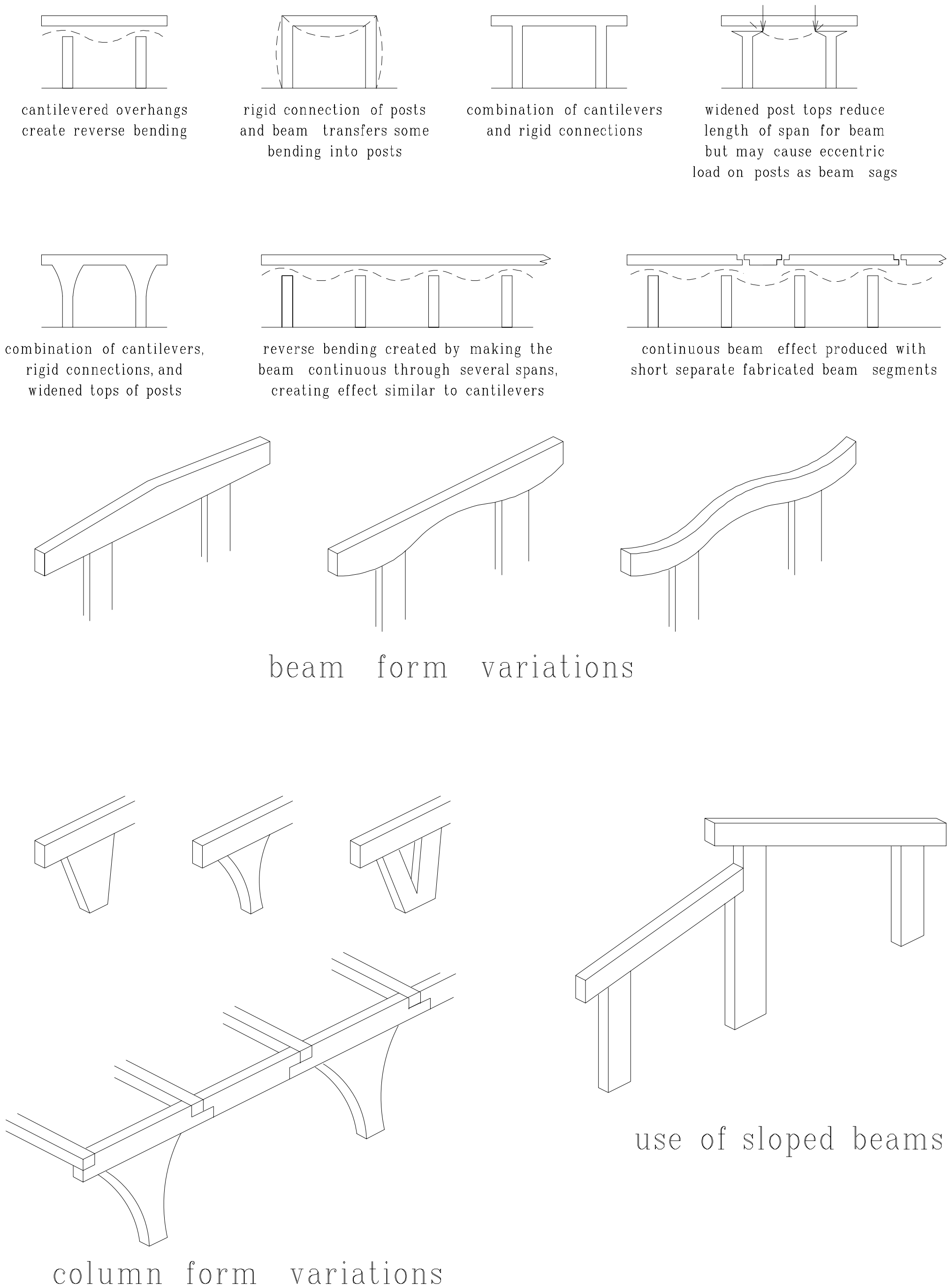

Figure 1.5: Variations in Post and Beams Configurations

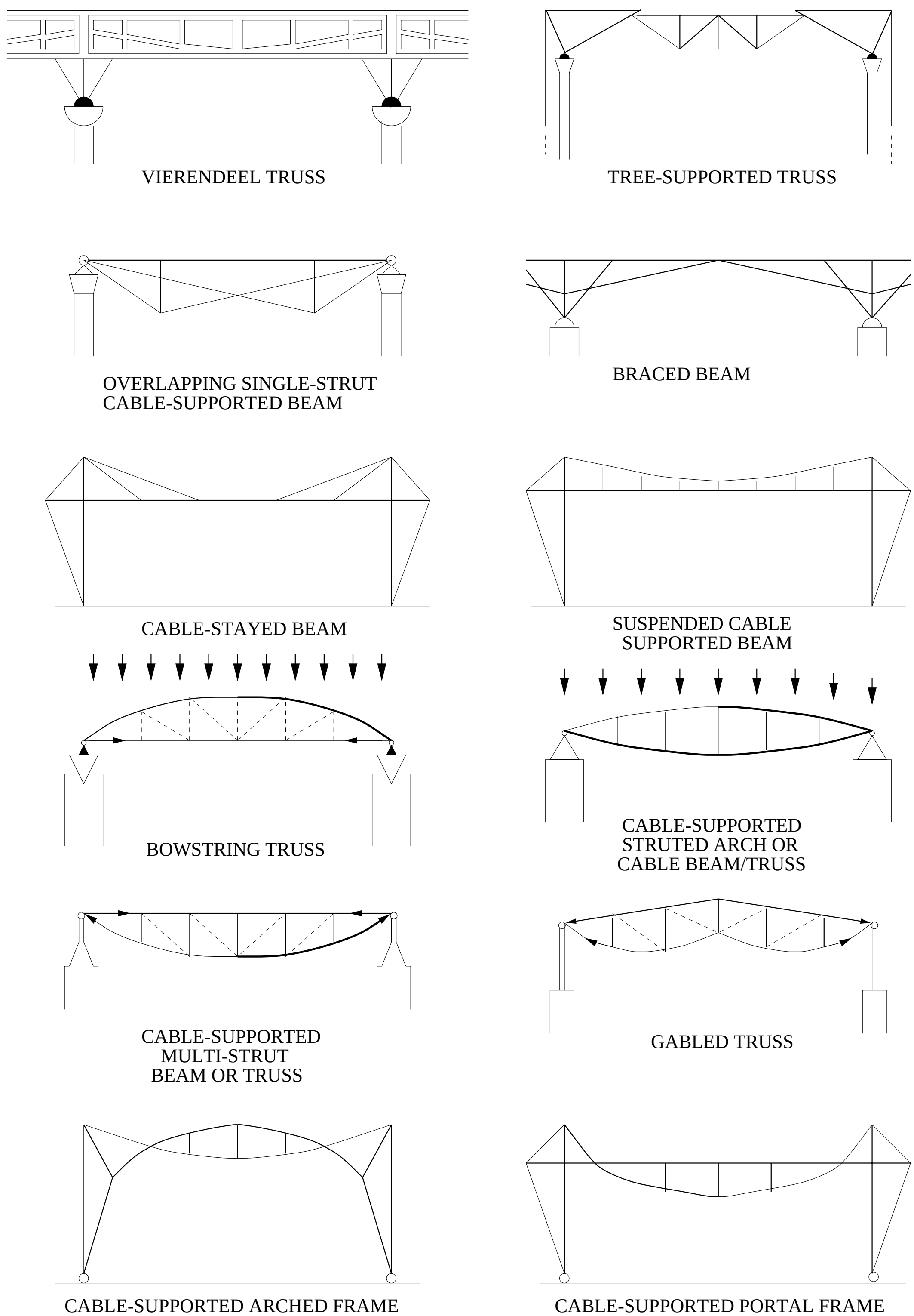

Figure 1.6: Different Beam Types

Beams: Shear, flexure and sometimes axial forces. Recall that $\sigma = \frac{Mc}{I}$ is applicable only for **shallow beams**, i.e. span/depth at least equal to five.

Whereas r/c beams are mostly rectangular or T shaped, steel beams are usually I shaped (if the top flanges are not properly stiffened, they may buckle, thus we must have **stiffeners**).

Frames: Load is co-planar with the structure. Axial, shear, flexure (with respect to one axis in 2D structures and with respect to two axis in 3D structures), torsion (only in 3D). The frame is composed of at least one horizontal member (beam) rigidly connected to vertical ones[1]. The vertical members can have different boundary conditions (which are usually governed by soil conditions). Frames are extensively used for houses and buildings, Fig. 1.7.

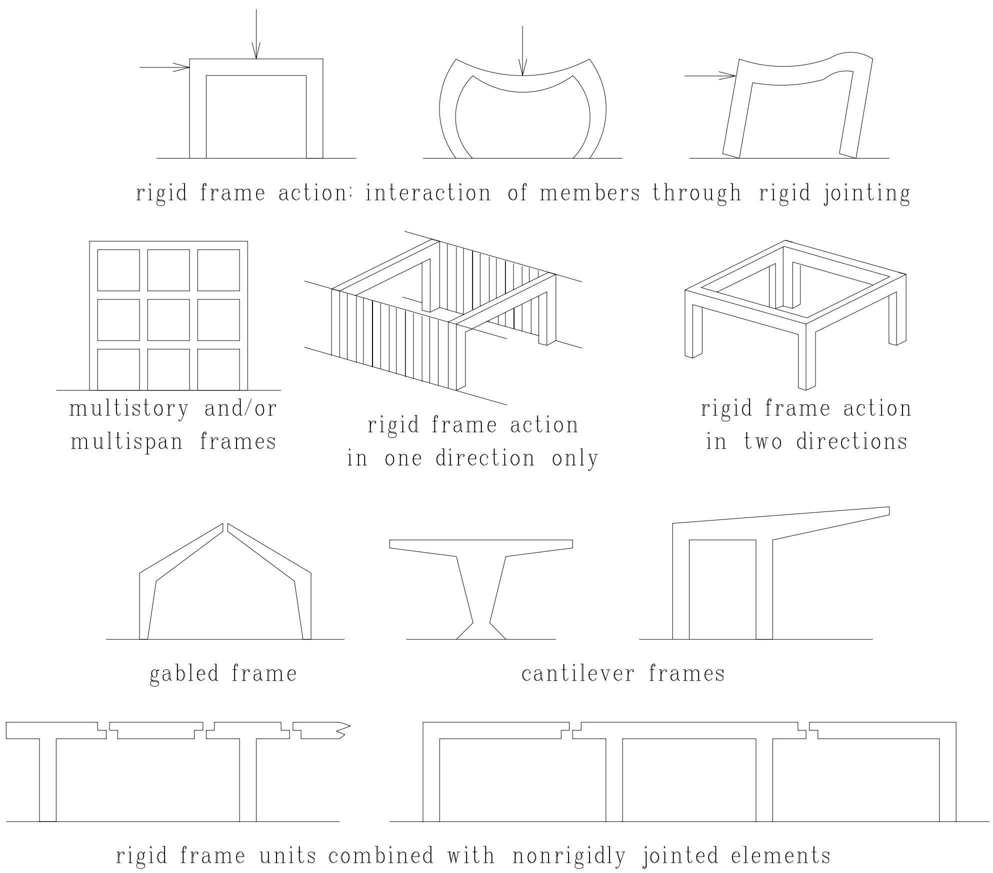

Figure 1.7: Basic Forms of Frames

Grids and Plates: Load is orthogonal to the plane of the structure. Flexure, shear, torsion.

In a grid, beams are at right angles resulting in a two-way dispersal of loads. Because of the rigid connections between the beams, additional stiffness is introduced by the torsional resistance of members.

Grids can also be **skewed** to achieve greater efficiency if the aspect ratio is not close to one.

Plates are flat, rigid, two dimensional structures which transmit vertical load to their supports. Used mostly for **floor slabs**.

Folded plates is a combination of transverse and longitudinal beam action. Used for long span roofs. Note that the plate may be folded circularly rather than longitudinally. Folded plates are used mostly as long span roofs. However, they can also be used as vertical walls to support both vertical and horizontal loads.

[1] The precursor of the frame structures were the **Post and Lintel** where the post is vertical member on which the lintel is simply posed.

Membranes: 3D structures composed of a flexible 2D surface resisting tension only. They are usually cable-supported and are used for tents and long span roofs Fig. 1.8.

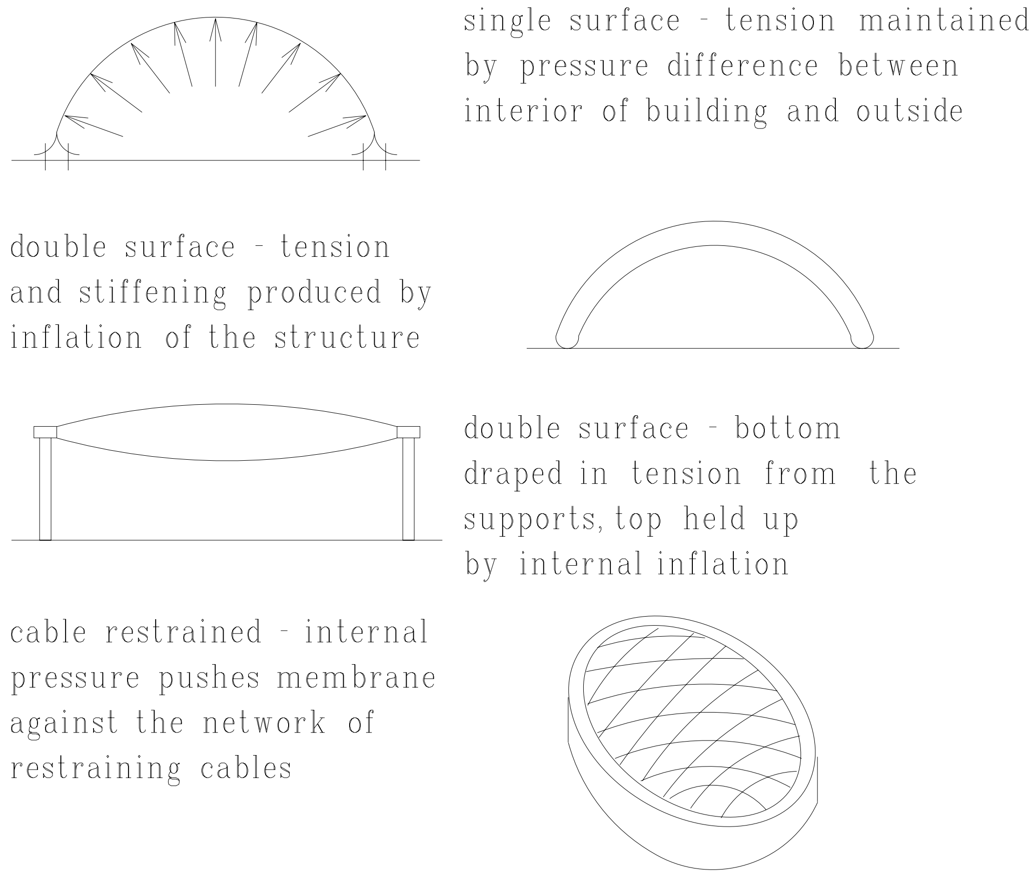

Figure 1.8: Examples of Air Supported Structures

Shells: 3D structures composed of a curved 2D surface, they are usually shaped to transmit compressive axial stresses only, Fig. 1.9.

Shells are classified in terms of their curvature.

1.10 Structural Engineering Courses

20 Structural engineering education can be approached from either one of two points of views:

Architectural: Start from overall design, and move toward detailed analysis.

Education: Elemental rather than global approach. Emphasis is on the individual structural elements and not always on the total system.

CVEN3525 will seek a balance between those two approaches.

21 This is only the third of a long series of courses which can be taken in Structural Engineering, Fig. 1.10

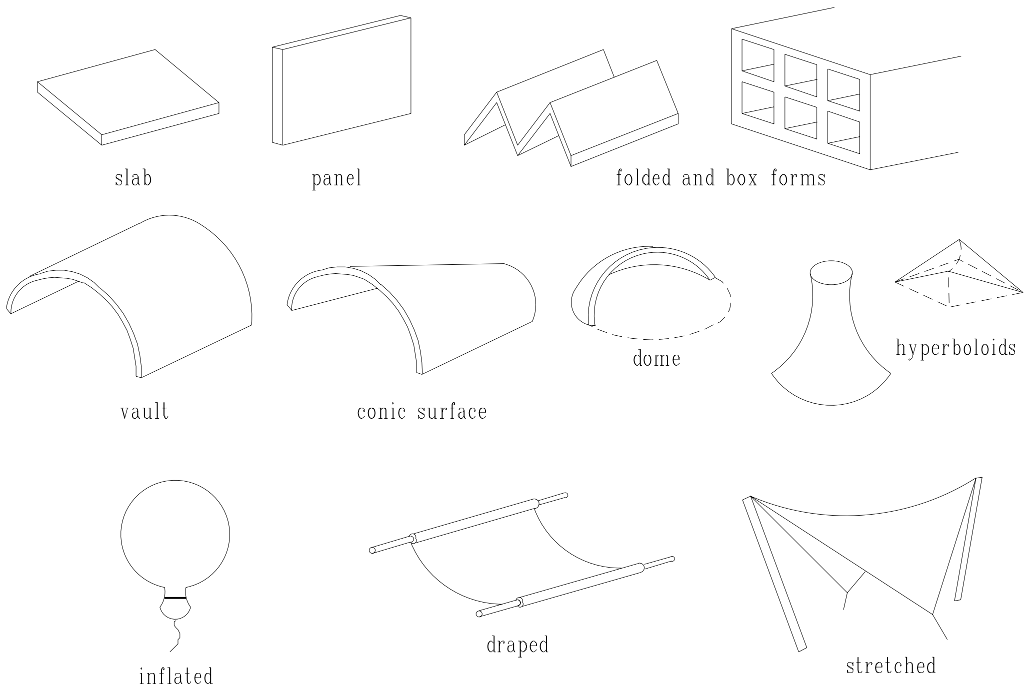

Figure 1.9: Basic Forms of Shells

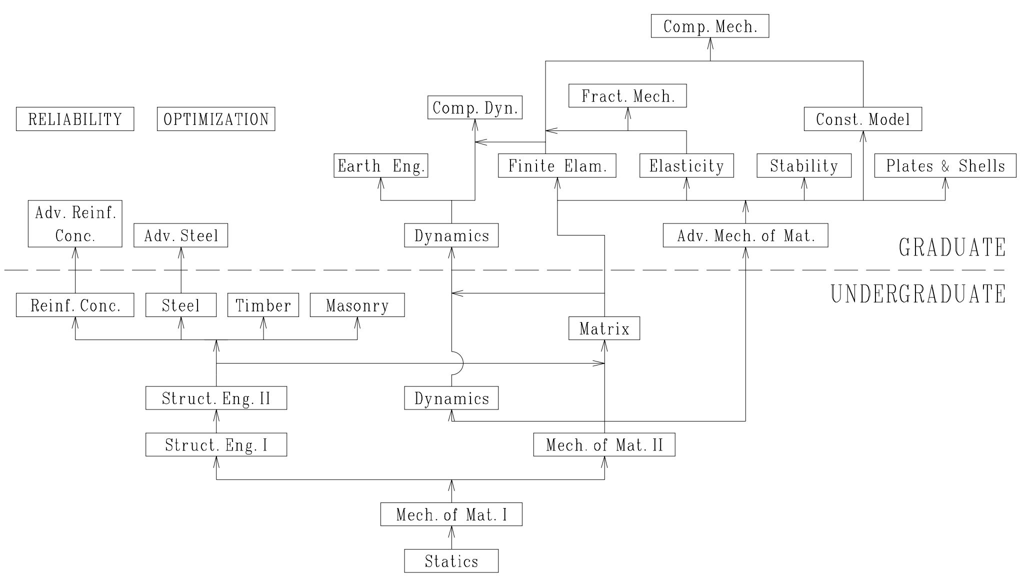

Figure 1.10: Sequence of Structural Engineering Courses

1.11 References

22 Following are some useful references for structural engineering, those marked by † were consulted, and "borrowed from" in preparing the Lecture Notes:

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1. Biggs, J.M., *Introduction to Structural Engineering; Analysis and Design*, Prentice Hall, 1986.
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1. Ambrose, J., *Building Structures*, second Ed. Wiley, 1993.
2. Salvadori, M. and Heller, R., *Structure in Architecture; The Building of Buildings*, Prentice Hall, Third Edition, 1986.
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4. Salvadori, M., *Why Buildings Stand Up; The Strength of Architecture*, Norton Paperack, 1990.
5. Lin, T.Y. and Stotesbury, S.D., *Structural Concepts and Systems for Architects and Engineers*, John Wiley, 1981.
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1. † Arbadi, F. *Structural Analysis and Behavior*, McGraw-Hill, Inc., 1991.
2. Hsieh, Y.Y., *Elementary Theory of Structures*, Third Edition, Prentice Hall, 1988.
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1. † Nilson, A., and Winter, G. *Design of Concrete Structures*, Eleventh Edition, McGraw Hill, 1991.
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1. ACI-318-89, *Building Code Requirements for Reinforced Concrete*, American Concrete Institute
2. *Load & Resistance Factor Design*, Manual of Steel Construction, American Institute of Steel Construction.
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4. *Minimum Design Loads in Buildings and Other Structures*, ANSI A58.1, American National Standards Institute, Inc., New York, 1972.

Chapter 2

EQUILIBRIUM & REACTIONS

To every action there is an equal and opposite reaction.

Newton's third law of motion

2.1 Introduction

1 In the analysis of structures (hand calculations), it is often easier (but not always necessary) to start by determining the reactions.

2 Once the reactions are determined, internal forces are determined next; finally, deformations (deflections and rotations) are determined last[1].

3 Reactions are necessary to determine **foundation load**.

4 Depending on the type of structures, there can be different types of support conditions, Fig. 2.1.

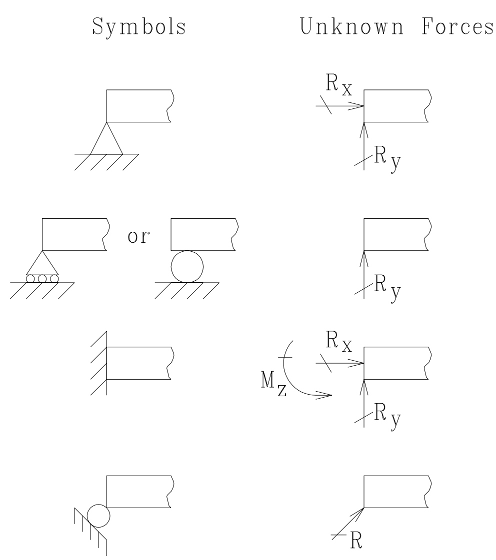


Figure 2.1: Types of Supports

[1]This is the sequence of operations in the **flexibility** method which lends itself to hand calculation. In the **stiffness** method, we determine displacements firsts, then internal forces and reactions. This method is most suitable to computer implementation.

Roller: provides a restraint in only one direction in a 2D structure, in 3D structures a roller may provide restraint in one or two directions. A roller will allow rotation.

Hinge: allows rotation but no displacements.

Fixed Support: will prevent rotation and displacements in all directions.

2.2 Equilibrium

5 Reactions are determined from the appropriate equations of static equilibrium.

6 Summation of forces and moments, **in a static system** must be equal to zero[2].

7 In a 3D cartesian coordinate system there are a total of 6 **independent** equations of equilibrium:

$$\boxed{\begin{array}{ccccccc} \Sigma F_x & = & \Sigma F_y & = & \Sigma F_z & = & 0 \\ \Sigma M_x & = & \Sigma M_y & = & \Sigma M_z & = & 0 \end{array}} \quad (2.1)$$

8 In a 2D cartesian coordinate system there are a total of 3 independent equations of equilibrium:

$$\boxed{\Sigma F_x = \Sigma F_y = \Sigma M_z = 0} \quad (2.2)$$

9 For reaction calculations, the externally applied load may be reduced to an equivalent force[3].

10 Summation of the moments can be taken with respect to **any** arbitrary point.

11 Whereas forces are represented by a vector, moments are also vectorial quantities and are represented by a curved arrow or a double arrow vector.

12 Not all equations are applicable to all structures, Table 2.1

Structure Type	**Equations**					
Beam, no axial forces		ΣF_y				ΣM_z
2D Truss, Frame, Beam	ΣF_x	ΣF_y				ΣM_z
Grid			ΣF_z	ΣM_x	ΣM_y	
3D Truss, Frame	ΣF_x	ΣF_y	ΣF_z	ΣM_x	ΣM_y	ΣM_z
Alternate Set						
Beams, no axial Force	ΣM_z^A	ΣM_z^B				
2 D Truss, Frame, Beam	ΣF_x	ΣM_z^A	ΣM_z^B			
	ΣM_z^A	ΣM_z^B	ΣM_z^C			

Table 2.1: Equations of Equilibrium

13 The three conventional equations of equilibrium in 2D: $\Sigma F_x, \Sigma F_y$ and ΣM_z can be replaced by the independent moment equations ΣM_z^A, ΣM_z^B, ΣM_z^C provided that A, B, and C **are not colinear**.

14 It is always preferable to **check** calculations by another equation of equilibrium.

15 Before you write an equation of equilibrium,

1. Arbitrarily decide which is the **+ve** direction
2. Assume a direction for the unknown quantities
3. The right hand side of the equation should be zero

[2] In a dynamic system $\Sigma F = ma$ where m is the mass and a is the acceleration.
[3] However for internal forces (shear and moment) we must use the actual load distribution.

If your reaction is negative, then it will be in a direction opposite from the one assumed.

16 Summation of all external forces (including reactions) is not necessarily zero (except at hinges and at points outside the structure).

17 Summation of external forces is equal and **opposite** to the internal ones. Thus the net force/moment is equal to zero.

18 The external forces give rise to the (non-zero) shear and moment diagram.

2.3 Equations of Conditions

19 If a structure has an **internal hinge** (which may connect two or more substructures), then this will provide an additional equation ($\Sigma M = 0$ at the hinge) which can be exploited to determine the reactions.

20 Those equations are often exploited in trusses (where each connection is a hinge) to determine reactions.

21 In an **inclined roller** support with S_x and S_y horizontal and vertical projection, then the reaction R would have, Fig. 2.2.

$$\boxed{\frac{R_x}{R_y} = \frac{S_y}{S_x}} \tag{2.3}$$

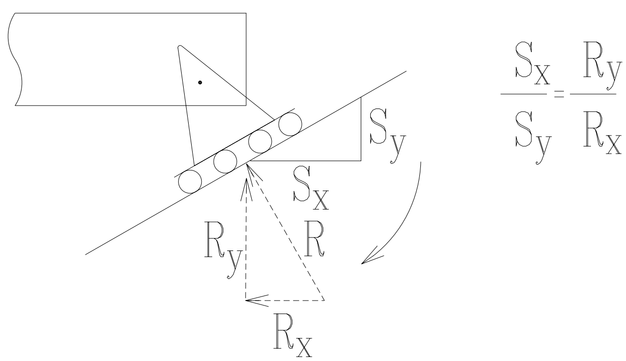


Figure 2.2: Inclined Roller Support

2.4 Static Determinacy

22 In statically determinate structures, reactions depend only on the geometry, boundary conditions and loads.

23 If the reactions can not be determined simply from the equations of static equilibrium (and equations of conditions if present), then the reactions of the structure are said to be **statically indeterminate**.

24 the **degree of static indeterminacy** is equal to the difference between the number of reactions and the number of equations of equilibrium, Fig. 2.3.

25 Failure of one support in a statically determinate system results in the collapse of the structures. Thus a statically indeterminate structure is **safer** than a statically determinate one.

26 For statically indeterminate structures[4], reactions depend also on the material properties (e.g. Young's and/or shear modulus) and element cross sections (e.g. length, area, moment of inertia).

[4] Which will be studied in CVEN3535.

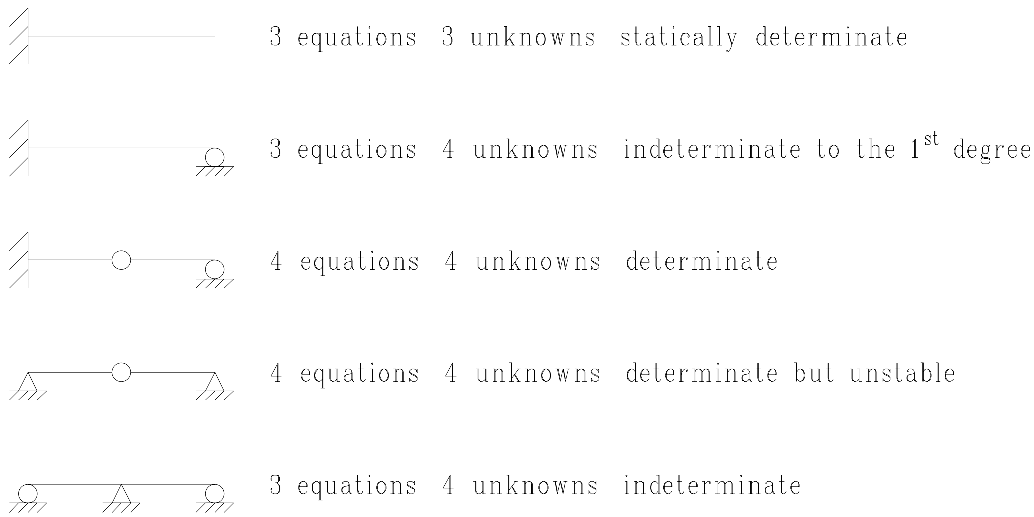
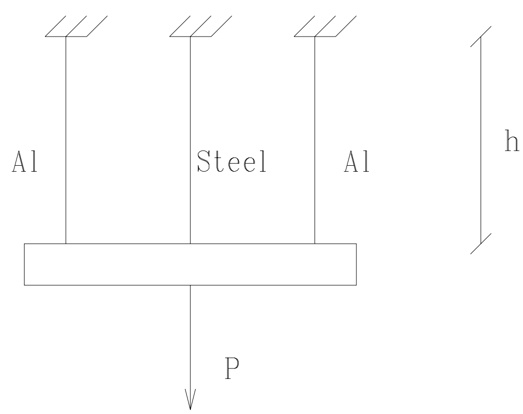


Figure 2.3: Examples of Static Determinate and Indeterminate Structures

■ **Example 2-1: Statically Indeterminate Cable Structure**

A rigid plate is supported by two aluminum cables and a steel one. Determine the force in each cable[5].



If the rigid plate supports a load P, determine the stress in each of the three cables. **Solution:**

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.

$$\begin{aligned} \Sigma M_z = 0; &\quad \Rightarrow \quad P_{Al}^{\text{left}} = P_{Al}^{\text{right}} \\ \Sigma F_y = 0; &\quad \Rightarrow \quad 2P_{Al} + P_{St} = P \end{aligned}$$

   Thus we effectively have two unknowns and one equation.

2. We need to have a third equation to solve for the three unknowns. This will be derived from the **compatibility of the displacements** in all three cables, i.e. all three displacements must be equal:

$$\left.\begin{array}{rcl} \sigma & = & \frac{P}{A} \\ \varepsilon & = & \frac{\Delta L}{L} \\ \varepsilon & = & \frac{\sigma}{E} \end{array}\right\} \Rightarrow \Delta L = \frac{PL}{AE}$$

[5]This example problem will be the only statically indeterminate problem analyzed in CVEN3525.

$$\underbrace{\frac{P_{Al}L}{E_{Al}A_{Al}}}_{\Delta_{Al}} = \underbrace{\frac{P_{St}L}{E_{St}A_{St}}}_{\Delta_{St}} \Rightarrow \frac{P_{Al}}{P_{St}} = \frac{(EA)_{Al}}{(EA)_{St}}$$

or $-(EA)_{St}P_{Al} + (EA)_{Al}P_{St} = 0$

3. Solution of this system of two equations with two unknowns yield:

$$\begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix} \begin{Bmatrix} P_{Al} \\ P_{St} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$
$$\Rightarrow \begin{Bmatrix} P_{Al} \\ P_{St} \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix}^{-1} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$
$$= \underbrace{\frac{1}{2(EA)_{Al} + (EA)_{St}}}_{\text{Determinant}} \begin{bmatrix} (EA)_{Al} & -1 \\ (EA)_{St} & 2 \end{bmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

■

2.5 Geometric Instability

27 The stability of a structure is determined not only by the number of reactions but also by their arrangement.

28 Geometric instability will occur if:

1. All **reactions are parallel** and a non-parallel load is applied to the structure.
2. All **reactions are concurrent**, Fig. **??**.

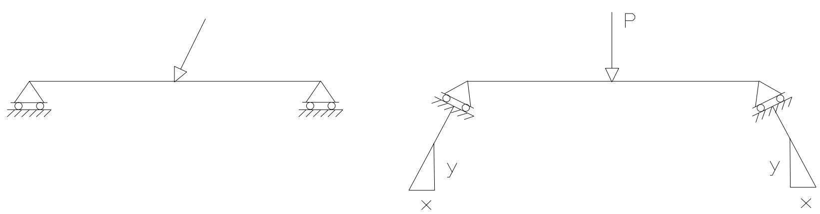


Figure 2.4: Geometric Instability Caused by Concurrent Reactions

3. The number of reactions is smaller than the number of equations of equilibrium, that is a **mechanism** is present in the structure.

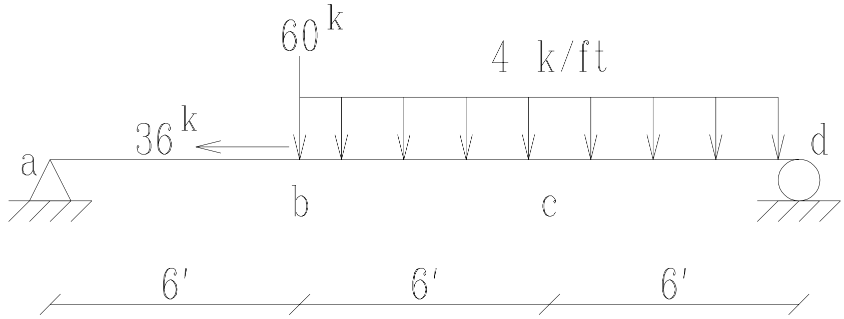
29 Mathematically, this can be shown if the **determinant** of the equations of equilibrium is equal to zero (or the equations are inter-dependent).

2.6 Examples

30 Examples of reaction calculation will be shown next. Each example has been carefully selected as it brings a different "twist" from the preceding one. Some of those same problems will be revisited later for the determination of the internal forces and/or deflections. Many of those problems are taken from Prof. Gerstle textbok *Basic Structural Analysis*.

■ **Example 2-2: Simply Supported Beam**

Determine the reactions of the simply supported beam shown below.



Solution:
The beam has 3 reactions, we have 3 equations of static equilibrium, hence it is statically determinate.

$$
\begin{aligned}
(+\rightarrow)\ \Sigma F_x = 0; &\quad \Rightarrow \quad R_{ax} - 36 \text{ k} = 0 \\
(+\uparrow)\ \Sigma F_y = 0; &\quad \Rightarrow \quad R_{ay} + R_{dy} - 60 \text{ k} - (4) \text{ k/ft}(12) \text{ ft} = 0 \\
(+\curvearrowleft)\ \Sigma M_z^c = 0; &\quad \Rightarrow \quad 12R_{ay} - 6R_{dy} - (60)(6) = 0
\end{aligned}
$$

or

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 12 & -6 \end{bmatrix} \begin{Bmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{Bmatrix} = \begin{Bmatrix} 36 \\ 108 \\ 360 \end{Bmatrix} \Rightarrow \begin{Bmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{Bmatrix} = \begin{Bmatrix} 36 \text{ k} \\ 56 \text{ k} \\ 52 \text{ k} \end{Bmatrix}
$$

Alternatively we could have used another set of equations:

$$
\begin{aligned}
(+\curvearrowright)\ \Sigma M_z^a = 0; &\quad (60)(6) + (48)(12) - (R_{dy})(18) = 0 \quad \Rightarrow \quad R_{dy} = 52 \text{ k} \uparrow \\
(+\curvearrowright)\ \Sigma M_z^d = 0; &\quad (R_{ay})(18) - (60)(12) - (48)(6) = 0 \quad \Rightarrow \quad R_{ay} = 56 \text{ k} \uparrow
\end{aligned}
$$

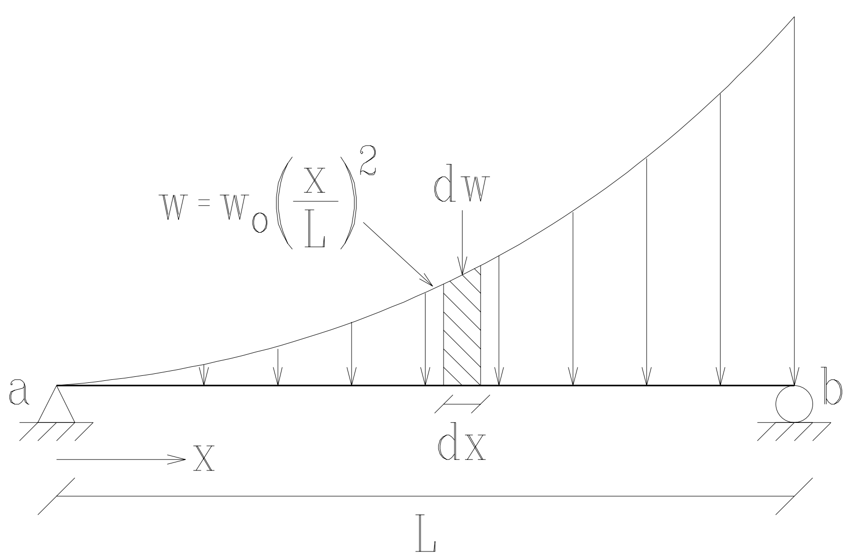
Check:

$$
(+\uparrow)\ \Sigma F_y = 0; ; 56 - 52 - 60 - 48 = 0\surd
$$

■

■ Example 2-3: Parabolic Load

Determine the reactions of a simply supported beam of length L subjected to a parabolic load $w = w_0 \left(\frac{x}{L}\right)^2$



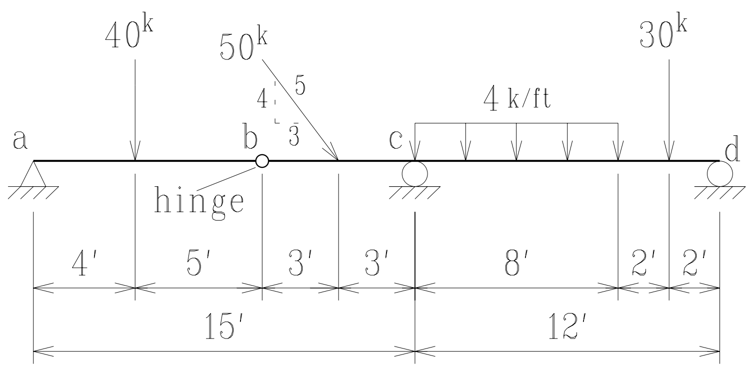
Solution:
Since there are no axial forces, there are two unknowns and two equations of equilibrium. Considering an infinitesimal element of length dx, weight dW, and moment dM:

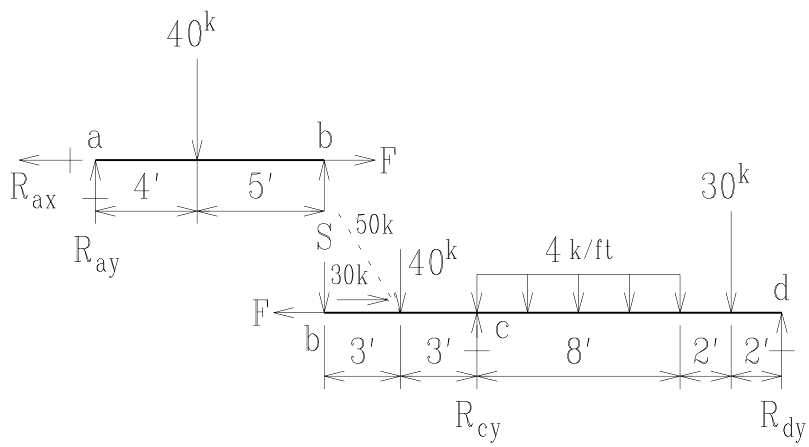
$$
\begin{aligned}
(+\circlearrowleft)\ \Sigma M_a = 0; \quad & \underbrace{\int_{x=0}^{x=L} \underbrace{\underbrace{\underbrace{w_0\left(\frac{x}{L}\right)^2}_{w} dx}_{dW} \times x}_{dM}}_{M} - (R_b)(L) && = 0 \\
& \Rightarrow R_b = \tfrac{1}{L} w_0 \left(\tfrac{L^4}{4L^2}\right) && = \tfrac{1}{4} w_0 L \\
(+\uparrow)\ \Sigma F_y = 0; \quad & R_a + \underbrace{\frac{1}{4} w_0 L}_{R_b} - \int_{x=0}^{x=L} w_0 \left(\frac{x}{L}\right)^2 dx && = 0 \\
& \Rightarrow R_a = \tfrac{w_0}{L^2}\tfrac{L^3}{3} - \tfrac{1}{4} w_0 L && = \tfrac{1}{12} w_0 L
\end{aligned}
$$

■

■ **Example 2-4: Three Span Beam**

Determine the reactions of the following three spans beam





Solution:
We have 4 unknowns (R_{ax}, R_{ay}, R_{cy} and R_{dy}), three equations of equilibrium and one equation of condition ($\Sigma M_b = 0$), thus the structure is statically determinate.

1. Isolating ab:

$$\begin{aligned} \Sigma M_{\curvearrowleft b} &= 0; & (9)(R_{ay}) - (40)(5) &= 0 & \Rightarrow R_{ay} &= 22.2 \text{ k} \uparrow \\ (+\curvearrowleft)\ \Sigma M_a &= 0; & (40)(4) - (S)(9) &= 0 & \Rightarrow S &= 17.7 \text{ k} \uparrow \\ \Sigma F_x &= 0; & & & \Rightarrow R_{ax} &= 30 \text{ k} \leftarrow \end{aligned}$$

2. Isolating bd:

$$\begin{aligned} (+\curvearrowleft)\ \Sigma M_d = 0; \quad & -(17.7)(18) - (40)(15) - (4)(8)(8) - (30)(2) + R_{cy}(12) = 0 \\ & \Rightarrow R_{cy} = \tfrac{1{,}236}{12} = 103 \text{ k} \uparrow \\ (+\curvearrowleft)\ \Sigma M_c = 0; \quad & -(17.7)(6) - (40)(3) + (4)(8)(4) + (30)(10) - R_{dy}(12) = 0 \\ & \Rightarrow R_{dy} = \tfrac{201.3}{12} = 16.7 \text{ k} \uparrow \end{aligned}$$

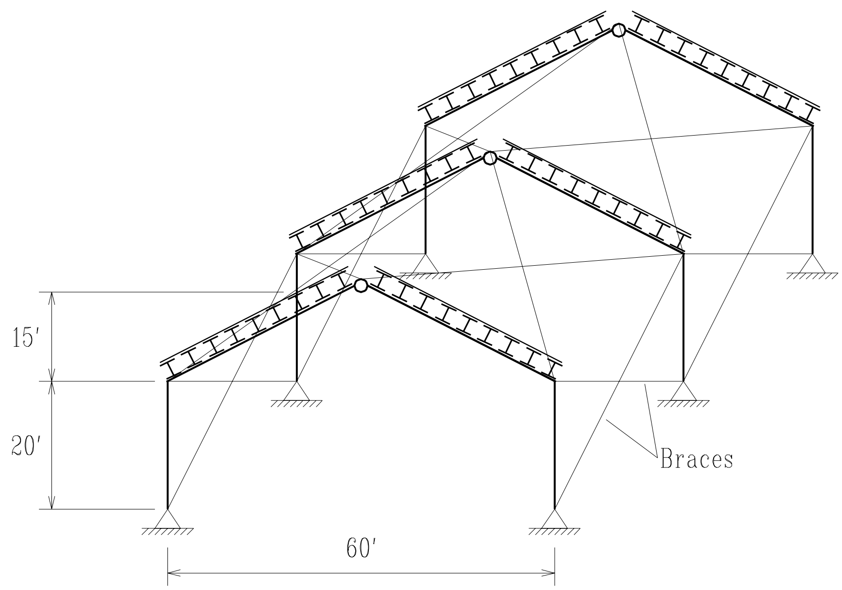
3. Check

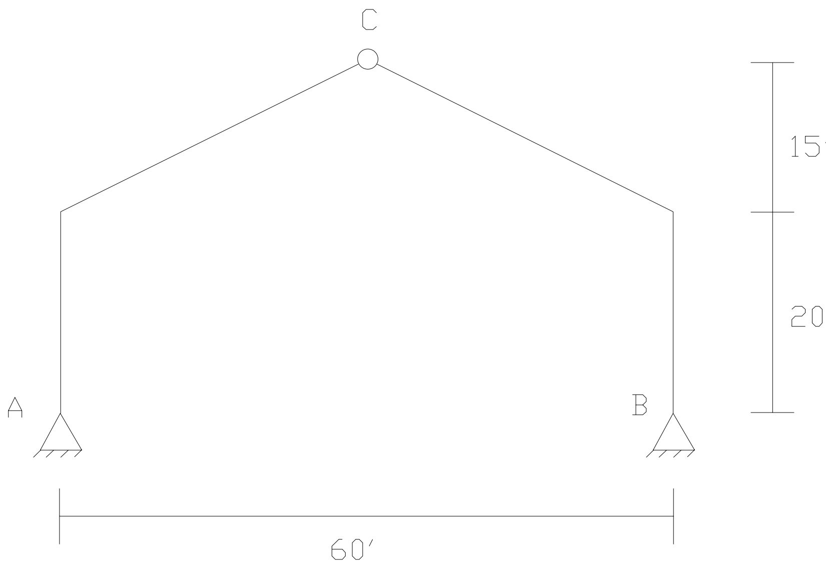
$$\Sigma F_y = 0; \uparrow; 22.2 - 40 - 40 + 103 - 32 - 30 + 16.7 = 0 \surd$$

■

■ Example 2-5: Three Hinged Gable Frame

The three-hinged gable frames spaced at 30 ft. on center. Determine the reactions components on the frame due to: 1) Roof dead load, of 20 psf of roof area; 2) Snow load, of 30 psf of horizontal projection; 3) Wind load of 15 psf of vertical projection. Determine the critical design values for the vertical and horizontal reactions.





Solution:

1. Due to symmetry, we will consider only the dead load on one side of the frame.
2. Due to symmetry, there is no vertical force transmitted by the hinge for snow and dead load.
3. Roof dead load per frame is
$$DL = (20)\text{ psf}(30)\text{ ft}\left(\sqrt{30^2+15^2}\right)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 20.2\text{ k}\downarrow$$
4. Snow load per frame is
$$SL = (30)\text{ psf}(30)\text{ ft}(30)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 27.\text{ k}\downarrow$$
5. Wind load per frame (ignoring the suction) is
$$WL = (15)\text{ psf}(30)\text{ ft}(35)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 15.75\text{ k}\rightarrow$$
6. There are 4 reactions, 3 equations of equilibrium and one equation of condition $\Rightarrow$ statically determinate.
7. The horizontal reaction H due to a vertical load V at midspan of the roof, is obtained by taking moment with respect to the hinge
$$(+\circlearrowleft)\,\Sigma M_C = 0;\quad 15(V) - 30(V) + 35(H) = 0 \quad\Rightarrow\quad H = \tfrac{15V}{35} = .429V$$
Substituting for roof dead and snow load we obtain
$$\begin{array}{lllll} V_{DL}^A = & V_{DL}^B & = & & 20.2\text{ k}\uparrow \\ H_{DL}^A = & H_{DL}^B & = & (.429)(20.2) & = & 8.66\text{ k}\rightarrow \\ V_{SL}^A = & V_{SL}^B & = & & 27.\text{ k}\uparrow \\ H_{SL}^A = & H_{SL}^B & = & (.429)(27.) & = & 11.58\text{ k}\rightarrow \end{array}$$
8. The reactions due to wind load are
$$\begin{array}{lll} (+\circlearrowleft)\,\Sigma M_A = 0; & (15.75)(\frac{20+15}{2}) - V_{WL}^B(60) = 0 & \Rightarrow V_{WL}^B = 4.60\text{ k}\uparrow \\ (+\circlearrowleft)\,\Sigma M_C = 0; & H_{WL}^B(35) - (4.6)(30) = 0 & \Rightarrow H_{WL}^B = 3.95\text{ k}\leftarrow \\ (+\rightarrow)\,\Sigma F_x = 0; & 15.75 - 3.95 - H_{WL}^A = 0 & \Rightarrow H_{WL}^A = 11.80\text{ k}\leftarrow \\ (+\uparrow)\,\Sigma F_y = 0; & V_{WL}^B - V_{WL}^A = 0 & \Rightarrow V_{WL}^A = -4.60\text{ k}\downarrow \end{array}$$

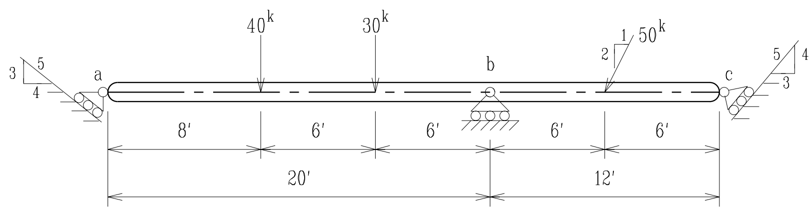
9. Thus supports should be designed for

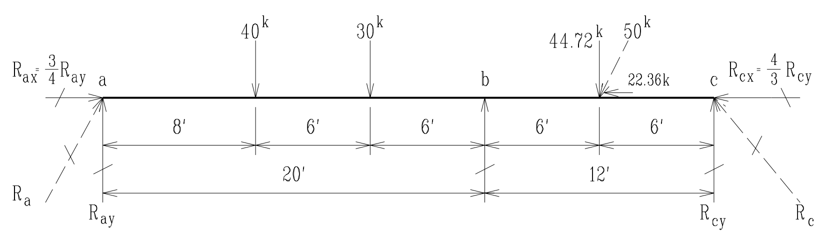
$$\begin{aligned} H = &\ 8.66 \text{ k} + 11.58 \text{ k} + 3.95 \text{ k} &= 24.19 \text{ k} \\ V = &\ 20.7 \text{ k} + 27.0 \text{ k} + 4.60 \text{ k} &= 52.3 \text{ k} \end{aligned}$$

■

■ Example 2-6: Inclined Supports

Determine the reactions of the following two spans beam resting on inclined supports.





Solution:

A priori we would identify 5 reactions, however we do have 2 equations of conditions (one at each inclined support), thus with three equations of equilibrium, we have a statically determinate system.

$$\begin{aligned} (+\circlearrowleft)\ \Sigma M_b = 0; &\quad (R_{ay})(20) - (40)(12) - (30)(6) + (44.72)(6) - (R_{cy})(12) = 0 \\ &\quad \Rightarrow 20R_{ay} = 12R_{cy} + 391.68 \\ (+ \rightarrow)\ \Sigma F_x = 0; &\quad \tfrac{3}{4}R_{ay} - 22.36 - \tfrac{4}{3}R_{cy} = 0 \\ &\quad \Rightarrow R_{cy} = 0.5625R_{ay} - 16.77 \end{aligned}$$

Solving for those two equations:

$$\begin{bmatrix} 20 & -12 \\ 0.5625 & -1 \end{bmatrix} \left\{ \begin{array}{c} R_{ay} \\ R_{cy} \end{array} \right\} = \left\{ \begin{array}{c} 391.68 \\ 16.77 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} R_{ay} \\ R_{cy} \end{array} \right\} = \left\{ \begin{array}{c} 14.37 \text{ k} \uparrow \\ -8.69 \text{ k} \downarrow \end{array} \right\}$$

The horizontal components of the reactions at a and c are

$$\begin{aligned} R_{ax} &= \tfrac{3}{4}14.37 &= 10.78 \text{ k} \rightarrow \\ R_{cx} &= \tfrac{4}{3}8.69 &= -11.59 \text{ k} \rightarrow \end{aligned}$$

Finally we solve for R_{by}

$$\begin{aligned} (+\circlearrowleft)\ \Sigma M_a = 0; &\quad (40)(8) + (30)(14) - (R_{by})(20) + (44.72)(26) + (8.69)(32) = 0 \\ &\quad \Rightarrow R_{by} = 109.04 \text{ k} \uparrow \end{aligned}$$

We check our results

$$\begin{aligned} (+\uparrow)\ \Sigma F_y = 0; &\quad 14.37 - 40 - 30 + 109.04 - 44.72 - 8.69 &= 0\surd \\ (+\rightarrow)\ \Sigma F_x = 0; &\quad 10.78 - 22.36 + 11.59 &= 0\surd \end{aligned}$$

■

2.7 Arches

31 See Section ??.

Chapter 3

TRUSSES

3.1 Introduction

3.1.1 Assumptions

1 Cables and trusses are 2D or 3D structures composed of an assemblage of simple one dimensional components which transfer only **axial** forces along their axis.

2 Cables can carry only tensile forces, trusses can carry tensile and compressive forces.

3 Cables tend to be **flexible**, and hence, they tend to oscillate and therefore must be stiffened.

4 Trusses are extensively used for bridges, long span roofs, electric tower, space structures.

5 For trusses, it is assumed that

1. Bars are **pin-connected**
2. Joints are frictionless hinges[1].
3. Loads are applied at the **joints only**.

6 A truss would typically be composed of triangular elements with the bars on the **upper chord** under compression and those along the **lower chord** under tension. Depending on the **orientation of the diagonals**, they can be under either tension or compression.

7 Fig. 3.1 illustrates some of the most common types of trusses.

8 It can be easily determined that in a Pratt truss, the diagonal members are under tension, while in a Howe truss, they are in compression. Thus, the Pratt design is an excellent choice for steel whose members are slender and long diagonal member being in tension are not prone to buckling. The vertical members are less likely to buckle because they are shorter. On the other hand the Howe truss is often preferred for for heavy timber trusses.

9 In a truss analysis or design, we seek to determine the internal force along each member, Fig. 3.2

3.1.2 Basic Relations

Sign Convention: Tension positive, compression negative. On a truss the axial forces are indicated as forces acting on the joints.

Stress-Force: $\sigma = \frac{P}{A}$

Stress-Strain: $\sigma = E\varepsilon$

[1]In practice the bars are riveted, bolted, or welded directly to each other or to gusset plates, thus the bars are not free to rotate and so-called **secondary bending moments** are developed at the bars. Another source of secondary moments is the dead weight of the element.

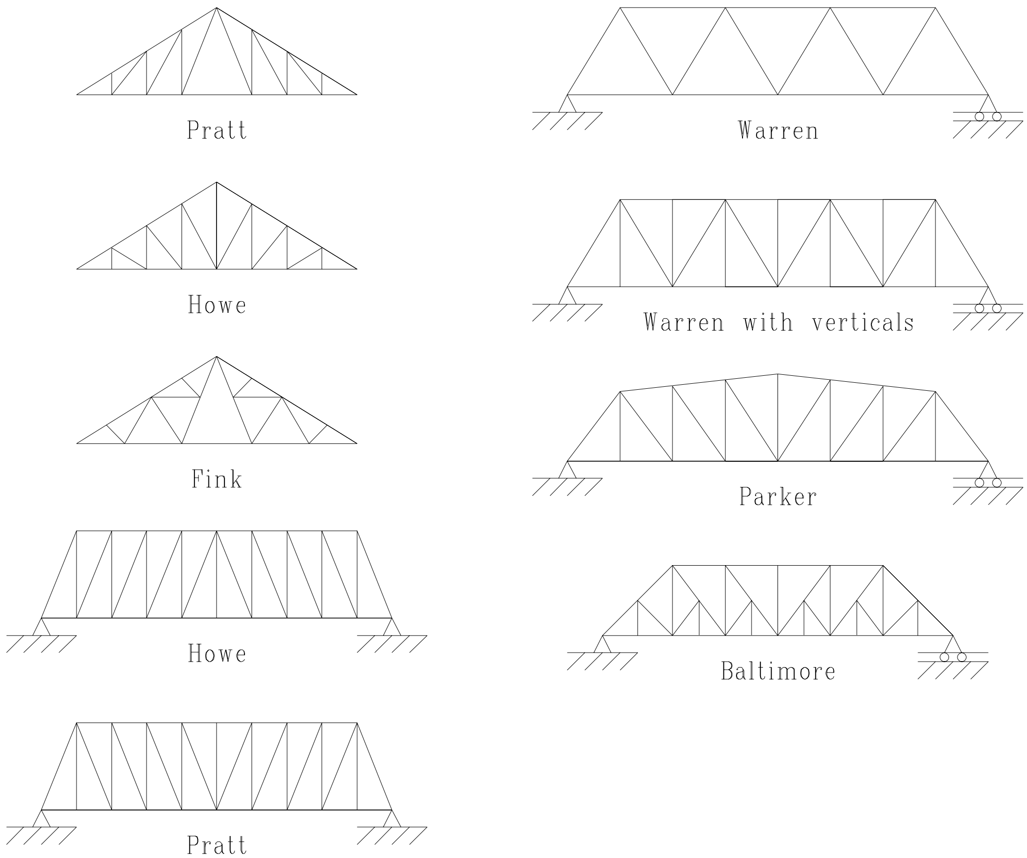


Figure 3.1: Types of Trusses

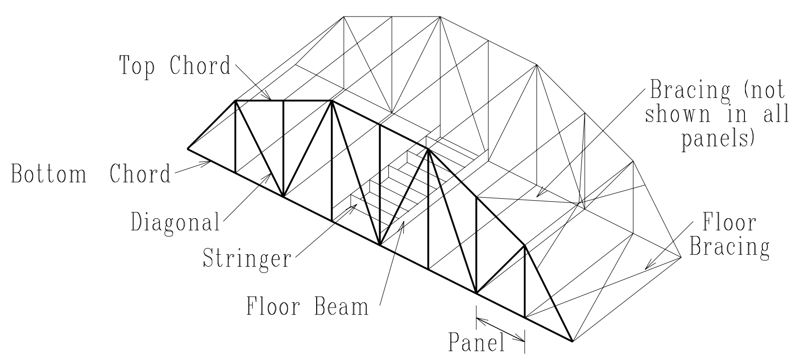


Figure 3.2: Bridge Truss

Force-Displacement: $\varepsilon = \frac{\Delta L}{L}$

Equilibrium: $\Sigma F = 0$

3.2 Trusses

3.2.1 Determinacy and Stability

10 Trusses are **statically determinate** when all the bar forces can be determined from the equations of **statics** alone. Otherwise the truss is **statically indeterminate**.

11 A truss may be statically/externally determinate or indeterminate with respect to the reactions (more than 3 or 6 reactions in 2D or 3D problems respectively).

12 A truss may be internally determinate or indeterminate, Table 3.1.

13 If we refer to j as the number of joints, R the number of reactions and m the number of members, then we would have a total of $m + R$ unknowns and $2j$ (or $3j$) equations of statics (2D or 3D at each joint). If we do not have enough equations of statics then the problem is indeterminate, if we have too many equations then the truss is unstable, Table 3.1.

	2D	3D
Static Indeterminacy		
External	$R > 3$	$R > 6$
Internal	$m + R > 2j$	$m + R > 3j$
Unstable	$m + R < 2j$	$m + R < 3j$

Table 3.1: Static Determinacy and Stability of Trusses

14 If $m < 2j - 3$ (in 2D) the truss is not internally stable, and it will not remain a rigid body when it is detached from its supports. However, when attached to the supports, the truss will be rigid.

15 Since each joint is pin-connected, we can apply $\Sigma M = 0$ at each one of them. Furthermore, summation of forces applied on a joint must be equal to zero.

16 For 2D trusses the external equations of equilibrium which can be used to determine the reactions are $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma M_Z = 0$. For 3D trusses the available equations are $\Sigma F_X = 0$, $\Sigma F_Y = 0$, $\Sigma F_Z = 0$ and $\Sigma M_X = 0$, $\Sigma M_Y = 0$, $\Sigma M_Z = 0$.

17 For a 2D truss we have 2 equations of equilibrium $\Sigma F_X = 0$ and $\Sigma F_Y = 0$ which can be applied at each joint. For 3D trusses we would have three equations: $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma F_Z = 0$.

18 Fig. 3.3 shows a truss with 4 reactions, thus it is externally indeterminate. This truss has 6 joints ($j = 6$), 4 reactions ($R = 4$) and 9 members ($m = 9$). Thus we have a total of $m + R = 9 + 4 = 13$ unknowns and $2 \times j = 2 \times 6 = 12$ equations of equilibrium, thus the truss is statically indeterminate.

19 There are two methods of analysis for statically determinate trusses

1. The Method of joints
2. The Method of sections

3.2.2 Method of Joints

20 The method of joints can be summarized as follows

1. Determine if the structure is statically determinate
2. Compute all reactions

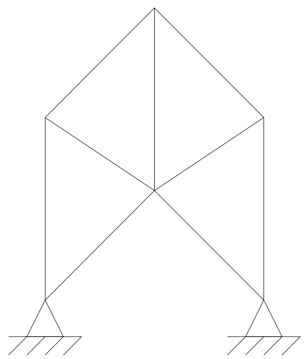


Figure 3.3: A Statically Indeterminate Truss

3. Sketch a free body diagram showing all joint loads (including reactions)
4. For each joint, and starting with the loaded ones, apply the appropriate equations of equilibrium (ΣF_x and ΣF_y in 2D; ΣF_x, ΣF_y and ΣF_z in 3D).
5. Because truss elements can only carry axial forces, the resultant force ($\vec{F} = \vec{F_x} + \vec{F_y}$) must be **along** the member, Fig. 3.4.

$$\boxed{\frac{F}{L} = \frac{F_x}{L_x} = \frac{F_y}{L_y}} \tag{3.1}$$

21 Always keep track of the x and y components of a member force (F_x, F_y), as those might be needed later on when considering the force equilibrium at another joint to which the member is connected.

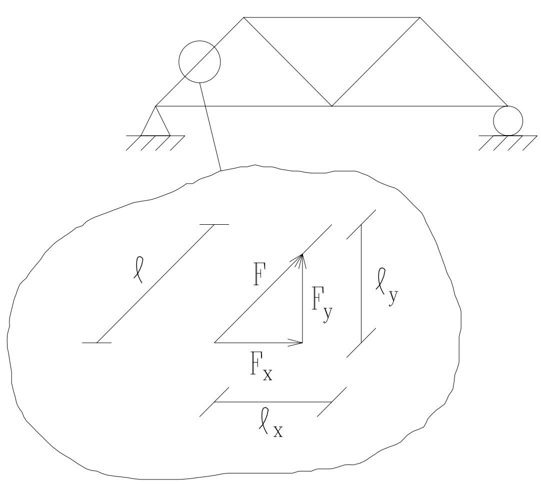


Figure 3.4: X and Y Components of Truss Forces

22 This method should be used when **all** member forces must be determined.

23 In truss analysis, there is **no sign convention**. A member is **assumed** to be under tension (or compression). If after analysis, the force is found to be negative, then this would imply that the wrong assumption was made, and that the member should have been under compression (or tension).

24 On a **free body diagram**, the internal forces are represented by arrow acting **on the joints** and not as end forces on the element itself. That is for tension, the arrow is pointing away from the joint, and for compression toward the joint, Fig. 3.5.

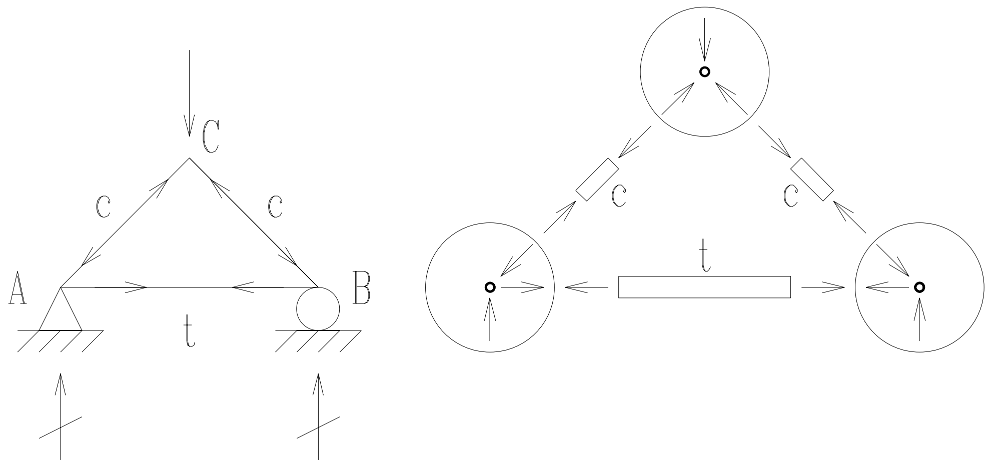
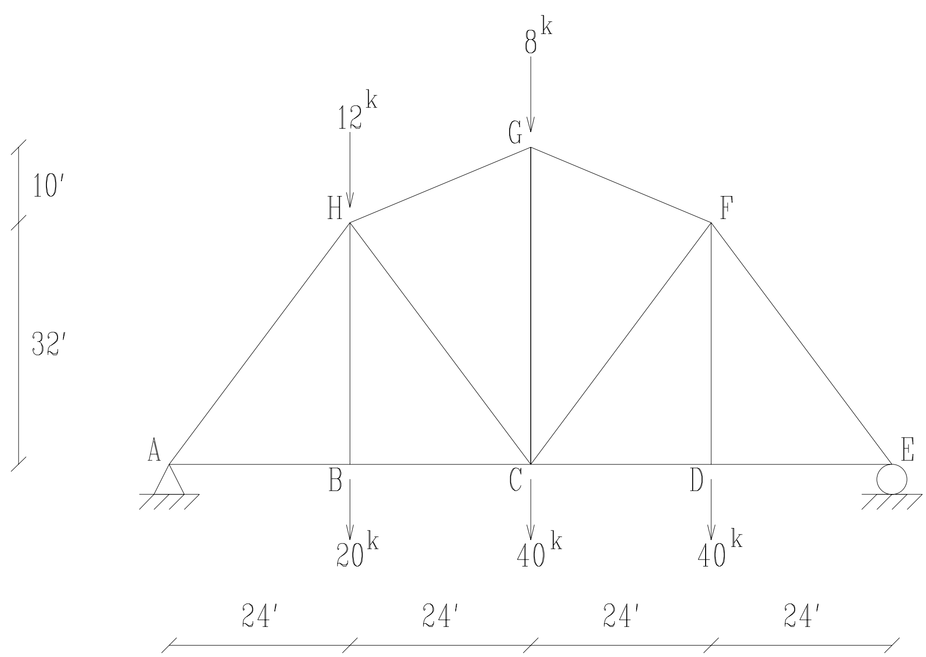


Figure 3.5: Sign Convention for Truss Element Forces

■ Example 3-1: Truss, Method of Joints

Using the method of joints, analyze the following truss



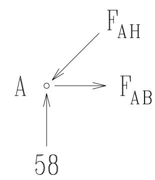
Solution:

1. $R = 3$, $m = 13$, $2j = 16$, and $m + R = 2j\surd$
2. We compute the reactions

$$
\begin{array}{rcl}
(+\circlearrowleft)\ \Sigma M_E = 0; & \Rightarrow & (20+12)(3)(24) + (40+8)(2)(24) + (40)(24) - R_{A_y}(4)(24) = 0 \\
 & \Rightarrow & R_{A_y} = \boxed{58 \text{ k} \uparrow} \\
(+\downarrow)\ \Sigma F_y = 0; & \Rightarrow & 20 + 12 + 40 + 8 + 40 - 58 - R_{E_y} = 0 \\
 & \Rightarrow & R_{E_y} = \boxed{62 \text{ k} \uparrow}
\end{array}
$$

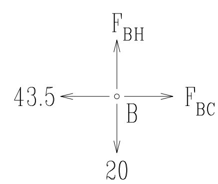
3. Consider each joint separately:

Node A: Clearly AH is under compression, and AB under tension.



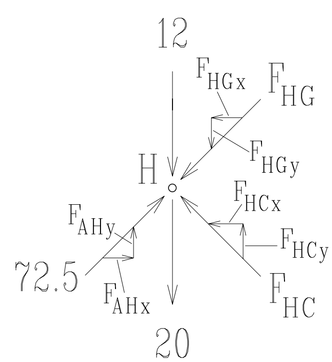
$$\begin{array}{rcll}
(+\uparrow)\ \Sigma F_y = 0; & \Rightarrow & -F_{AH_y} + 58 = 0 & \\
 & & F_{AH} = \frac{l}{l_y}(F_{AH_y}) & \\
 & & l_y = 32 & l = \sqrt{32^2+24^2} = 40 \\
 & \Rightarrow & F_{AH} = \frac{40}{32}(58) = \boxed{72.5 \text{ k Compression}} & \\
(+\rightarrow)\ \Sigma F_x = 0; & \Rightarrow & -F_{AH_x} + F_{AB} = 0 & \\
 & & F_{AB} = \frac{L_x}{L_y}(F_{AH_y}) = \frac{24}{32}(58) = \boxed{43.5 \text{ k Tension}} &
\end{array}$$

Node B:



$$\begin{array}{rclcl}
(+\rightarrow)\ \Sigma F_x = 0; & \Rightarrow & F_{BC} & = & \boxed{43.5 \text{ k Tension}} \\
(+\uparrow)\ \Sigma F_y = 0; & \Rightarrow & F_{BH} & = & \boxed{20 \text{ k Tension}}
\end{array}$$

Node H:



$$\begin{array}{rcl}
(+\rightarrow)\ \Sigma F_x = 0; & \Rightarrow & F_{AH_x} - F_{HC_x} - F_{HG_x} = 0 \\
 & & 43.5 - \frac{24}{\sqrt{24^2+32^2}}(F_{HC}) - \frac{24}{\sqrt{24^2+10^2}}(F_{HG}) = 0 \\
(+\uparrow)\ \Sigma F_y = 0; & \Rightarrow & F_{AH_y} + F_{HC_y} - 12 - F_{HG_y} - 20 = 0 \\
 & & 58 + \frac{32}{\sqrt{24^2+32^2}}(F_{HC}) - 12 - \frac{10}{\sqrt{24^2+10^2}}(F_{HG}) - 20 = 0
\end{array}$$

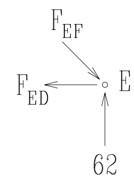
This can be most conveniently written as

$$\begin{bmatrix} 0.6 & 0.921 \\ -0.8 & 0.385 \end{bmatrix} \left\{ \begin{array}{c} F_{HC} \\ F_{HG} \end{array} \right\} = \left\{ \begin{array}{c} -7.5 \\ 52 \end{array} \right\} \tag{3.2}$$

Solving we obtain

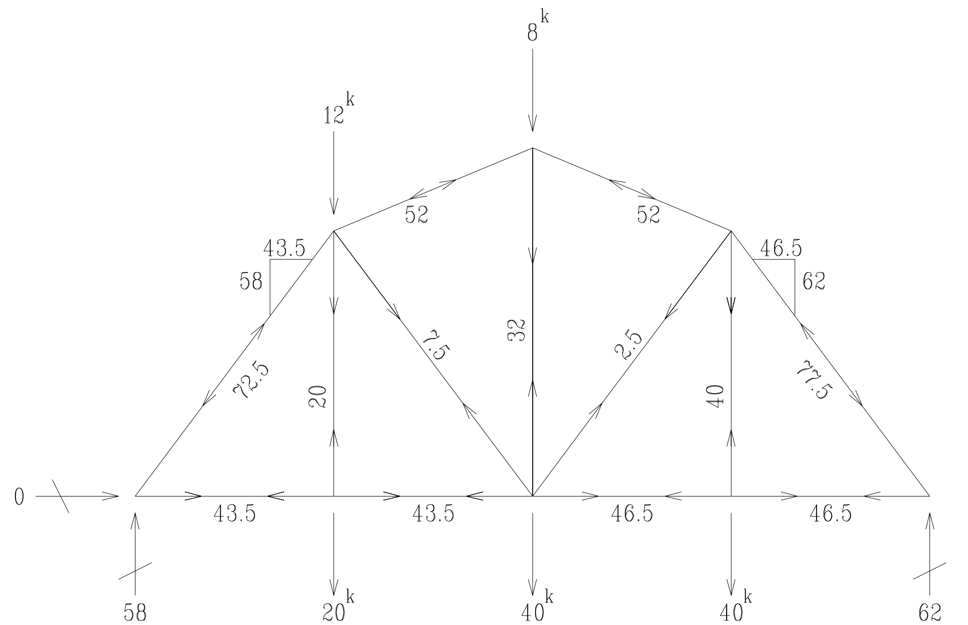
$$\begin{array}{rcl}
F_{HC} & = & \boxed{-7.5 \text{ k Tension}} \\
F_{HG} & = & \boxed{52 \text{ k Compression}}
\end{array}$$

Node E:



$$
\begin{array}{lllll}
\Sigma F_y = 0; & \Rightarrow & F_{EF_y} = 62 & \Rightarrow F_{EF} = \frac{\sqrt{24^2+32^2}}{32}(62) & = \boxed{77.5 \text{ k}} \\
\Sigma F_x = 0; & \Rightarrow & F_{ED} = F_{EF_x} & \Rightarrow F_{ED} = \frac{24}{32}(F_{EF_y}) = \frac{24}{32}(62) & = \boxed{46.5 \text{ k}}
\end{array}
$$

The results of this analysis are summarized below



4. We could check our calculations by verifying equilibrium of forces at a node not previously used, such as D

■

3.2.2.1 Matrix Method

25 This is essentially the method of joints cast in matrix form[2].

26 We seek to determine the **Statics Matrix** $[B]$ such that

$$
\begin{bmatrix} B_{FF} & B_{FR} \\ B_{RF} & B_{RR} \end{bmatrix} \left\{ \frac{F}{R} \right\} = \left\{ \frac{P}{0} \right\} \tag{3.3}
$$

27 This method can be summarized as follows

1. Select a coordinate system
2. Number the joints and the elements separately

[2]Writing a computer program for this method, would be an acceptable project.

3. Assume
   (a) All member forces to be positive (i.e. tension)
   (b) All reactions to be positive
4. Compute the **direction cosines** at each node j and for each element e, Fig. 3.6

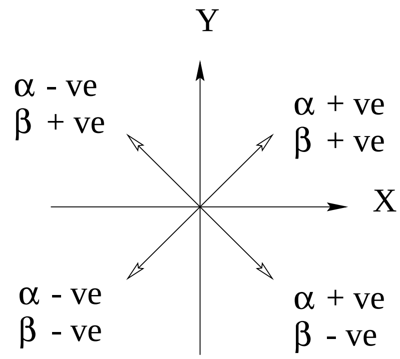


Figure 3.6: Direction Cosines

$$\begin{aligned} \alpha_j^e &= \frac{L_x}{L} \\ \beta_j^e &= \frac{L_y}{L} \end{aligned}$$

5. Write the two equations of equilibrium at each joint j in terms of the unknowns (member forces and reactions), Fig. 3.7

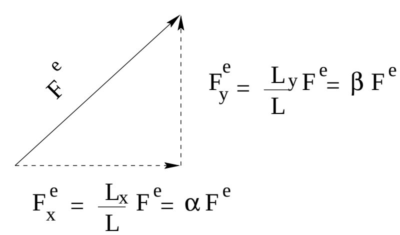


Figure 3.7: Forces Acting on Truss Joint

$$\begin{aligned} (+ \rightarrow)\, \Sigma F_x = 0; \quad &\Rightarrow \quad \Sigma_{e=1}^{\#of elements} \alpha_j^e F_e + R_{x_j} + P_{x_j} = 0 \\ (+ \uparrow)\, \Sigma F_y = 0; \quad &\Rightarrow \quad \Sigma_{e=1}^{\#of elements} \beta_j^e F_e + R_{y_j} + P_{y_j} = 0 \end{aligned}$$

6. Invert the matrix to compute $\{F\}$ and $\{R\}$

28 The advantage of this method, is that once the $[B]$ matrix has been inverted, we can readily reanalyze the same structure for **different load cases**. With the new design codes in which dead loads and live loads are separately factored (Chapter **??**), this method can save substantial reanalysis effort. Furthermore, when deflections are determined by the virtual force method (Chapter 7), two analysis with two different loads are required.

29 This method may be the only one appropriate to analyze statically determinate trusses which solution's defy the previous two methods, Fig. 3.8.

30 Element e connecting joint i to j will have $\alpha_i^e = -\alpha_j^e$, and $\beta_i^e = -\beta_j^e$

31 The matrix $[B]$ will have $2j$ rows and $m + r$ columns. It can only be inverted if ti is symmetric (i.e $2j = m + r$, statically determinate).

32 An algorithm to implement this method in simple computer programs:

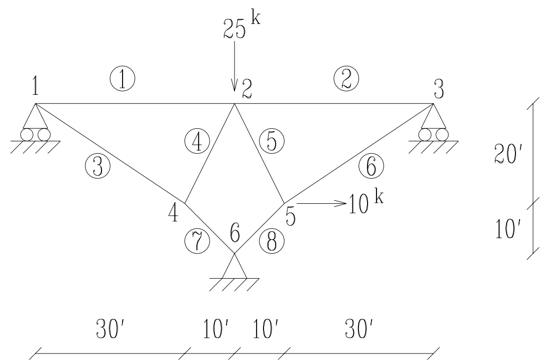


Figure 3.8: Complex Statically Determinate Truss

1. Prepare input data:

   (a) Nodal information:

Node	$x(1)$	$x(2)$	$P(1)$	$P(2)$	$R(1)$	$R(2)$
1	0.	0.	0.	0.	1	1
2	10.	0.	0.	0.	0	1
3	5.	5.	0.	-10.	0	0

   where $x(1)$, $x(2)$, $P(1)$, and $P(2)$ are the x and y coordinates; the x and y component of applied nodal load. $R(1)$, $R(2)$ correspond to the x and y boundary conditions, they will be set to 1 if there is a corresponding reaction, and 0 otherwise.

   (b) Element Connectivity

Element	Node(1)	Node(2)
1	1	2
2	2	3
3	3	1

2. Determine the size of the matrix (2 times the number of joints) and initialize a square matrix of this size to zero.

3. Assemble the first submnatrix of the Statics matrix

   (a) Loop over each element (e), determine L_x^i, L_y^i (as measured from the first node), L, $\alpha_i^e = \frac{L_x^i}{L}$ $\beta_i^e = \frac{L_y^i}{L}$, $\alpha_j^e = -\alpha_i^e$, $\beta_j^e = -\beta_i^e$, where i and j are the end nodes of element e.

   (b) Store α_i^e in row $2i-1$ column e, β_i^e in row $2i$ column e, α_j^e in row $2j-1$ column e, β_j^e in row $2j$ column e.

4. Loop over each node:

   (a) Store R_x^i (reaction boundary condition for node i along x) in row $2i-1$ column $e+2i-1$. Similarly, store R_y^i (reaction boundary condition for node i along y) in row $2i$ column $e+2i$.

   (b) Store P_x^i (Load at node i along x axis) in vector **P** row $2i-1$, similarly, store P_y^i in row $2i$.

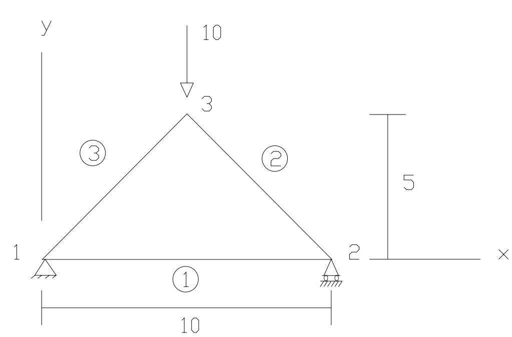
5. Invert the matrix **B**, multiply it with the load vector **P** and solve for the unknown member forces and reactions.

```
%
%  Initialize the statics matrix
%
b(1:2*npoin,1:2*npoin)=0.;
%
```

```
% Determine direction cosines and insert them in b
%
for ielem=1:nelem
  nod1=lnods(ielem,1);
  nod2=lnods(ielem,2);
  lx=coord(nod2,1)-coord(nod1,1);
  ly=coord(nod2,2)-coord(nod1,2);
  l_elem(ielem)=
  alpha_i=...
  beta_i=...
  b(...,...)=...;
  b(...,...)=...;
  alpha_j=...
  beta_j=...
  b(...,...)=...
  b(...,...)=...
end
%
% Boundary conditions
%
nbc=0;
for inode=1:npoin
  for ibc=1:2
    if id(inode,ibc)==1
         nbc=nbc+1;
         b(...,...)=1.;
    end
  end
end
```

■ **Example 3-2: Truss I, Matrix Method**

Determine all member forces for the following truss



Solution:

1. We first determine the directions cosines

Member	Nodes	α	β	α	β
1	1-2	Node 1		Node 2	
		$\alpha_1^1 = 1$	$\beta_1^1 = 0$	$\alpha_2^1 = -1$	$\beta_2^1 = 0$
2	2-3	Node 2		Node 3	
		$\alpha_2^2 = -.7071$	$\beta_2^2 = .707$	$\alpha_3^2 = .707$	$\beta_3^2 = -.707$
3	3-1	Node 3		Node 1	
		$\alpha_3^3 = -.707$	$\beta_3^3 = -.707$	$\alpha_1^3 = .707$	$\beta_1^3 = .707$

2. Next we write the equations of equilibrium

$$
\begin{array}{lll}
 & & \begin{array}{cccccc} F_1 & F_2 & F_3 & R_{1x} & R_{1y} & R_{2y} \end{array} \\
\text{Node } 1 & \Sigma F_x = 0 & \\
 & \Sigma F_y = 0 & \\
\text{Node } 2 & \Sigma F_x = 0 & \\
 & \Sigma F_y = 0 & \\
\text{Node } 3 & \Sigma F_x = 0 & \\
 & \Sigma F_y = 0 &
\end{array}
\begin{bmatrix}
\alpha_1^1 & 0 & \alpha_1^3 & 1 & 0 & 0 \\
\beta_1^1 & 0 & \beta_1^3 & 0 & 1 & 0 \\
\alpha_2^1 & \alpha_2^2 & 0 & 0 & 0 & 0 \\
\beta_2^1 & \beta_2^2 & 0 & 0 & 0 & 1 \\
0 & \alpha_3^2 & \alpha_3^3 & 0 & 0 & 0 \\
0 & \beta_3^2 & \beta_3^3 & 0 & 0 & 0
\end{bmatrix}
\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ R_{1_x} \\ R_{1_y} \\ R_{2_y} \end{Bmatrix}}_{\{F\}}
+
\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{Bmatrix}}_{\{P\}}
= 0
$$

Or

$$
\begin{array}{ll}
\text{Node } 1 & \Sigma F_x = 0 \\
 & \Sigma F_y = 0 \\
\text{Node } 2 & \Sigma F_x = 0 \\
 & \Sigma F_y = 0 \\
\text{Node } 3 & \Sigma F_x = 0 \\
 & \Sigma F_y = 0
\end{array}
\underbrace{\left[\begin{array}{c|c|c||c|c|c}
1 & 0 & .707 & 1 & 0 & 0 \\ \hline
0 & 0 & .707 & 0 & 1 & 0 \\ \hline
-1 & -.707 & 0 & 0 & 0 & 0 \\ \hline
0 & .707 & 0 & 0 & 0 & 1 \\ \hline
0 & .707 & -.707 & 0 & 0 & 0 \\ \hline
0 & -.707 & -.707 & 0 & 0 & 0
\end{array}\right]}_{[B]}
\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ R_{1_x} \\ R_{1_y} \\ R_{2_y} \end{Bmatrix}}_{\{F\}}
+
\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{Bmatrix}}_{\{P\}}
= 0
$$

Inverting $[B]$ we obtain from $F = [B]^{-1}P$

$$
\left\{\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ \hline R_{1_x} \\ R_{1_y} \\ R_{2_y} \end{array}\right\}
=
\begin{Bmatrix} 5 \\ -7.07 \\ -7.07 \\ 0 \\ 5 \\ 5 \end{Bmatrix}
$$

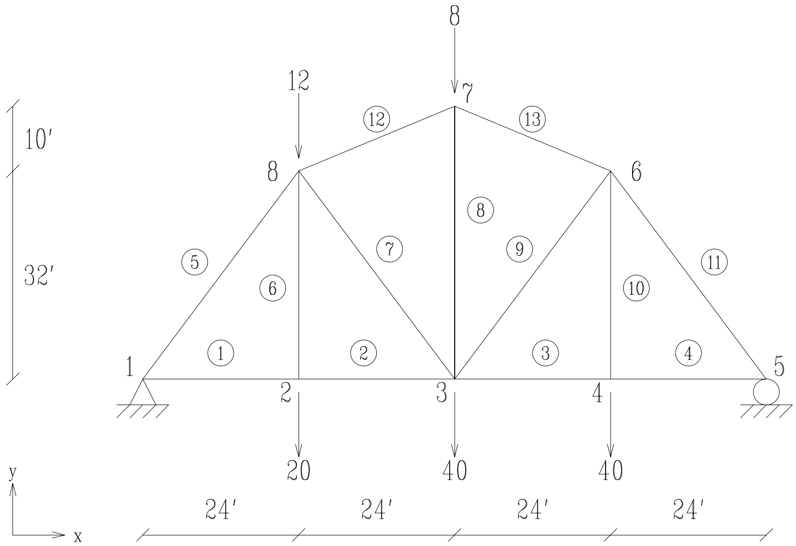
■

■ Example 3-3: Truss II, Matrix Method

Set up the statics matrix for the truss of Example 3-1 using the matrix method

Solution:

First we number the joints and the elements as shown below.



Then we determine the direction cosines

$$
\begin{array}{lll}
\frac{24}{\sqrt{24^2+32^2}} & = \frac{3}{5} & = 0.6 \\
\frac{32}{\sqrt{24^2+32^2}} & = \frac{4}{5} & = 0.8 \\
\frac{10}{\sqrt{24^2+10^2}} & = \frac{10}{26} & = 0.38 \\
\frac{24}{\sqrt{24^2+10^2}} & = \frac{24}{26} & = 0.92
\end{array}
$$

$$
\begin{array}{ll}
\#\ 1 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 2 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 3 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 4 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 5 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 6 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 7 & \Sigma F_x \\ & \Sigma F_y \\
\#\ 8 & \Sigma F_x \\ & \Sigma F_y
\end{array}
\underbrace{\left[\begin{array}{ccccccccccccc|ccc}
1 & & & & .6 & & & & & & & & & 1 & & \\
 & & & & .8 & & & & & & & & & & 1 & \\
-1 & 1 & & & & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & \\
 & -1 & 1 & & & & -.6 & & .6 & & & & & & & \\
 & & & & & & .8 & 1 & .8 & & & & & & & \\
 & & -1 & 1 & & & & & & & & & & & & \\
 & & & & & & & & & 1 & & & & & & \\
 & & & -1 & & & & & & & -.6 & & & & & \\
 & & & & & & & & & & .8 & & & & & 1 \\
 & & & & & & & & -.6 & & .6 & & -.92 & & & \\
 & & & & & & & & -.8 & -1 & -.8 & & .38 & & & \\
 & & & & & & & & & & & -.92 & .92 & & & \\
 & & & & & & & -1 & & & & -.38 & -.38 & & & \\
 & & & & -.6 & & .6 & & & & & .92 & & & & \\
 & & & & -.8 & -1 & -.8 & & & & & .38 & & & &
\end{array}\right]}_{[B]}
\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \\ \hline R_{1x} \\ R_{1y} \\ R_{5y} \end{Bmatrix}}_{?}
+
\underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -20 \\ 0 \\ -40 \\ 0 \\ -40 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -8 \\ 0 \\ -12 \end{Bmatrix}}_{\surd}
= 0
$$

Using *Mathematica* we would have

```
b={
{ 1, 0., 0.,  0.,0.6,  0.,  0., 0.,  0.,  0.,  0.,   0.,   0.,  1, 0., 0.},
{0., 0., 0.,  0.,0.8,  0.,  0., 0.,  0.,  0.,  0.,   0.,   0., 0., 1,  0.},
{-1, 1,  0.,  0.,  0.,  0.,  0., 0.,  0.,  0.,  0.,   0.,   0., 0., 0., 0.},
{0.,0.,  0.,  0.,  0.,   1,  0., 0.,  0.,  0.,  0.,   0.,   0., 0., 0., 0.},
{0.,-1,   1,  0.,  0.,  0.,-0.6, 0.,  0.6, 0.,  0.,   0.,   0., 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0., 0.8, 1.,  0.8, 0.,  0.,   0.,   0., 0., 0., 0.},
{0., 0., -1,   1,  0.,  0.,  0., 0.,   0., 0.,  0.,   0.,   0., 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0.,  0., 0.,   0., 1.,  0.,   0.,   0., 0., 0., 0.},
{0., 0., 0.,  -1,  0.,  0.,  0., 0.,   0., 0.,-0.6,   0.,   0., 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0.,  0., 0.,   0., 0., 0.8,   0.,   0., 0., 0., 1.},
{0., 0., 0.,  0.,  0.,  0.,  0., 0., -0.6, 0., 0.6,   0.,-0.92, 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0.,  0., 0., -0.8,-1.,-0.8,   0., 0.38, 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0.,  0., 0.,   0., 0.,  0.,-0.92, 0.92, 0., 0., 0.},
{0., 0., 0.,  0.,  0.,  0.,  0., -1,   0., 0.,  0.,-0.38,-0.38, 0., 0., 0.},
{0., 0., 0.,  0., -0.6, 0., 0.6, 0.,   0., 0.,  0., 0.92,   0., 0., 0., 0.},
{0., 0., 0.,  0.,-0.8, -1,-0.8, 0.,   0., 0.,  0., 0.38,   0., 0., 0., 0.}
}
p={0, 0, 0, -20,  0, -40 ,  0, -40,  0, 0,  0, 0, 0, -8,  0, -12}
m=Inverse[b].p
```

And the result will be

```
Out[3]= {-43.5, -43.5, -46.5, -46.5, 72.5, -20., -7.66598, -31.7344,

                                                       -14
>    -2.66598, -40., 77.5, 52.2822, 52.2822, -1.22125 10   , -58., -62.}
```

which correspond to the unknown element internal forces and external reactions. ■

3.2.3 Method of Sections

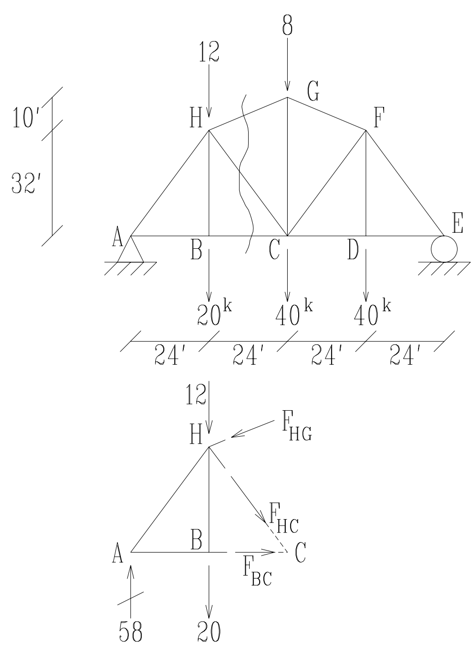
33 When only forces in **selected** members (away from loaded joints) is to be determined, this method should be used.

34 This method can be summarized as follows

1. "Cut" the truss into two substructures by an imaginary line (not necessarily straight) such that it will at least intersect the member for which force is to be determined.
2. Consider either one of the two substructures as the free body
3. Each substructure must remain in equilibrium. Apply the equations of equilibrium
   (a) Summation of moments about a particular point (usually the intersection of 2 cut members) would permit the determination of other member forces
   (b) Summation of forces is usually used to determine forces in inclined members

■ **Example 3-4: Truss, Method of Sections**

Determine F_{BC} and F_{HG} in the previous example.
Solution:
Cutting through members HG, HC and BC, we first take the summation of forces with respect to H:



$$
\begin{aligned}
(+\circlearrowleft)\ \Sigma M_H = 0 \quad &\Rightarrow \quad R_{A_y}(24) - F_{BC}(32) = 0 \\
& \qquad F_{BC} = \tfrac{24}{32}(58) = \boxed{43.5 \text{ k Tension}} \\
(+\circlearrowleft)\ \Sigma M_C = 0; \quad &\Rightarrow \quad (58)(24)(2) - (20+12)(24) - F_{HG_x}(32) - F_{HG_y}(24) = 0 \\
& \qquad 2770 - 768 - (32)(F_{HG})\tfrac{24}{\sqrt{24^2+10^2}} - (24)(F_{HG})\tfrac{10}{\sqrt{24^2+10^2}} = 0 \\
& \qquad 2,000 - (29.5)F_{HG} - (9.2)F_{HG} = 0 \\
&\Rightarrow \quad F_{HG} = \boxed{52 \text{ k Compression}}
\end{aligned}
$$

■

3.3 Case Study: Stadium

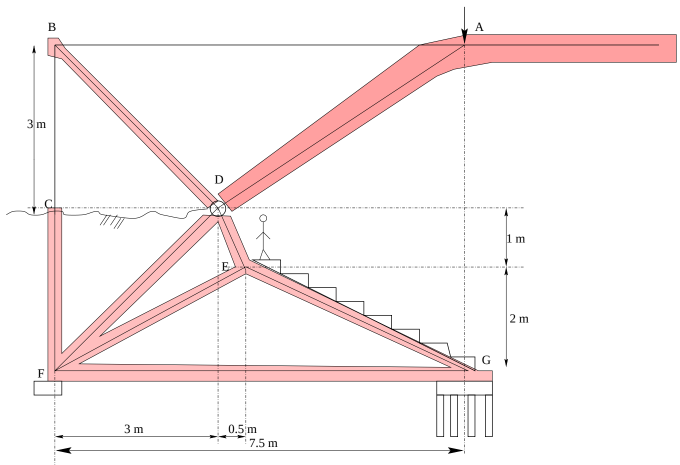


Figure 3.9: Florence Stadium, Pier Luigi Nervi (?)

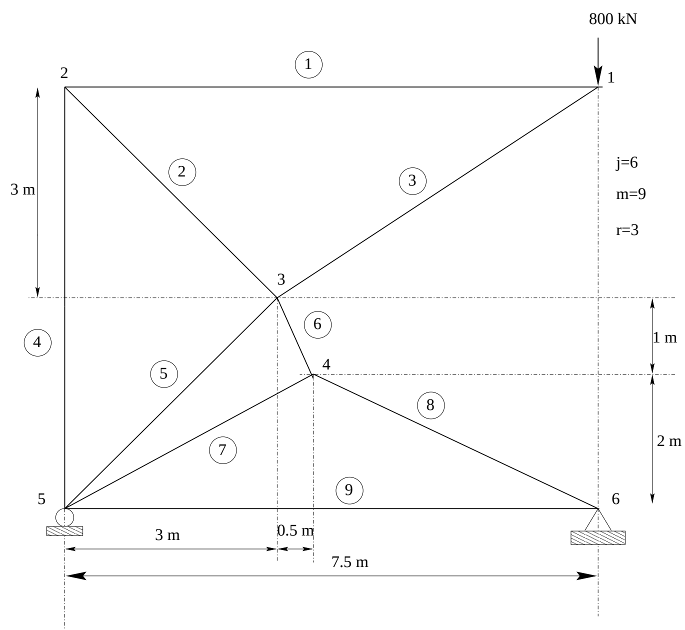


Figure 3.10: Florence Stadioum, Pier Luigi Nervi (?)

Chapter 4

CABLES

4.1 Funicular Polygons

1 A cable is a slender flexible member with zero or negligible flexural stiffness, thus it can only transmit **tensile** forces[1].

2 The tensile force at any point acts in the direction of the tangent to the cable (as any other component will cause bending).

3 Its strength stems from its ability to undergo extensive changes in slope at the point of load application.

4 Cables resist vertical forces by undergoing **sag** (h) and thus developing tensile forces. The horizontal component of this force (H) is called **thrust**.

5 The distance between the cable supports is called the **chord**.

6 The sag to span ratio is denoted by

$$r = \frac{h}{l} \tag{4.1}$$

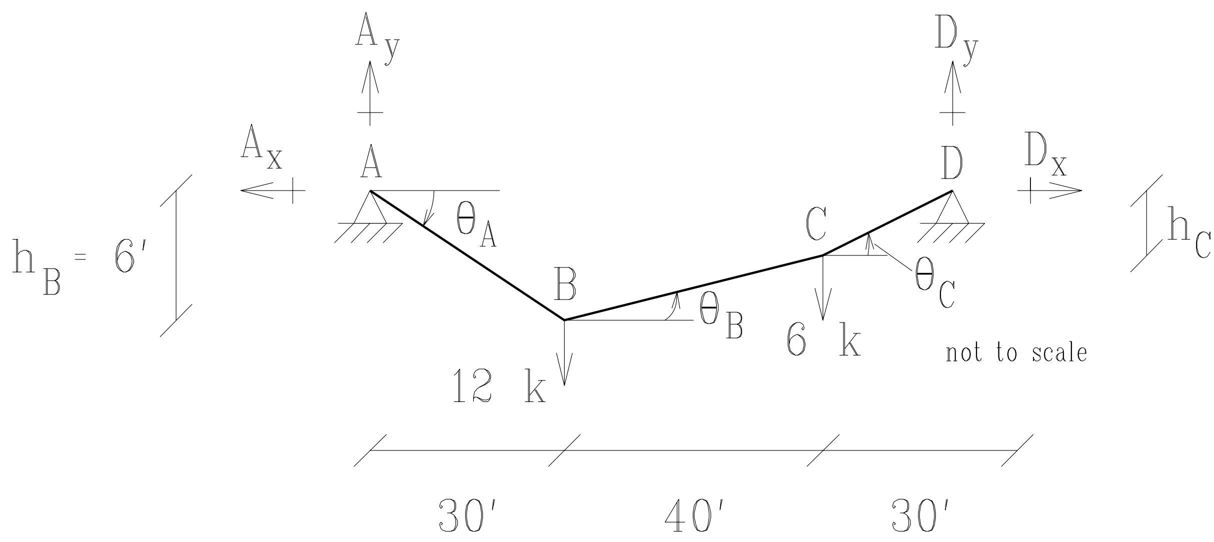
7 When a set of concentrated loads is applied to a cable of negligible weight, then the cable deflects into a series of linear segments and the resulting shape is called the **funicular polygon**.

8 If a cable supports vertical forces only, then the horizontal component H of the cable tension T remains constant.

■ **Example 4-1: Funicular Cable Structure**

Determine the reactions and the tensions for the cable structure shown below.

[1]Due to the zero flexural rigidity it will buckle under axial compressive forces.



Solution:
We have 4 external reactions, however the horizontal ones are equal and we can use any one of a number of equations of conditions in addition to the three equations of equilibrium. First, we solve for A_y, D_y and $H = A_x = D_x$. For this problem we could use the following 3 equations of static equilibrium $\Sigma F_x = \Sigma F_y = \Sigma M = 0$, however since we do not have any force in the x direction, the second equation is of no avail. Instead we will consider the following set $\Sigma F_y = \Sigma M_A = \Sigma M_D = 0$

1. First we solve for D_y

$$(+\circlearrowleft)\ \Sigma M_A = 0; \Rightarrow 12(30) + 6(70) - D_y(100) = 0 \Rightarrow D_y = \boxed{7.8\text{ k}} \tag{4.2}$$

2. Then we solve for A_y

$$(+\uparrow)\ \Sigma F_y = 0; \Rightarrow A_y - 12 - 6 + 7.8 = 0 \Rightarrow A_y = \boxed{10.2\text{ k}} \tag{4.3}$$

3. Solve for the horizontal force

$$(+\circlearrowleft)\ \Sigma M_B = 0; \Rightarrow A_y(30) - H(6) = 0 \Rightarrow H = \boxed{51\text{ k}} \tag{4.4}$$

4. Now we can solve for the sag at point C

$$(+\circlearrowleft)\ \Sigma M_C = 0 \Rightarrow -D_Y(30) + H(h_c) = 0 \Rightarrow h_c = \frac{30D_y}{H} = \frac{30(7.8)}{51} = 4.6\text{ ft} \tag{4.5}$$

5. We now solve for the cable internal forces or tractions in this case

$$T_{AB}; \qquad \tan\theta_A = \frac{6}{30} = 0.200 \Rightarrow \theta_A = 11.31\,\text{deg} \tag{4.6-a}$$

$$= \frac{H}{\cos\theta_A} = \frac{51}{0.981} = \boxed{51.98\text{ k}} \tag{4.6-b}$$

$$T_{BC}; \qquad \tan\theta_B = \frac{6-4.6}{40} = 0.035 \Rightarrow \theta_B = 2\,\text{deg} \tag{4.6-c}$$

$$= \frac{H}{\cos\theta_B} = \frac{51}{0.999} = \boxed{51.03\text{ k}} \tag{4.6-d}$$

$$T_{CD}; \qquad \tan\theta_C = \frac{4.6}{30} = 0.153 \Rightarrow \theta_C = 8.7\,\text{deg} \tag{4.6-e}$$

$$= \frac{H}{\cos\theta_C} = \frac{51}{0.988} = \boxed{51.62\text{ k}} \tag{4.6-f}$$

■

4.2 Uniform Load

4.2.1 qdx; Parabola

9 Whereas the forces in a cable can be determined from statics alone, its configuration must be derived from its deformation. Let us consider a cable with distributed load $p(x)$ **per unit horizontal projection** of the cable length[2]. An infinitesimal portion of that cable can be assumed to be a straight line, Fig. 4.1 and in the absence of any horizontal load we have H =constant. Summation of the vertical

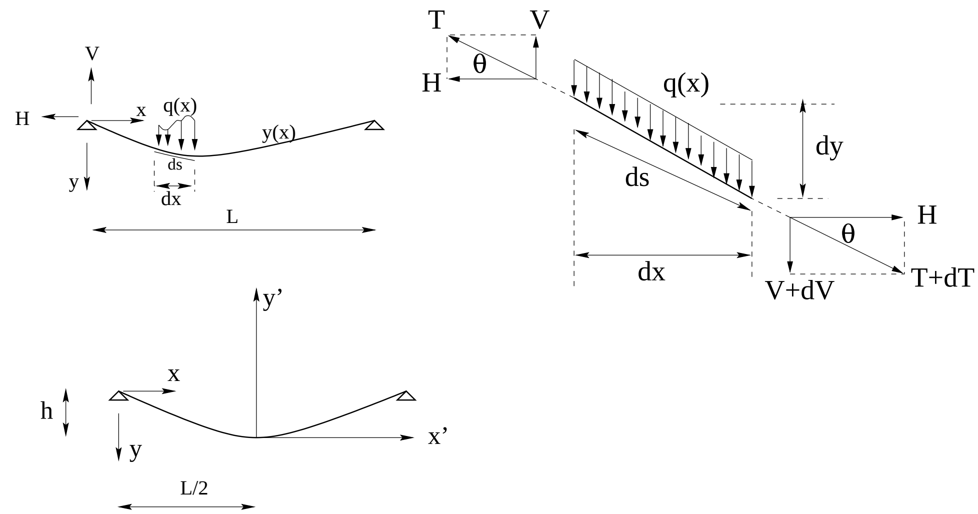


Figure 4.1: Cable Structure Subjected to $q(x)$

forces yields

$$(+\downarrow)\ \Sigma F_y = 0 \Rightarrow -V + qdx + (V + dV) = 0 \qquad (4.7\text{-a})$$
$$dV + qdx = 0 \qquad (4.7\text{-b})$$

where V is the vertical component of the cable tension at x[3]. Because the cable must be tangent to T, we have

$$\tan\theta = \frac{V}{H} \qquad (4.8)$$

10 Substituting into Eq. 4.7-b yields

$$d(H\tan\theta) + qdx = 0 \qquad (4.9)$$

or

$$-\frac{d}{dx}(H\tan\theta) = q \qquad (4.10)$$

Since H is constant (no horizontal load is applied), this last equation can be rewritten as

$$-H\frac{d}{dx}(\tan\theta) = q \qquad (4.11)$$

11 Written in terms of the vertical displacement y, $\tan\theta = \frac{dy}{dx}$ which when substituted in Eq. 4.11 yields the **governing equation for cables**

$$\boxed{\frac{d^2y}{dx^2} = -\frac{q}{H}} \qquad (4.12)$$

[2] Thus neglecting the weight of the cable

[3] Note that if the cable was subjected to its own weight then we would have qds instead of pdx.

12 For a cable subjected to a uniform load p. we can determine its shape by double integration of Eq. 4.12

$$\begin{aligned} -Hy' &= qx + C_1 && \text{(4.13-a)} \\ -Hy &= \frac{qx^2}{2} + C_1 x + C_2 && \text{(4.13-b)} \end{aligned}$$

To solve for C_1 and C_2 this last equation must satisfy the boundary conditions: $y = 0$ at $x = 0$ and at $x = L \Rightarrow C_2 = 0$ and $C_1 = -\frac{qL}{2}$. Thus

$$Hy = \frac{q}{2}x(L-x) \tag{4.14}$$

13 This equation gives the shape $y(x)$ in terms of the horizontal force H, it can be rewritten in terms of the maximum sag h which occurs at midspan, hence at $x = \frac{L}{2}$ we would have[4]

$$\boxed{Hh = \frac{qL^2}{8}} \tag{4.15}$$

14 This relation clearly shows that the horizontal force is inversely proportional to the sag h, as $h \searrow H \nearrow$. Furthermore, this equation can be rewritten as

$$\frac{qL}{H} = \frac{8h}{L} \tag{4.16}$$

and combining this equation with Eq. 4.1 we obtain

$$\frac{qL}{H} = 8r \tag{4.17}$$

15 Combining Eq. 4.14 and 4.15 we obtain

$$y = \frac{4hx}{L^2}(L-x) \tag{4.18}$$

16 If we shift the origin to midspan, and reverse y, then

$$\boxed{y = \frac{4h}{L^2}x^2} \tag{4.19}$$

Thus the cable assumes a parabolic shape (as the moment diagram of the applied load).

17 The maximum tension occurs at the support where the vertical component is equal to $V = \frac{qL}{2}$ and the horizontal one to H, thus

$$T_{max} = \sqrt{V^2 + H^2} = \sqrt{\left(\frac{qL}{2}\right)^2 + H^2} = H\sqrt{1 + \left(\frac{qL/2}{H}\right)^2} \tag{4.20}$$

Combining this with Eq. 4.17 we obtain[5].

$$\boxed{T_{max} = H\sqrt{1 + 16r^2} \approx H(1 + 8r^2)} \tag{4.21}$$

[4]Note the analogy between this equation and the maximum moment in a simply supported uniformly loaded beam $M = \frac{qL^2}{8}$.

[5]Recalling that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdot$ or $(1+b)^n = 1 + nb + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \cdots$; Thus for $b^2 << 1$, $\sqrt{1+b} = (1+b)^{\frac{1}{2}} \approx 1 + \frac{b}{2}$

4.2.2 † qds; Catenary

18 Let us consider now the case where the cable is subjected to its own weight (plus ice and wind if any). We would have to replace qdx by qds in Eq. 4.7-b

$$dV + qds = 0 \tag{4.22}$$

The differential equation for this new case will be derived exactly as before, but we substitute qdx by qds, thus Eq. 4.12 becomes

$$\boxed{\frac{d^2y}{dx^2} = -\frac{q}{H}\frac{ds}{dx}} \tag{4.23}$$

19 But $ds^2 = dx^2 + dy^2$, hence:

$$\frac{d^2y}{dx^2} = -\frac{q}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{4.24}$$

solution of this differential equation is considerably more complicated than for a parabola.

20 We let $dy/dx = p$, then

$$\frac{dp}{dx} = -\frac{q}{H}\sqrt{1+p^2} \tag{4.25}$$

rearranging

$$\int \frac{dp}{\sqrt{1+p^2}} = -\int \frac{q}{H}dx \tag{4.26}$$

From Mathematica (or handbooks), the left hand side is equal to

$$\int \frac{dp}{\sqrt{1+p^2}} = \log_e(p + \sqrt{1+p^2}) \tag{4.27}$$

Substituting, we obtain

$$\begin{aligned}
\log_e(p + \sqrt{1+p^2}) &= \underbrace{-\frac{qx}{H} + C_1}_{A} && \text{(4.28-a)}\\
p + \sqrt{1+p^2} &= e^A && \text{(4.28-b)}\\
\sqrt{1+p^2} &= -p + e^A && \text{(4.28-c)}\\
1 + p^2 &= p^2 - 2pe^A + e^{2A} && \text{(4.28-d)}\\
p &= \frac{e^{2A}-1}{2e^A} = \frac{e^A - e^{-A}}{2} = \sinh A && \text{(4.28-e)}\\
&= \frac{dy}{dx} = \sinh\left(-\frac{qx}{H} + C_1\right) && \text{(4.28-f)}\\
y &= \int \sinh\left(-\frac{qx}{H} + C_1\right)dx = -\frac{H}{q}\cosh\left(-\frac{qx}{H} + C_1\right) + C_2 && \text{(4.28-g)}
\end{aligned}$$

21 To determine the two constants, we set

$$\begin{aligned}
\frac{dy}{dx} &= 0 \quad \text{at} \;\; x = \frac{L}{2} && \text{(4.29-a)}\\
\frac{dy}{dx} &= -\frac{q}{H}\frac{H}{q}\sinh\left(-\frac{qx}{H} + C_1\right) && \text{(4.29-b)}\\
\Rightarrow 0 &= \sinh\left(-\frac{q}{H}\frac{L}{2} + C_1\right) \Rightarrow C_1 = \frac{q}{H}\frac{L}{2} && \text{(4.29-c)}\\
\Rightarrow y &= -\frac{H}{q}\cosh\left[\frac{q}{H}\left(\frac{L}{2} - x\right)\right] + C_2 && \text{(4.29-d)}
\end{aligned}$$

At midspan, the sag is equal to h, thus

$$h = -\frac{H}{q}\cosh\left[\frac{q}{H}\left(\frac{L}{2}-\frac{L}{2}\right)\right]+C_2 \quad (4.30\text{-a})$$

$$C_2 = h+\frac{H}{q} \quad (4.30\text{-b})$$

22 If we move the origin at the lowest point along the cable at $x' = x - L/2$ and $y' = h - y$, we obtain

$$\boxed{\frac{q}{H}y = \cosh\left(\frac{q}{H}x\right)-1} \quad (4.31)$$

23 This equation is to be contrasted with 4.19, we can rewrite those two equations as:

$$\frac{q}{H}y = \frac{1}{2}\left(\frac{q}{H}x\right)^2 \quad \text{Parabola} \quad (4.32\text{-a})$$

$$\frac{q}{H}y = \cosh\left(\frac{q}{H}x\right)-1 \quad \text{Catenary} \quad (4.32\text{-b})$$

The hyperbolic cosine of the catenary can be expanded into a Taylor power series as

$$\frac{qy}{H} = \frac{1}{2}\left(\frac{qx}{H}\right)^2+\frac{1}{24}\left(\frac{qx}{H}\right)^4+\frac{1}{720}\left(\frac{qx}{H}\right)^6+... \quad (4.33)$$

The first term of this development is identical as the formula for the parabola, and the other terms constitute the difference between the two. The difference becomes significant only for large qx/H, that is for large sags in comparison with the span, Fig. 4.3.

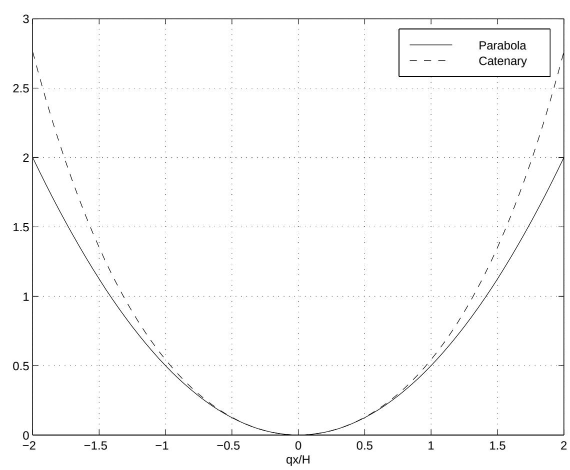


Figure 4.2: Catenary versus Parabola Cable Structures

4.2.2.1 Historical Note

24 It should be mentioned that solution of this problem constitued one of the major mathematical/Mechanics challenges of the early 18th century. Around 1684, differential and integral calculus took their first effective forms, and those powerful new techniques allowed scientists to tackle complex problems for the first time, (Penvenuto 1991). One of these problems was the solution to the catenary problem as presented by Jakob Bernouilli. Immediately thereafter, Leibniz presented a solution based on infinitesimal calculus, another one was presented by Huygens. Finally, the brother of the challenger, Johann Bernoulli did also present a solution.

25 Huygens solution was complex and relied on geometrical arguments. The one of Leibniz was ellegant and correct ($y/a = (b^{x/a} + b^{-x/a})/2$ (we recognize Eq. 4.31 albeit written in slightly different form, Fig. 4.3. Finally, Bernoulli presented two correct solution, and in his solution he did for the first time express

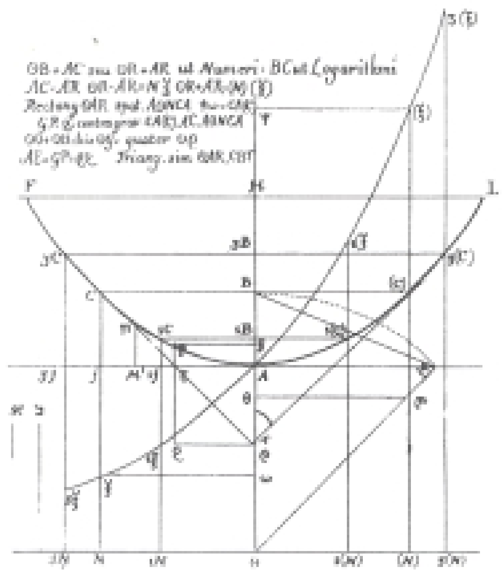
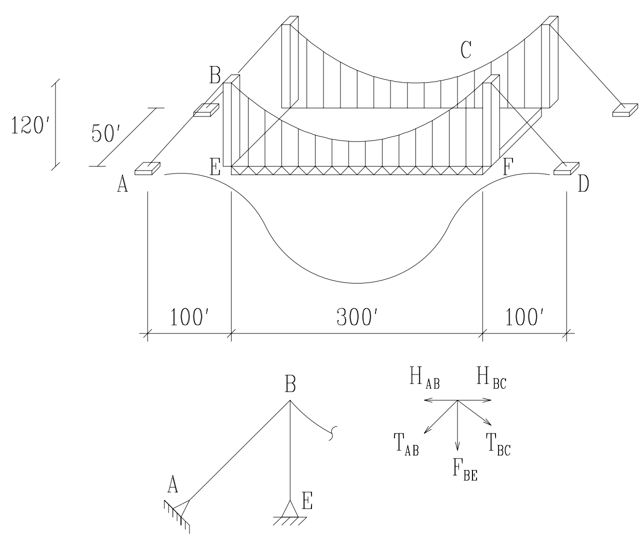


Figure 4.3: Leipniz's Figure of a catenary, 1690

equations of equilibrium in differential form.

■ **Example 4-2: Design of Suspension Bridge**

Design the following 4 lanes suspension bridge by selecting the cable diameters assuming an allowable cable strength σ_{all} of 190 ksi. The bases of the tower are hinged in order to avoid large bending moments.



The total dead load is estimated at 200 psf. Assume a sag to span ratio of $\frac{1}{5}$
Solution:

1. The dead load carried by each cable will be one half the total dead load or $p_1 = \frac{1}{2}(200)$ psf(50) ft$\frac{1}{1,000} =$ 5.0 k/ft

2. Using the HS 20 truck, the uniform additional load per cable is $p_2 = (2)\text{lanes/cable}(.64)\text{k/ft/lane} = 1.28$ k/ft/cable. Thus, the total design load is $p_1 + p_2 = 5 + 1.28 = 6.28$ k/ft

3. The thrust H is determined from Eq. 4.15

$$H = \frac{pl^2}{8h} \tag{4.34-a}$$
$$= \frac{(6.28)\text{ k/ft}(300)^2\text{ ft}^2}{(8)\left(\frac{300}{5}\text{ ft}\right)} \tag{4.34-b}$$
$$= \boxed{1,177\text{ k}} \tag{4.34-c}$$

4. From Eq. 4.21 the maximum tension is

$$T_{max} = H\sqrt{1+16r^2} \tag{4.35-a}$$
$$= (1,177)\text{ k}\sqrt{1+(16)\left(\frac{1}{5}\right)^2} \tag{4.35-b}$$
$$= \boxed{1,507\text{ k}} \tag{4.35-c}$$

5. Note that if we used the approximate formula in Eq. 4.21 we would have obtained

$$T_{max} = H(1+8r^2) \tag{4.36-a}$$
$$= 1,177\left(1+8\left(\frac{1}{5}\right)^2\right) \tag{4.36-b}$$
$$= \boxed{1,554\text{ k}} \tag{4.36-c}$$

or 3% difference!

6. The required cross sectional area of the cable along the main span should be equal to

$$A = \frac{T_{max}}{\sigma_{all}} = \frac{1,507\text{ k}}{190\text{ ksi}} = 7.93\text{ in}^2$$

which corresponds to a diameter

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{(4)(7.93)}{\pi}} = \boxed{3.18\text{ in}}$$

7. We seek next to determine the cable force in AB. Since the pylon can not take any horizontal force, we should have the horizontal component of T_{max} (H) equal and opposite to the horizontal component of T_{AB} or $\frac{T_{AB}}{H} = \frac{\sqrt{(100)^2+(120)^2}}{100}$ thus

$$T_{AB} = H\frac{\sqrt{(100)^2+(120)^2}}{100} = (1,177)(1.562) = \boxed{1,838\text{ k}} \tag{4.37}$$

the cable area should be

$$A = \frac{1,838\text{ k}}{190\text{ ksi}} = 9.68\text{ in}^2$$

which corresponds to a diameter

$$d = \sqrt{\frac{(4)(9.68)}{\pi}} = 3.51\text{ in}$$

8. To determine the vertical load acting on the pylon, we must add the vertical components of T_{max} and of T_{AB} (V_{BC} and V_{AB} respectively). We can determine V_{BC} from H and T_{max}, thus

$$P = \frac{120}{100}(1,177) + \sqrt{(1,507)^2 - (1,177)^2} = 1,412 + 941 = \boxed{2,353\text{ k}} \tag{4.38}$$

Using A36 steel with an allowable stress of 21 ksi, the cross sectional area of the tower should be $A = \frac{2,353}{21} = 112\text{ in}^2$. Note that buckling of such a high tower might govern the final dimensions.

9. If the cables were to be anchored to a concrete block, the volume of the block should be at least equal to
$$V = \frac{(1,412)\text{ k}(1,000)}{150\text{ lbs/ft}^3} = 9,413\text{ ft}^3$$
or a cube of approximately 21 ft

■

4.3 Case Study: George Washington Bridge

`Adapted from (Billington and Mark 1983)`

26 The George Washington bridge, is a suspension bridge spanning the Hudson river from New York City to New Jersey. It was completed in 1931 with a central span of 3,500 ft (at the time the world's longest span). The bridge was designed by O.H. Amman, who had emigrated from Switzerland. In 1962 the deck was stiffened with the addition of a lower deck.

4.3.1 Geometry

27 A longitudinal and plan elevation of the bridge is shown in Fig. 4.4. For simplicity we will assume in

??

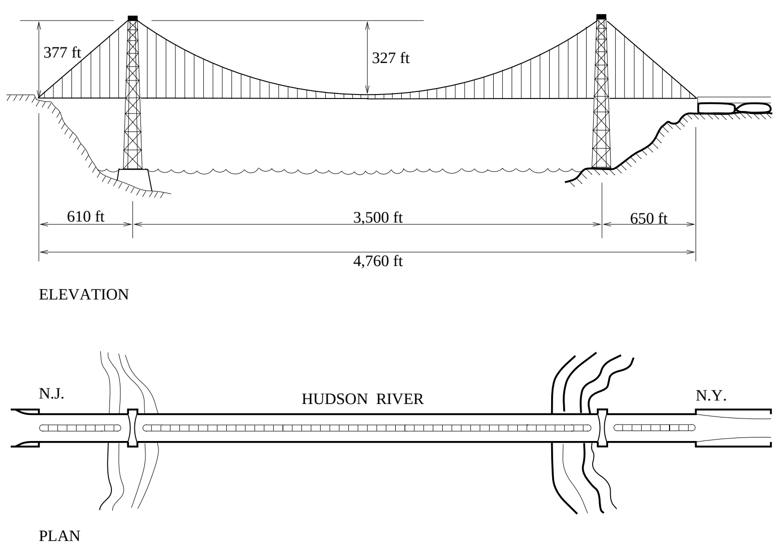


Figure 4.4: Longitudinal and Plan Elevation of the George Washington Bridge

our analysis that the two approaching spans are equal to 650 ft.

28 There are two cables of three feet diameter on each side of the bridge. The centers of each pair are 9 ft apart, and the pairs themselves are 106 ft apart. We will assume a span width of 100 ft.

29 The cables are idealized as supported by rollers at the top of the towers, hence the horizontal components of the forces in each side of the cable must be equal (their vertical components will add up).

30 The cables support the road deck which is hung by suspenders attached at the cables. The cables are made of 26,474 steel wires, each 0.196 inch in diameter. They are continuous over the tower supports and are firmly anchored in both banks by huge blocks of concrete, the anchors.

31 Because the cables are much longer than they are thick (small $\frac{I}{L}$), they can be idealized as perfectly flexible members with no shear/bending resistance but with high axial strength.

32 The towers are 578 ft tall and rest on concrete caissons in the river. Because of our assumption regarding the roller support for the cables, the towers will be subjected only to axial forces.

4.3.2 Loads

33 The dead load is composed of the weight of the deck and the cables and is estimated at 390 and 400 psf respectively for the central and side spans respectively. Assuming an average width of 100 ft, this would be equivalent to

$$DL = (390) \text{ psf}(100) \text{ ft}\frac{\text{k}}{(1,000) \text{ lbs}} = 39 \text{ k/ft} \tag{4.39}$$

for the main span and 40 k/ft for the side ones.

34 For highway bridges, design loads are given by the AASHTO (Association of American State Highway Transportation Officials). The HS-20 truck is often used for the design of bridges on main highways, Fig. 4.5. Either the design truck with specified axle loads and spacing must be used **or** the equivalent uniform load and concentrated load. This loading must be placed such that maximum stresses are produced.

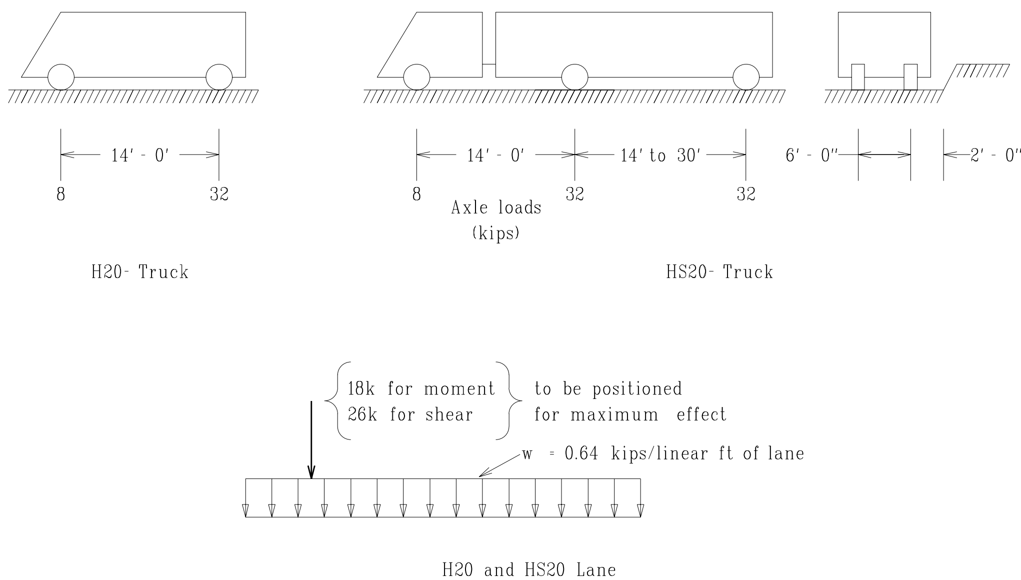


Figure 4.5: Truck Load

35 With two decks, we estimate that there is a total of 12 lanes or

$$LL = (12)\text{Lanes}(.64) \text{ k/ ft/Lane} = 7.68 \text{ k/ft} \approx 8 \text{ k/ft} \tag{4.40}$$

We do not consider earthquake, or wind loads in this analysis.

36 Final DL and LL are, Fig. 4.6: $TL = 39 + 8 = 47$ k/ft

4.3.3 Cable Forces

37 The thrust H (which is the horizontal component of the cable force) is determined from Eq. 4.15

$$H = \frac{wL_{\text{cs}}^2}{8h} \tag{4.41-a}$$

$$= \frac{(47) \text{ k/ft}(3,500)^2 \text{ ft}^2}{(8)(327) \text{ ft}} \tag{4.41-b}$$

$$= 220,000 \text{ k} \tag{4.41-c}$$

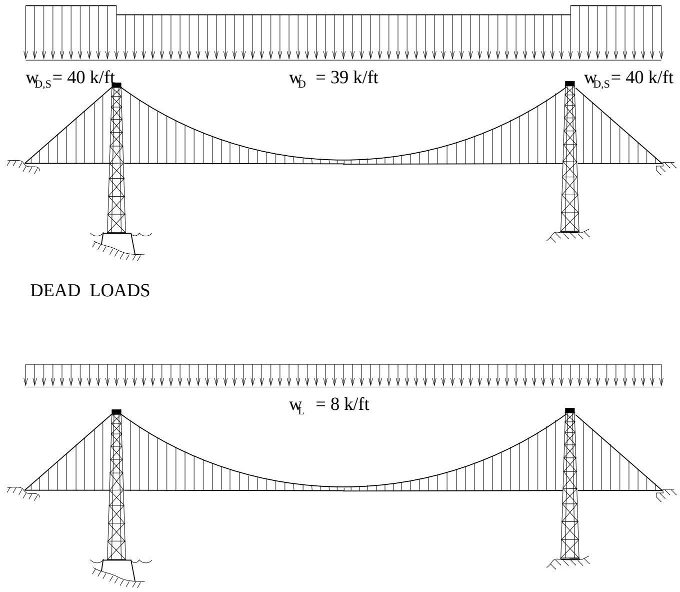


Figure 4.6: Dead and Live Loads

From Eq. 4.21 the maximum tension is

$$r = \frac{h}{L_{\text{cs}}} = \frac{327}{3,500} = 0.0934 \tag{4.42-a}$$

$$T_{\max} = H\sqrt{1+16r^2} \tag{4.42-b}$$

$$= (220,000) \text{ k}\sqrt{1+(16)(0.0934)^2} \tag{4.42-c}$$

$$= (220,000) \text{ k}(1.0675) = \boxed{235{,}000 \text{ k}} \tag{4.42-d}$$

4.3.4 Reactions

38 Cable reactions are shown in Fig. 4.7.

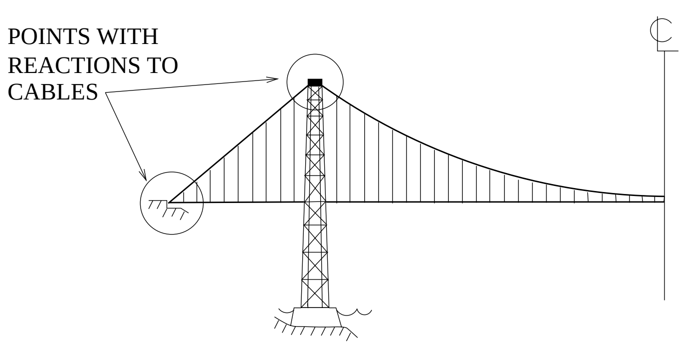


Figure 4.7: Location of Cable Reactions

39 The vertical force in the columns due to the central span (cs) is simply the support reaction, 4.8

$$V_{\text{cs}} = \frac{1}{2}wL_{\text{cs}} = \frac{1}{2}(47) \text{ k/ft}(3,500) \text{ ft} = 82,250 \text{ k} \tag{4.43}$$

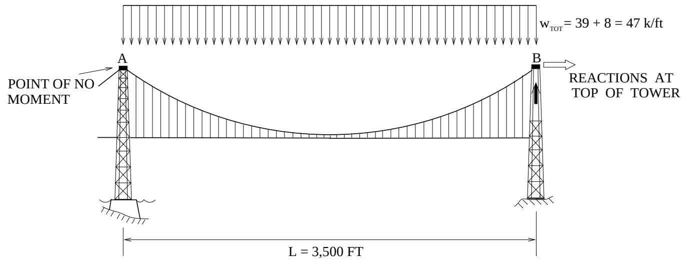


Figure 4.8: Vertical Reactions in Columns Due to Central Span Load

Note that we can check this by determining the vector sum of H and V which should be equal to $T_{\max}$:

$$\sqrt{V_{\rm CS}^2 + H^2} = \sqrt{(82,250)^2 + (220,000)^2} = 235,000 \text{ k}\surd \tag{4.44}$$

40 Along the side spans (ss), the total load is $TL = 40 + 8 = 48$ k/ft. We determine the vertical reaction by taking the summation of moments with respect to the anchor:

$$\Sigma M_D = 0; \circlearrowleft +; h_{\rm SS}H + (w_{\rm SS}L_{\rm SS})\frac{L_{\rm SS}}{2} - V_{\rm SS}L_{\rm SS} = 0 \tag{4.45-a}$$

$$= (377) \text{ k}(220,000) \text{ k} + (48) \text{ k/ft}(650) \text{ ft}\frac{(650) \text{ ft}}{2} - 650V_{\rm SS} = 0 \tag{4.45-b}$$

$$V_{\rm SS} = 143,200 \text{ k} \tag{4.45-c}$$

41 **Note:** that we have used equilibrium to determine the vertical component of the cable force. It would have been wrong to determine V_{ss} from $V_{ss} = 220,000\frac{377}{650}$ as we did in the previous example, because the cable is now loaded. We would have to determine the shape of the cable and the tangent at the support. Beginiing with Eq. 4.13-b:

$$-Hy = \frac{1}{2}wx^2 + C_1x + C_2 \tag{4.46-a}$$

$$-y = \frac{w}{H}\frac{x^2}{2} + \frac{C_1}{H}x + \frac{C_2}{H} \tag{4.46-b}$$

$$\tag{4.46-c}$$

At $x = 0$, $y = 0$, thus $C_2 = 0$; and at $x = 650$, $y = -377$; with $H = 220,000$ k and $w = 48$

$$y = -\frac{(48) \text{ k}}{\text{ft}(220,000) \text{ k}}x^2 - \frac{C_1}{(220,000) \text{ k}}x \tag{4.47-a}$$

$$= -1.091 \times 10^{-4}x^2 - 4.545 \times 10^{-6}C_1x \tag{4.47-b}$$

$$y|_{x=650} = -377 = -1.091 \times 10^{-4}(650)^2 - 4.545 \times 10^{-6}C_1(650) \tag{4.47-c}$$

$$C_1 = 112,000 \tag{4.47-d}$$

$$y = -1.091 \times 10^{-4}x^2 - 0.501x \tag{4.47-e}$$

$$\frac{dy}{dx} = -2.182 \times 10^{-4}x - 0.501 \tag{4.47-f}$$

$$\left.\frac{dy}{dx}\right|_{x=650} = -0.1418 - 0.501 = -0.6428 = \frac{V}{H} \tag{4.47-g}$$

$$V = (220,000)(0.6428) = 141,423 \text{ k} \tag{4.47-h}$$

which is only 1% different.

42 Hence the total axial force applied on the column is

$$V = V_{\rm CS} + V_{\rm SS} = (82,250) \text{ k} + (143,200) \text{ k} = \boxed{225{,}450 \text{ k}} \tag{4.48}$$

43 The vertical reaction at the anchor is given by summation of the forces in the y direction, Fig. 4.9:

$$
\begin{aligned}
(+\uparrow)\ \Sigma F_y = 0; (w_{\rm SS}L_{\rm SS}) + V_{\rm SS} + R_{\rm anchor} &= 0 && (4.49\text{-a})\\
-(48)\ \text{k/ft}(650)\ \text{ft} + (143,200)\ \text{k} + R_{\rm anchor} &= 0 && (4.49\text{-b})\\
R_{\rm anchor} &= \boxed{112{,}000\ \text{k}\downarrow} && (4.49\text{-c})\\
& && (4.49\text{-d})
\end{aligned}
$$

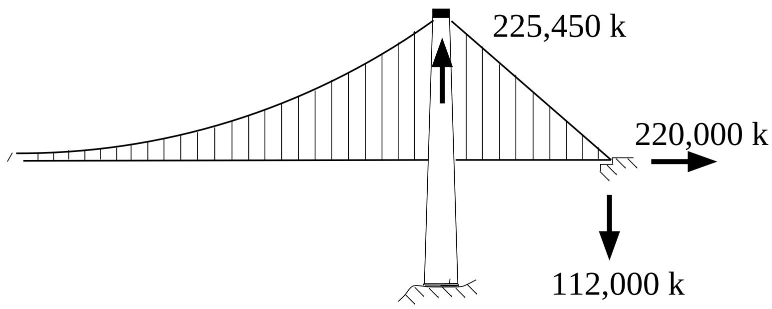


Figure 4.9: Cable Reactions in Side Span

44 The axial force in the side cable is determined the vector sum of the horizontal and vertical reactions.

$$
\begin{aligned}
T^{\rm SS}_{\rm anchor} &= \sqrt{R^2_{\rm anchor} + H^2} = \sqrt{(112,000)^2 + (220,000)^2} = 247,000\ \text{k} && (4.50\text{-a})\\
T^{\rm SS}_{\rm tower} &= \sqrt{V^2_{\rm SS} + H^2} = \sqrt{(143,200)^2 + (220,000)^2} = \boxed{262{,}500\ \text{k}} && (4.50\text{-b})
\end{aligned}
$$

45 The cable stresses are determined last, Fig. 4.10:

$$
\begin{aligned}
A_{wire} &= \frac{\pi D^2}{4} = \frac{(3.14)(0.196)^2}{4} = 0.03017\ \text{in}^2 && (4.51\text{-a})\\
A_{total} &= (4)\text{cables}(26,474)\text{wires/cable}(0.03017)\ \text{in}^2/\text{wire} = 3,200\ \text{in}^2 && (4.51\text{-b})\\
\text{Central Span}\ \ \sigma &= \frac{H}{A} = \frac{(220,000)\ \text{k}}{(3,200)\ \text{in}^2} = 68.75\ \text{ksi} && (4.51\text{-c})\\
\text{Side Span Tower}\ \ \sigma^{\rm SS}_{\rm tower} &= \frac{T^{\rm SS}_{\rm tower}}{A} = \frac{(262,500)\ \text{in}^2}{(3,200)\ \text{in}^2} = \boxed{82\ \text{ksi}} && (4.51\text{-d})\\
\text{Side Span Anchor}\ \ \sigma^{\rm SS}_{\rm tower} &= \frac{T^{\rm SS}_{\rm anchor}}{A} = \frac{(247,000)\ \text{in}^2}{(3,200)\ \text{in}^2} = 77.2\ \text{ksi} && (4.51\text{-e})
\end{aligned}
$$

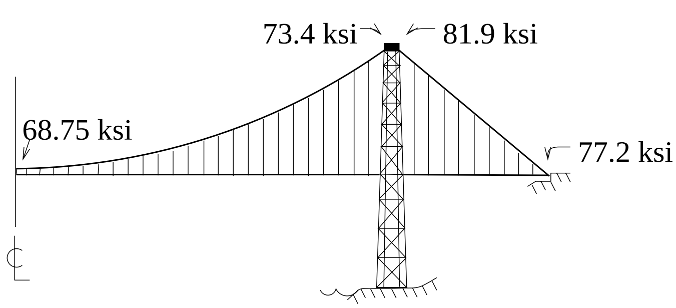


Figure 4.10: Cable Stresses

46 If the cables were to be anchored to a concrete block, the volume of the block should be at least equal to $V = \frac{(112{,}000)\ \text{k}(1{,}000)\ \text{lbs/ k}}{150\ \text{lbs/ft}^3} = 747{,}000\ \text{ft}^3$ or a cube of approximately 91 ft

47 The deck, for all practical purposes can be treated as a continuous beam supported by elastic springs with stiffness $K = AL/E$ (where L is the length of the supporting cable). This is often idealized as a beam on elastic foundations, and the resulting shear and moment diagrams for this idealization are shown in Fig. 4.11.

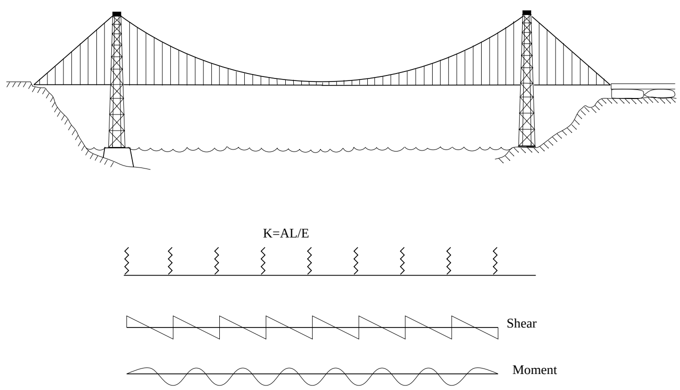


Figure 4.11: Deck Idealization, Shear and Moment Diagrams

Chapter 5

INTERNAL FORCES IN STRUCTURES

1 This chapter will start as a review of shear and moment diagrams which you have studied in both *Statics* and *Strength of Materials*, and will proceed with the analysis of statically determinate frames, arches and grids.

2 By the end of this lecture, you should be able to draw the shear, moment and torsion (when applicable) diagrams for each member of a structure.

3 Those diagrams will subsequently be used for member design. For instance, for flexural design, we will consider the section subjected to the highest moment, and make sure that the internal moment is equal and opposite to the external one. For the ASD method, the basic beam equation (derived in Strength of Materials) $\sigma = \frac{MC}{I}$, (where M would be the design moment obtained from the moment diagram) would have to be satisfied.

4 Some of the examples first analyzed in chapter 2 (Reactions), will be revisited here. Later on, we will determine the deflections of those same problems.

5.1 Design Sign Conventions

5 Before we (re)derive the Shear-Moment relations, let us *arbitrarily* define a sign convention.

6 The sign convention adopted here, is the one commonly used for design purposes[1].

7 With reference to Fig. 5.1

2D:

Load Positive along the beam's local y axis (assuming a right hand side convention), that is positive upward.

Axial: tension positive.

Flexure A positive moment is one which causes tension in the lower fibers, and compression in the upper ones. Alternatively, moments are drawn on the compression side (useful to keep in mind for frames).

Shear A positive shear force is one which is "up" on a negative face, or "down" on a positive one. Alternatively, a pair of positive shear forces will cause clockwise rotation.

Torsion Counterclockwise positive

3D: Use double arrow vectors (and NOT curved arrows). Forces and moments (including torsions) are defined with respect to a right hand side coordinate system, Fig. **??**.

[1]Later on, in more advanced analysis courses we will use a different one.

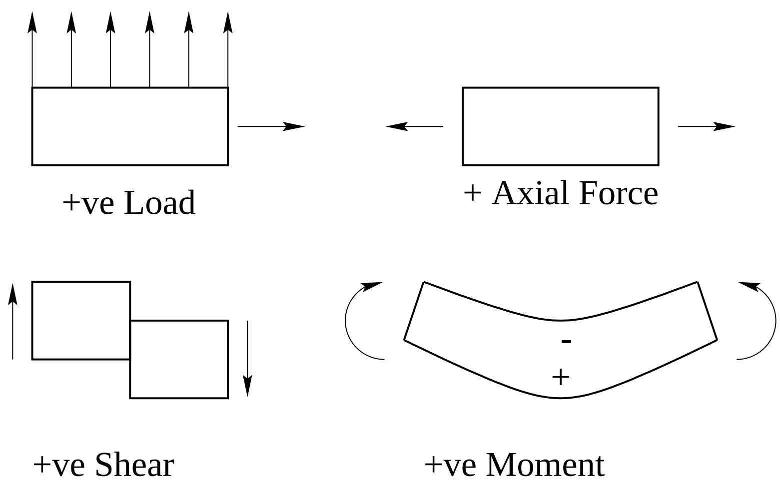


Figure 5.1: Shear and Moment Sign Conventions for Design

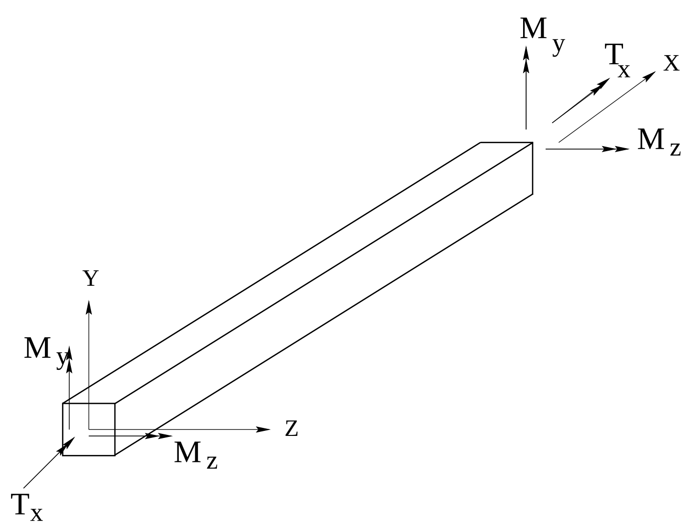


Figure 5.2: Sign Conventions for 3D Frame Elements

5.2 Load, Shear, Moment Relations

8 Let us (re)derive the basic relations between load, shear and moment. Considering an infinitesimal length dx of a beam subjected to a positive load[2] $w(x)$, Fig. 5.3. The infinitesimal section must also be

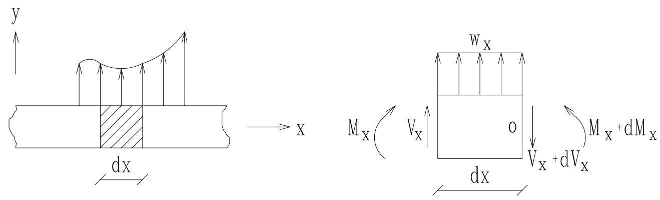


Figure 5.3: Free Body Diagram of an Infinitesimal Beam Segment

in equilibrium.

9 There are no axial forces, thus we only have two equations of equilibrium to satisfy $\Sigma F_y = 0$ and $\Sigma M_z = 0$.

10 Since dx is infinitesimally small, the small variation in load along it can be neglected, therefore we assume $w(x)$ to be constant along dx.

11 To denote that a small change in shear and moment occurs over the length dx of the element, we add the differential quantities dV_x and dM_x to V_x and M_x on the right face.

12 Next considering the first equation of equilibrium

$$(+\uparrow)\ \Sigma F_y = 0 \Rightarrow V_x + w_x dx - (V_x + dV_x) = 0$$

or

$$\boxed{\frac{dV}{dx} = w(x)} \tag{5.1}$$

> The slope of the shear curve at any point along the axis of a member is given by the load curve at that point.

13 Similarly

$$(+\curvearrowleft)\ \Sigma M_O = 0 \Rightarrow M_x + V_x dx - w_x dx \frac{dx}{2} - (M_x + dM_x) = 0$$

Neglecting the dx^2 term, this simplifies to

$$\boxed{\frac{dM}{dx} = V(x)} \tag{5.2}$$

> The slope of the moment curve at any point along the axis of a member is given by the shear at that point.

14 Alternative forms of the preceding equations

$$\boxed{\begin{array}{rcll} V &=& \int w(x)dx & (5.3) \\ \Delta V_{21} &=& V_{x_2} - V_{x_1} = \int_{x_1}^{x_2} w(x)dx & (5.4) \end{array}}$$

[2] In this derivation, as in all other ones we should assume all quantities to be positive.

The change in shear between 1 and 2, ΔV_{1-2}, is equal to the area under the load between x_1 and x_2.

and

$$\boxed{\begin{aligned} M &= \int V(x)dx & (5.5) \\ \Delta M_{21} &= M_2 - M_1 = \int_{x_1}^{x_2} V(x)dx & (5.6) \end{aligned}}$$

The change in moment between 1 and 2, ΔM_{21}, is equal to the area under the shear curve between x_1 and x_2.

15 Note that we still need to have V_1 and M_1 in order to obtain V_2 and M_2.

16 Similar relations will be determined later (Chapter 6) between curvature $\frac{M}{EI}$ and rotation θ (Eq. 6.30).

17 Fig. 5.4 and 5.5 further illustrates the variation in internal shear and moment under uniform and concentrated forces/moment.

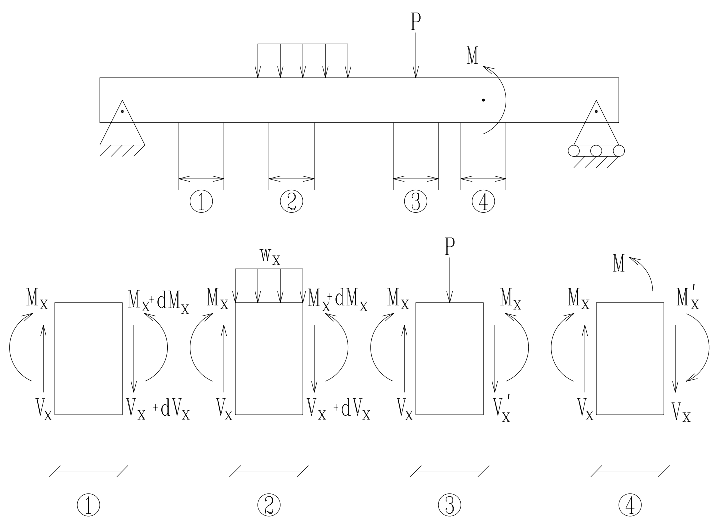


Figure 5.4: Shear and Moment Forces at Different Sections of a Loaded Beam

5.3 Moment Envelope

18 For design, we often must consider different load combinations.

19 For each load combination, we should draw the shear, moment diagrams. and then we should use the **Moment envelope** for design purposes.

5.4 Examples

5.4.1 Beams

■ **Example 5-1: Simple Shear and Moment Diagram**

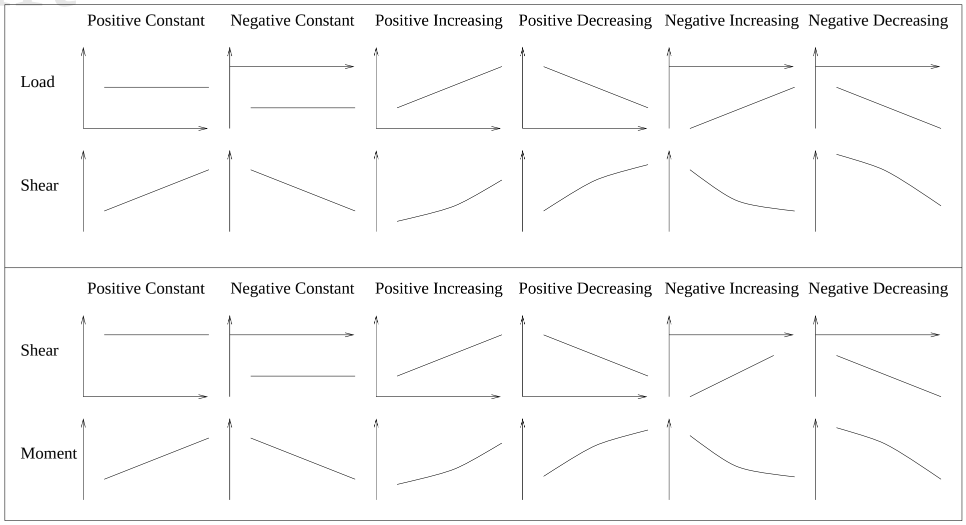
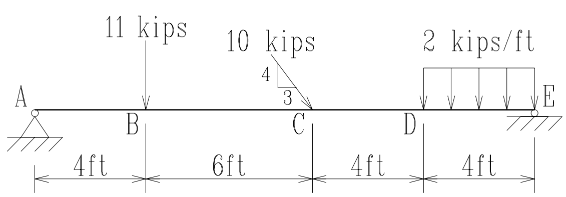
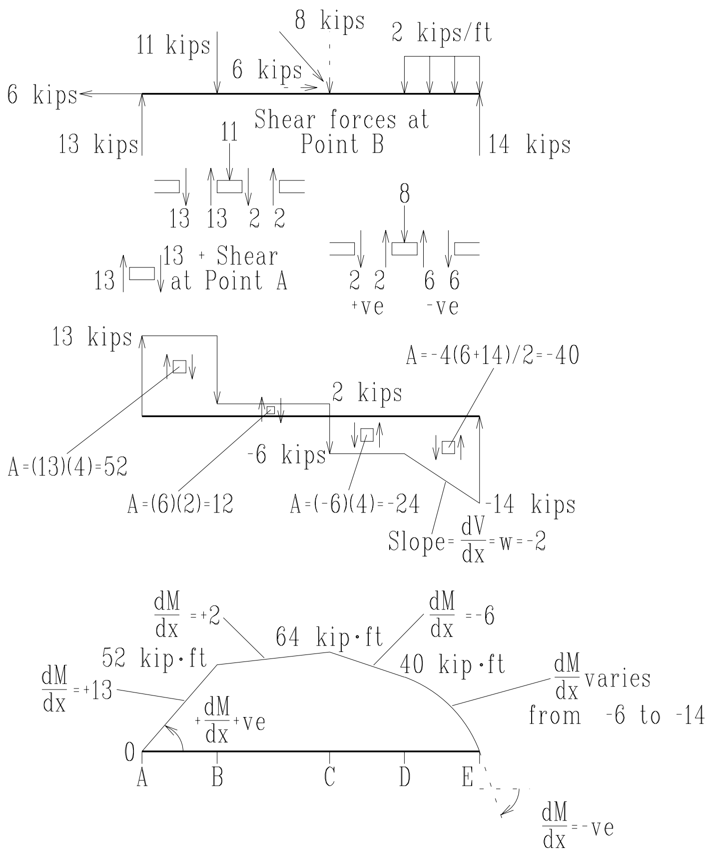


Figure 5.5: Slope Relations Between Load Intensity and Shear, or Between Shear and Moment

Draw the shear and moment diagram for the beam shown below



Solution:
The free body diagram is drawn below



Reactions are determined from the equilibrium equations

$$
\begin{array}{rll}
(+ \rightarrow) \Sigma F_x = 0; & \Rightarrow & -R_{A_x} + 6 = 0 \Rightarrow R_{A_x} = 6 \text{ k} \\
(+ \circlearrowleft) \Sigma M_A = 0; & \Rightarrow & (11)(4) + (8)(10) + (4)(2)(14+2) - R_{F_y}(18) = 0 \Rightarrow R_{F_y} = 14 \text{ k} \\
(+ \uparrow) \Sigma F_y = 0; & \Rightarrow & R_{A_y} - 11 - 8 - (4)(2) + 14 = 0 \Rightarrow R_{A_y} = 13 \text{ k}
\end{array}
$$

Shear are determined next.

1. At A the shear is equal to the reaction and is positive.
2. At B the shear drops (negative load) by 11 k to 2 k.
3. At C it drops again by 8 k to -6 k.
4. It stays constant up to D and then it decreases (constant negative slope since the load is uniform and negative) by 2 k per linear foot up to -14 k.
5. As a check, -14 k is also the reaction previously determined at F.

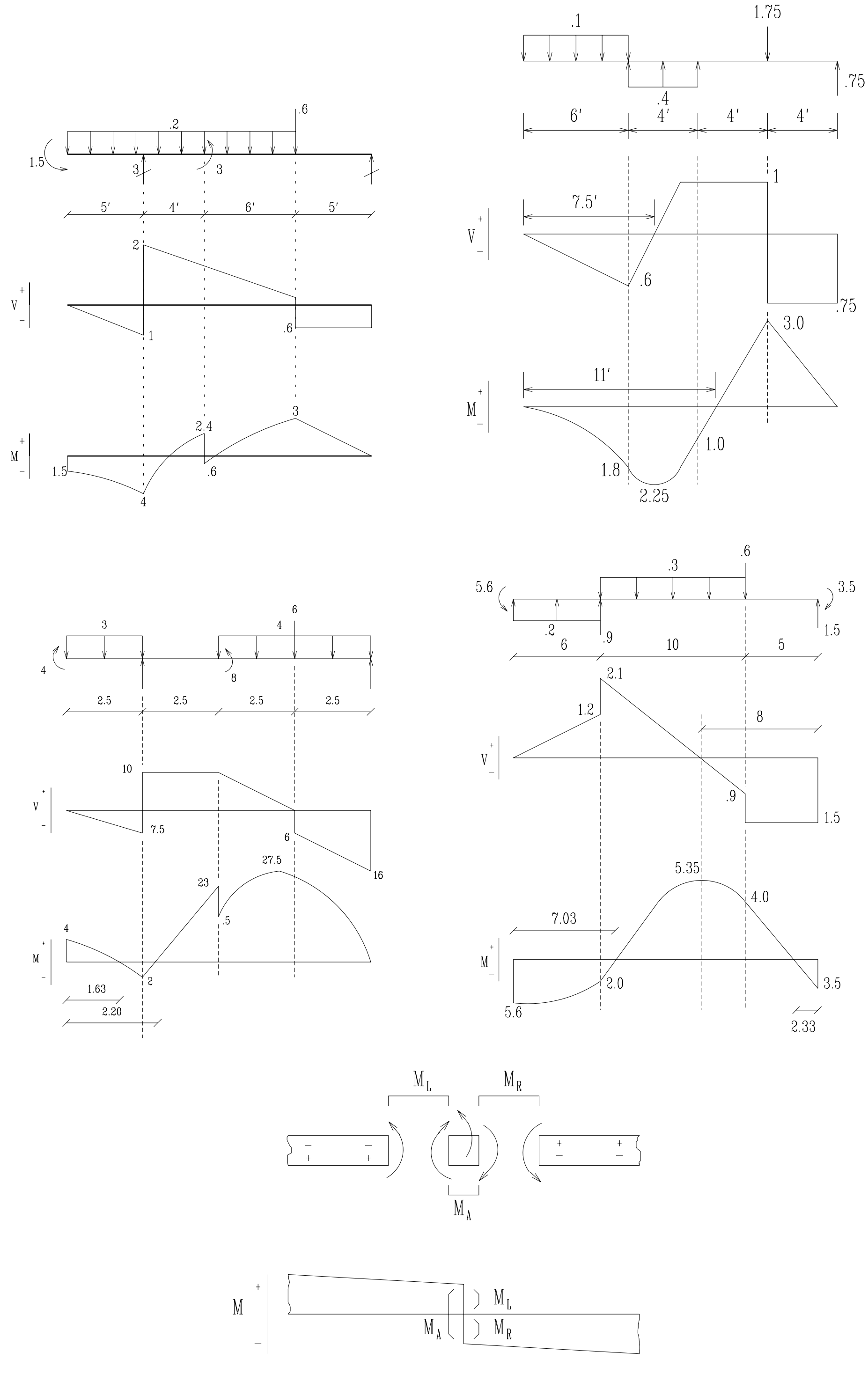
Moment is determined last:

1. The moment at A is zero (hinge support).
2. The change in moment between A and B is equal to the area under the corresponding shear diagram, or $\Delta M_{B-A} = (13)(4) = 52$.
3. etc...

■

■ **Example 5-2: Sketches of Shear and Moment Diagrams**

For each of the following examples, sketch the shear and moment diagrams.



5.4.2 Frames

20 Inclined loads on inclined members are often mishandled. With reference to Fig. 5.6 we would have the following relations

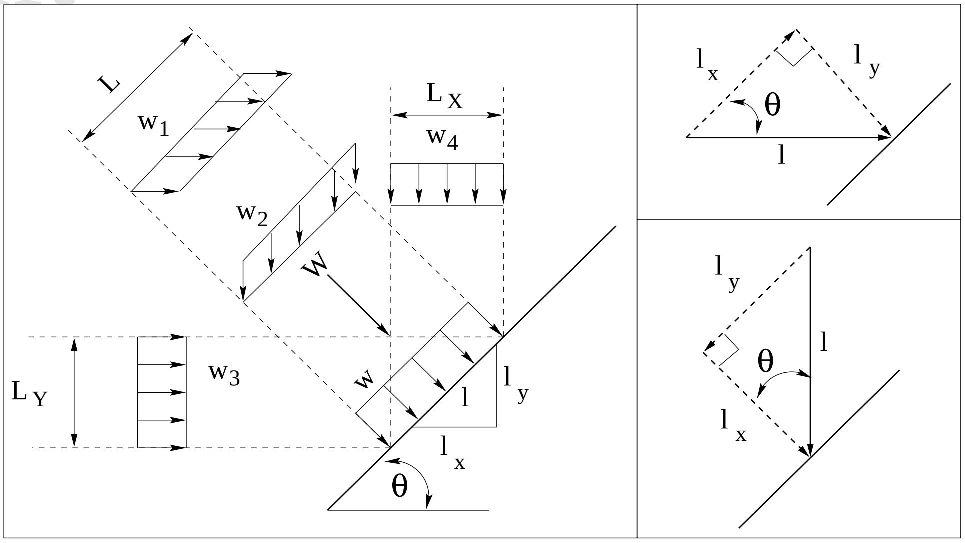


Figure 5.6: Inclined Loads on Inclined Members

$$w = w_1\frac{l_y}{l} + w_2\frac{lx}{l} + w_3\frac{l_y}{l} + w_4\frac{l_x}{l} \tag{5.7}$$

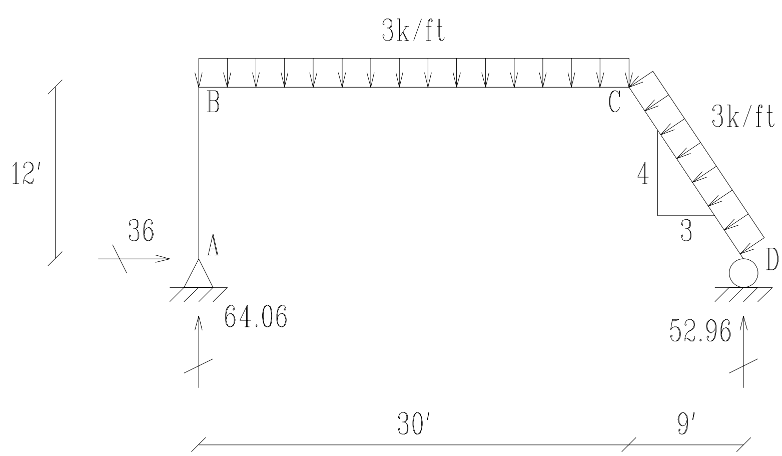
$$W = w_1\frac{l_y}{l}L + w_2\frac{lx}{l}L + w_3\frac{l_y}{l}L_Y + w_4\frac{l_x}{l}L_X \tag{5.8}$$

$$\frac{lx}{l} = \cos\theta \tag{5.9}$$

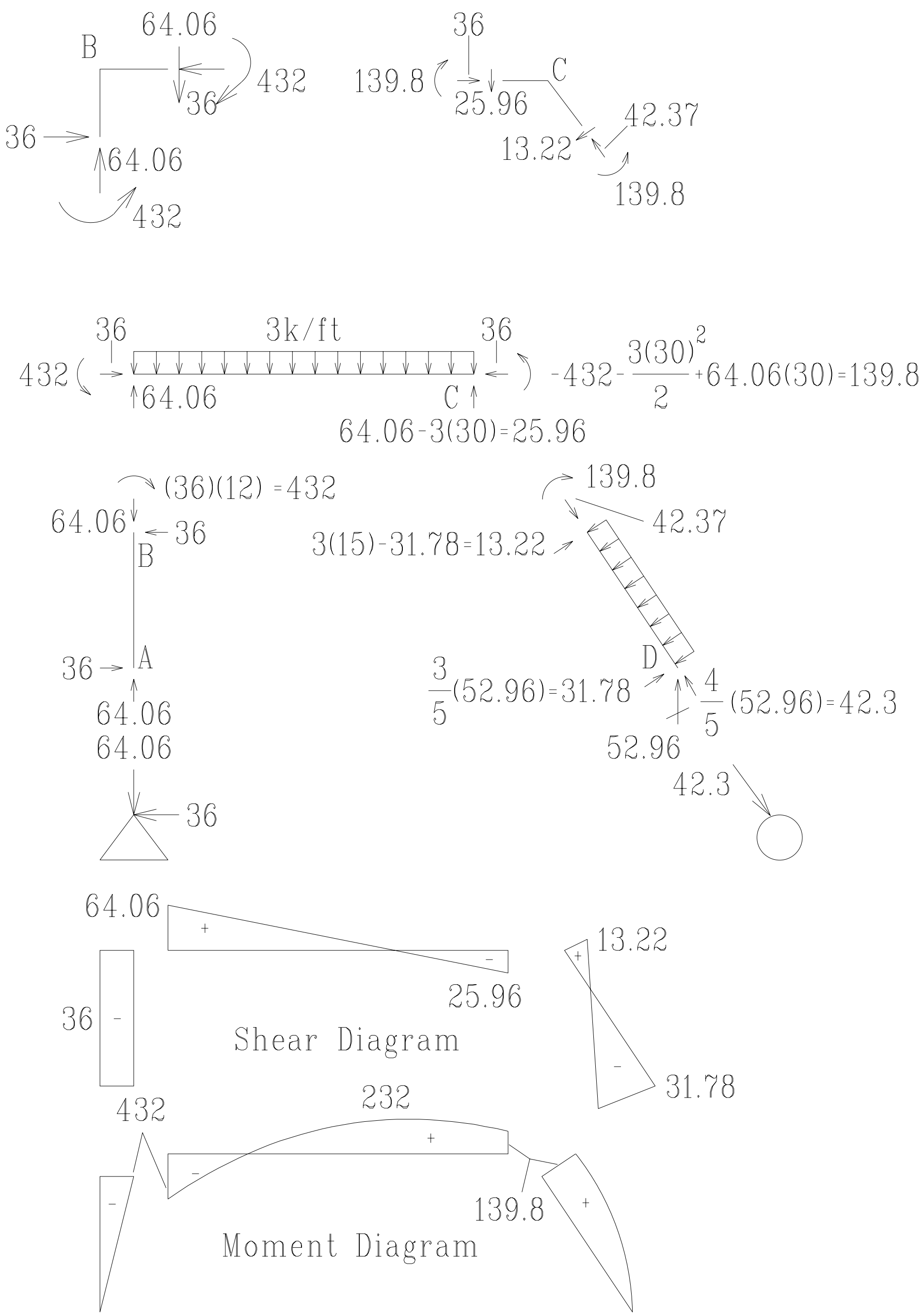
$$\frac{l_y}{l} = \sin\theta \tag{5.10}$$

■ Example 5-3: Frame Shear and Moment Diagram

Draw the shear and moment diagram of the following frame



Solution:



Reactions are determined first

$$
\begin{aligned}
(+ \rightarrow)\ \Sigma F_x = 0; \quad & \Rightarrow \quad R_{A_x} - \underbrace{\frac{4}{5}(3)(15)}_{\text{load}} = 0 \\
& \Rightarrow \quad R_{A_x} = 36 \text{ k} \\
(+ \circlearrowleft)\ \Sigma M_A = 0; \quad & \Rightarrow \quad (3)(30)(\tfrac{30}{2}) + \underbrace{\frac{3}{5}(3)(15)\left(30 + \frac{9}{2}\right)}_{CD_Y} - \underbrace{\frac{4}{5}(3)(15)\frac{12}{2}}_{CD_X} - 39R_{D_y} = 0 \\
& \Rightarrow \quad R_{D_y} = 52.96 \text{ k} \\
(+ \uparrow)\ \Sigma F_y = 0; \quad & \Rightarrow \quad R_{A_y} - (3)(30) - \tfrac{3}{5}(3)(15) + 52.96 = 0 \\
& \Rightarrow \quad R_{A_y} = 64.06 \text{ k}
\end{aligned}
$$

Shear:

1. For $A-B$, the shear is constant, equal to the horizontal reaction at A and negative according to our previously defined sign convention, $V_A = -36$ k
2. For member $B-C$ at B, the shear must be equal to the vertical force which was transmitted along $A-B$, and which is equal to the vertical reaction at A, $V_B = 64.06$.
3. Since $B-C$ is subjected to a uniform negative load, the shear along $B-C$ will have a slope equal to -3 and in terms of x (measured from B to C) is equal to
$$V_{B-C}(x) = 64.06 - 3x$$
4. The shear along $C-D$ is obtained by decomposing the vertical reaction at D into axial and shear components. Thus at D the shear is equal to $\frac{3}{5}52.96 = 31.78$ k and is negative. Based on our sign convention for the load, the slope of the shear must be equal to -3 along $C-D$. Thus the shear at point C is such that $V_c - \frac{5}{3}9(3) = -31.78$ or $V_c = 13.22$. The equation for the shear is given by (for x going from C to D)
$$V = 13.22 - 3x$$
5. We check our calculations by verifying equilibrium of node C
$$\begin{array}{lcl}(+ \rightarrow) \Sigma F_x = 0 & \Rightarrow & \frac{3}{5}(42.37) + \frac{4}{5}(13.22) = 25.42 + 10.58 = 36\surd \\ (+ \uparrow) \Sigma F_y = 0 & \Rightarrow & \frac{4}{5}(42.37) - \frac{3}{5}(13.22) = 33.90 - 7.93 = 25.97\surd\end{array}$$

Moment:

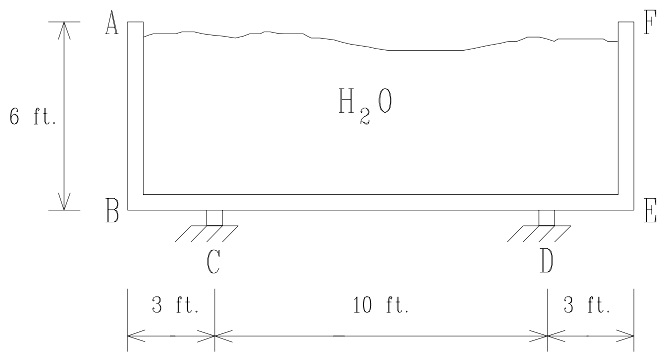
1. Along $A-B$, the moment is zero at A (since we have a hinge), and its slope is equal to the shear, thus at B the moment is equal to $(-36)(12) = -432$ k.ft
2. Along $B-C$, the moment is equal to
$$\begin{aligned} M_{B-C} &= M_B + \int_0^x V_{B-C}(x)dx = -432 + \int_0^x (64.06 - 3x)dx \\ &= -432 + 64.06x - 3\frac{x^2}{2}\end{aligned}$$
which is a parabola. Substituting for $x = 30$, we obtain at node C: $M_C = -432 + 64.06(30) - 3\frac{30^2}{2} = 139.8$ k.ft
3. If we need to determine the maximum moment along $B-C$, we know that $\frac{dM_{B-C}}{dx} = 0$ at the point where $V_{B-C} = 0$, that is $V_{B-C}(x) = 64.06 - 3x = 0 \Rightarrow x = \frac{64.06}{3} = 25.0$ ft. In other words, maximum moment occurs where the shear is zero.
Thus $M_{B-C}^{max} = -432 + 64.06(25.0) - 3\frac{(25.0)^2}{2} = -432 + 1,601.5 - 937.5 = 232$ k.ft
4. Finally along $C-D$, the moment varies quadratically (since we had a linear shear), the moment first increases (positive shear), and then decreases (negative shear). The moment along $C-D$ is given by
$$\begin{aligned} M_{C-D} &= M_C + \int_0^x V_{C-D}(x)dx = 139.8 + \int_0^x (13.22 - 3x)dx \\ &= 139.8 + 13.22x - 3\frac{x^2}{2}\end{aligned}$$
which is a parabola.
Substituting for $x = 15$, we obtain at node C $M_C = 139.8 + 13.22(15) - 3\frac{15^2}{2} = 139.8 + 198.3 - 337.5 = 0\surd$

■

■ **Example 5-4: Frame Shear and Moment Diagram; Hydrostatic Load**

The frame shown below is the structural support of a flume. Assuming that the frames are spaced 2 ft apart along the length of the flume,

1. Determine all internal member end actions
2. Draw the shear and moment diagrams
3. Locate and compute maximum internal bending moments
4. If this is a reinforced concrete frame, show the location of the reinforcement.



Density of water=62.4 lb/ft^3

Spacing of frames=2 ft.

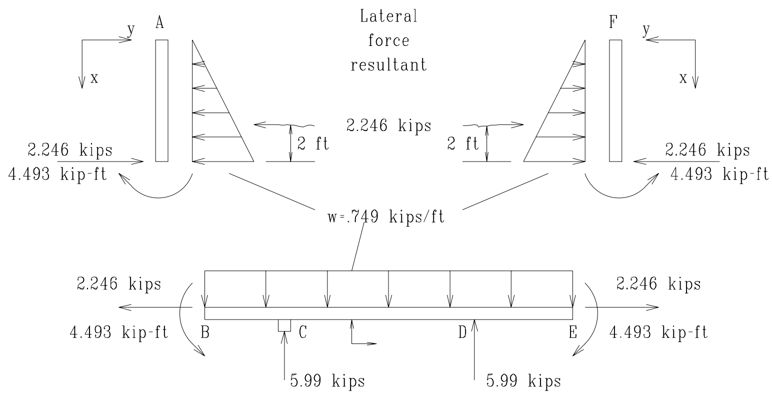
Solution:

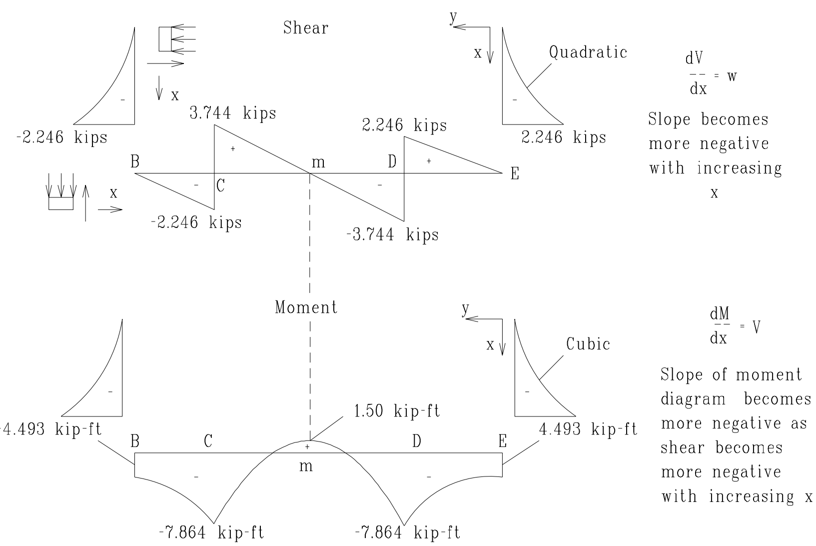
The hydrostatic pressure causes lateral forces on the vertical members which can be treated as cantilevers fixed at the lower end.

The pressure is linear and is given by $p = \gamma h$. Since each frame supports a 2 ft wide slice of the flume, the equation for w (pounds/foot) is

$$\begin{aligned} w &= (2)(62.4)(h) \\ &= 124.8h \text{ lbs/ft} \end{aligned}$$

At the base $w = (124.8)(6) = 749$ lbs/ft $= .749$ k/ft Note that this is both the lateral pressure on the end walls as well as the uniform load on the horizontal members.





End Actions

1. Base force at B is $F_{Bx} = (.749)\frac{6}{2} = 2.246$ k
2. Base moment at B is $M_B = (2.246)\frac{6}{3} = 4.493$ k.ft
3. End force at B for member $B-E$ are equal and opposite.
4. Reaction at C is $R_{Cy} = (.749)\frac{16}{2} = 5.99$ k

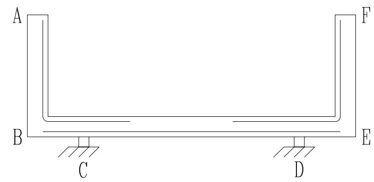
Shear forces

1. Base at B the shear force was determined earlier and was equal to 2.246 k. Based on the orientation of the $x-y$ axis, this is a negative shear.
2. The vertical shear at B is zero (neglecting the weight of $A-B$)
3. The shear to the left of C is $V = 0 + (-.749)(3) = -2.246$ k.
4. The shear to the right of C is $V = -2.246 + 5.99 = 3.744$ k

Moment diagrams

1. At the base: B $M = 4.493$ k.ft as determined above.
2. At the support C, $M_c = -4.493 + (-.749)(3)(\frac{3}{2}) = -7.864$ k.ft
3. The maximum moment is equal to $M_{max} = -7.864 + (.749)(5)(\frac{5}{2}) = 1.50$ k.ft

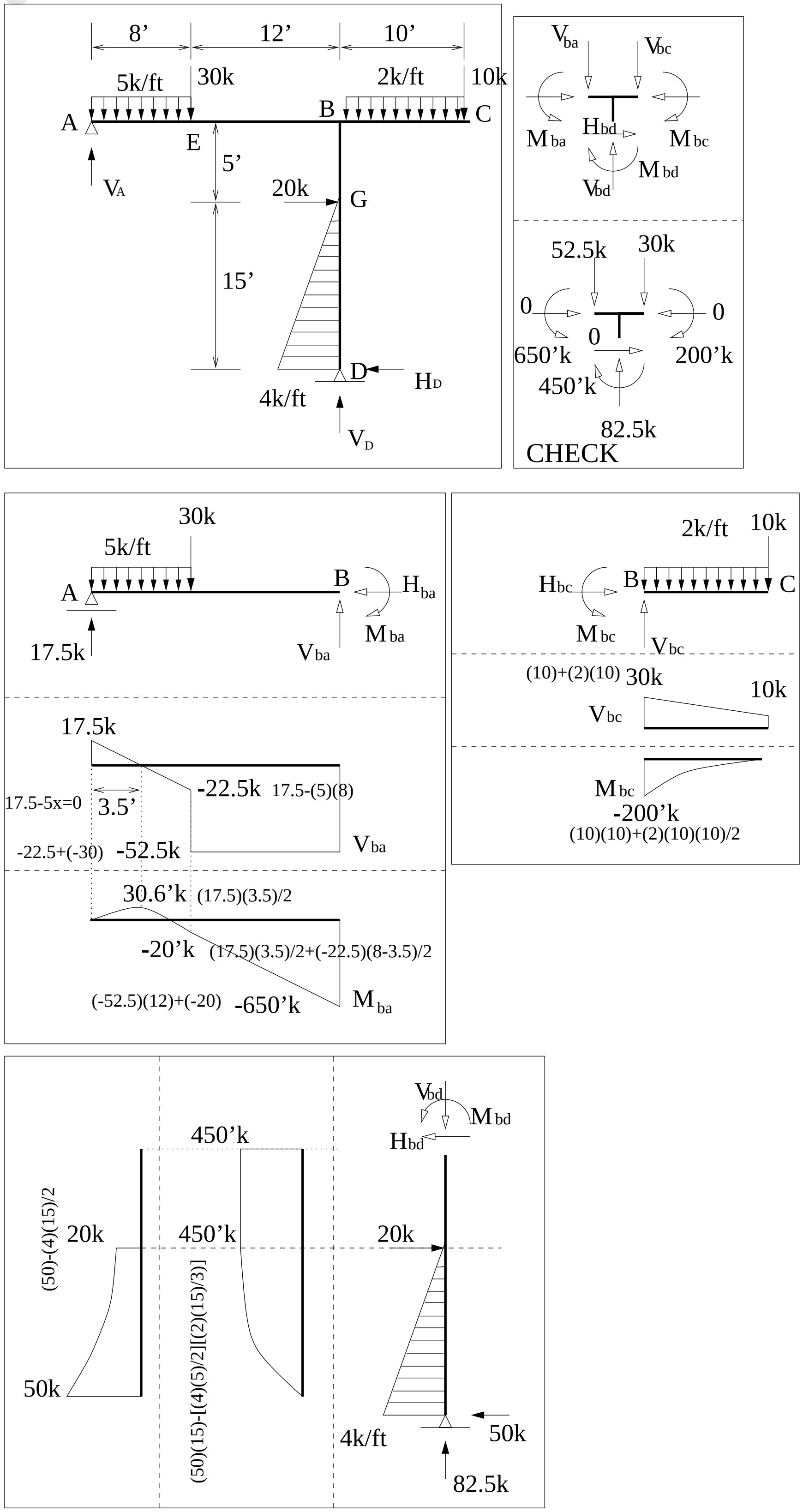
Design: Reinforcement should be placed along the fibers which are under tension, that is on the side of the negative moment[3]. The figure below schematically illustrates the location of the flexural[4] reinforcement.



■ Example 5-5: Shear Moment Diagrams for Frame

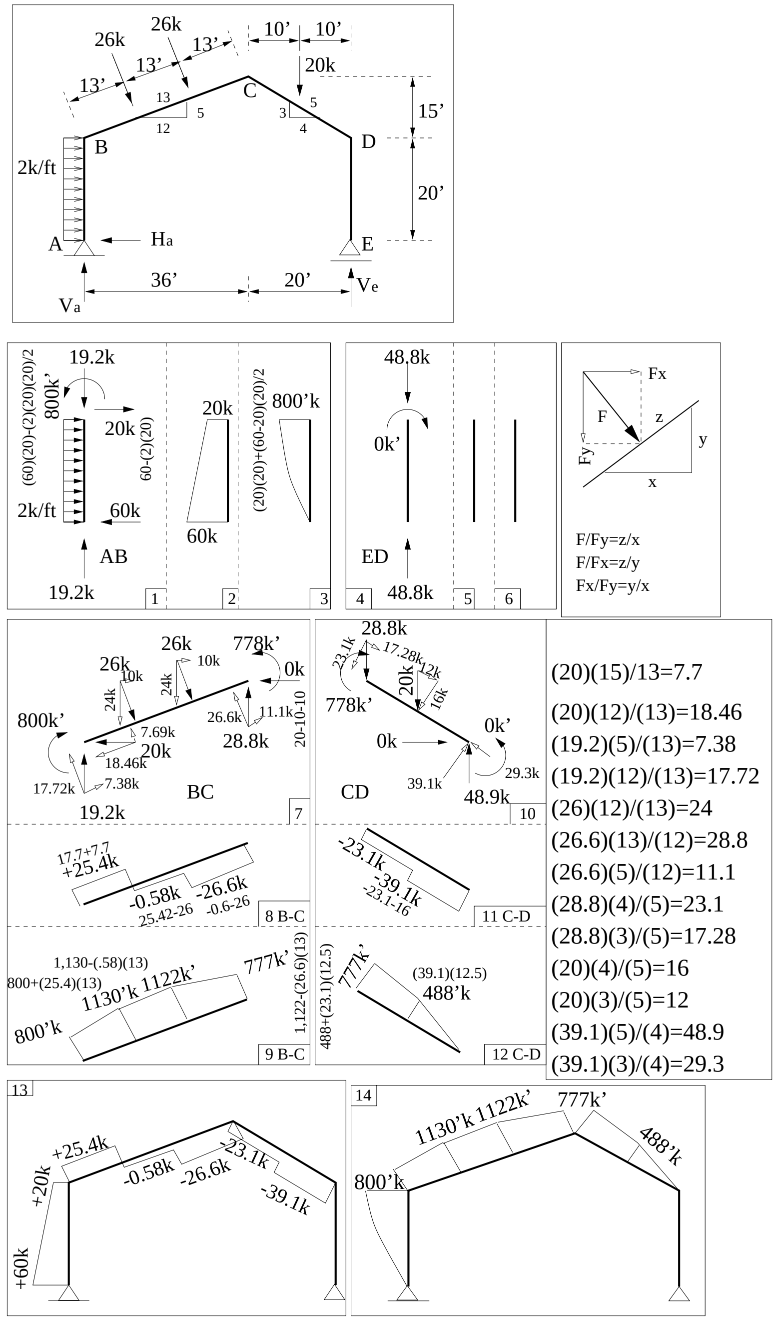
[3] That is why in most European countries, the sign convention for design moments is the opposite of the one commonly used in the U.S.A.; Reinforcement should be placed where the moment is "postive".

[4] Shear reinforcement is made of a series of vertical stirrups.



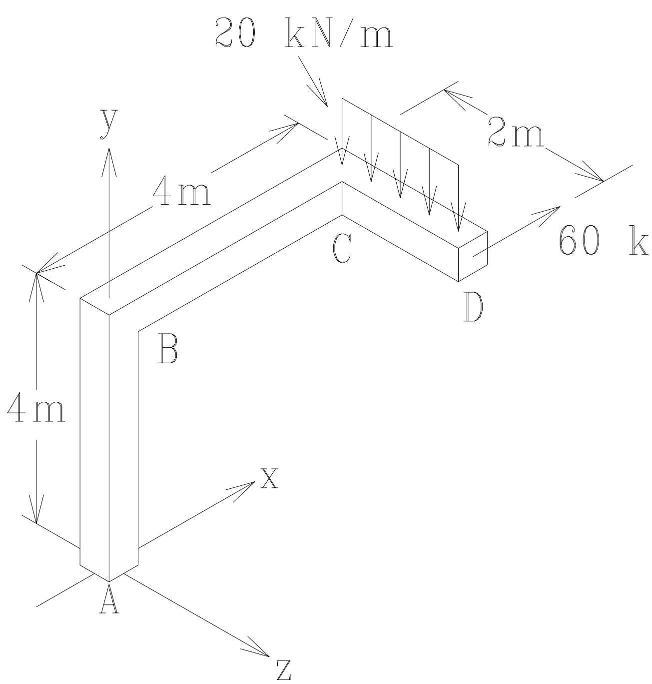
■

■ **Example 5-6: Shear Moment Diagrams for Inclined Frame**



5.4.3 3D Frame

Example 5-7: 3D Frame



1. The frame has a total of 6 reactions (3 forces and 3 moments) at the support, and we have a total of 6 equations of equilibrium, thus it is statically determinate.

2. Each member has the following internal forces (defined in terms of the *local coordinate system* of each member $x' - y' - z'$)

Member	Internal Forces					
Member	Axial	Shear		Moment		Torsion
	$N_{x'}$	$V_{y'}$	$V_{z'}$	$M_{y'}$	$M_{z'}$	$T_{x'}$
$C-D$		$\surd$	$\surd$	$\surd$	$\surd$	
$B-C$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$
$A-B$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$

3. The numerical calculations for the analysis of this three dimensional frame are quite simple, however the main complexity stems from the difficulty in visualizing the inter-relationships between internal forces of adjacent members.

4. In this particular problem, rather than starting by determining the reactions, it is easier to determine the internal forces at the end of each member starting with member $C-D$. Note that temporarily we adopt a sign convention which is compatible with the local coordinate systems.

C-D

$$
\begin{array}{rclllll}
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^C = (20)(2) & = +40\text{kN} & \\
\Sigma F_{z'} = 0 & \Rightarrow & V_{z'}^C & = +60\text{kN} & \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^C & = -(60)(2) & = -120\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^C & = (20)(2)\frac{2}{2} & = +40\text{kN.m}
\end{array}
$$

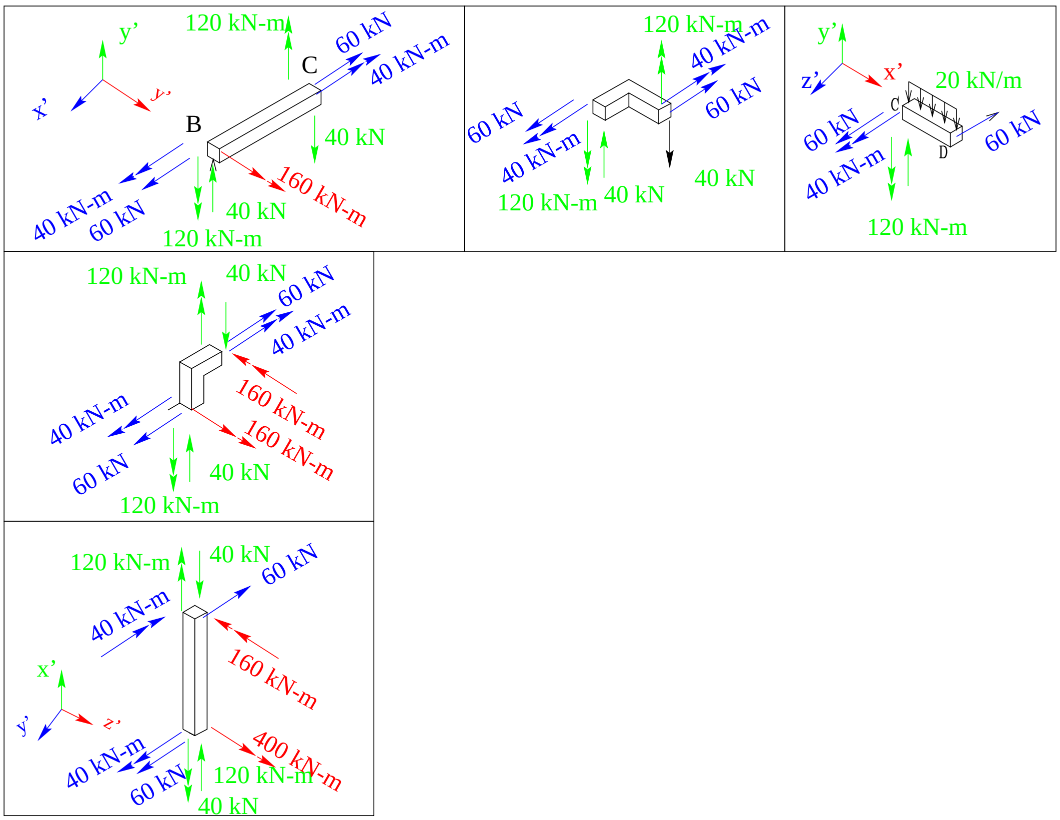
B-C

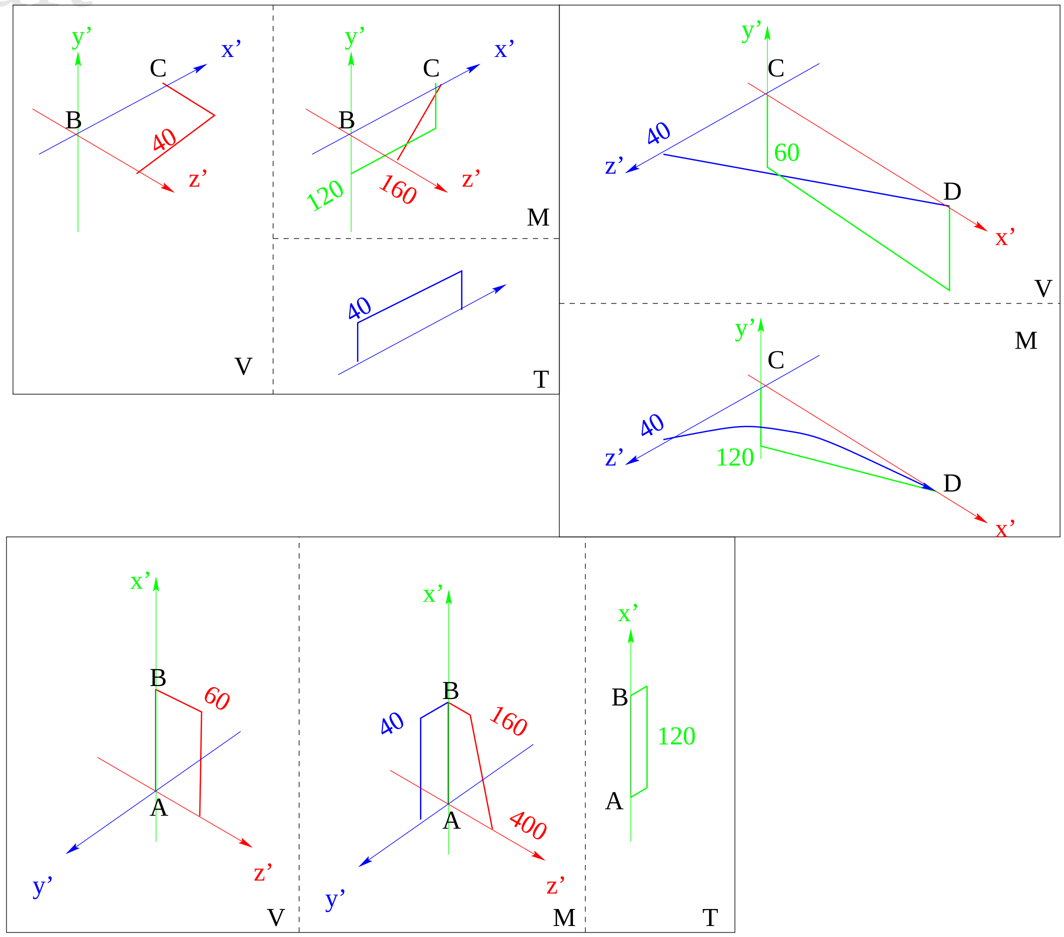
$$
\begin{array}{rcllll}
\Sigma F_{x'} = 0 & \Rightarrow & N_{x'}^B & = V_{z'}^C & & = -60\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^B & = V_{y'}^C & & = +40\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^B & = M_{y'}^C & & = -120\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^B & = V_y'^C(4) & = (40)(4) & = +160\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T_{x'}^B & = -M_{z'}^C & & = -40\text{kN.m}
\end{array}
$$

A-B

$$
\begin{array}{llllr}
\Sigma F_{x'} = 0 & \Rightarrow & N^A_{x'} & = V^B_{y'} & = +40\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V^A_{y'} & = N^B_{x'} & = +60\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M^A_{y'} & = T^B_{x'} & = +40\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M^A_{z'} & = M^B_{z'} + N^B_{x'}(4) = 160 + (60)(4) & = +400\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T^A_{x'} & = M^B_{y'} & = -120\text{kN.m}
\end{array}
$$

The interaction between axial forces N and shear V as well as between moments M and torsion T is clearly highlighted by this example.





■

5.5 Arches

21 See section ??.

Chapter 6

DEFLECTION of STRUCTRES; Geometric Methods

1 Deflections of structures must be determined in order to satisfy serviceability requirements i.e. limit deflections under *service* loads to acceptable values (such as $\frac{\Delta}{L} \leq 360$).

2 Later on, we will see that deflection calculations play an important role in the analysis of statically indeterminate structures.

3 We shall focus on flexural deformation, however the end of this chapter will review axial and torsional deformations as well.

4 Most of this chapter will be a *review* of subjects covered in *Strength of Materials*.

5 This chapter will examine deflections of structures based on geometric considerations. Later on, we will present a more pwerful method based on energy considerations.

6.1 Flexural Deformation

6.1.1 Curvature Equation

6 Let us consider a segment (between point 1 and point 2), Fig. 6.1 of a beam subjected to flexural loading.

7 The **slope** is denoted by θ, the change in slope per unit length is the **curvature** κ, the **radius of curvature** is ρ.

8 From *Strength of Materials* we have the following relations

$$ds = \rho d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho} \tag{6.1}$$

9 We also note by extension that $\Delta s = \rho \Delta\theta$

10 As a first order approximation, and with $ds \approx dx$ and $\frac{dy}{dx} = \theta$ Eq. 6.1 becomes

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \tag{6.2}$$

11 † Next, we shall (re)derive the **exact** expression for the curvature. From Fig. 6.1, we have

$$\tan\theta = \frac{dy}{dx} \tag{6.3}$$

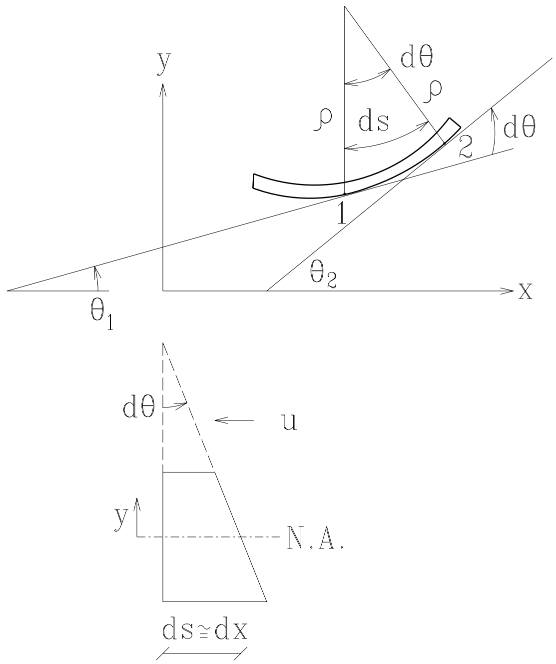


Figure 6.1: Curvature of a flexural element

Defining t as

$$t = \frac{dy}{dx} \tag{6.4}$$

and combining with Eq. 6.3 we obtain

$$\theta = \tan^{-1} t \tag{6.5}$$

12 Applying the chain rule to $\kappa = \frac{d\theta}{ds}$ we have

$$\kappa = \frac{d\theta}{dt}\frac{dt}{ds} \tag{6.6}$$

ds can be rewritten as

$$\left.\begin{array}{rcl} ds & = & \sqrt{dx^2 + dy^2} \\ & = & \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ t & = & \frac{dy}{dx} \end{array}\right\} ds = \sqrt{1+t^2}dx \tag{6.7}$$

13 Next combining Eq. 6.6 and 6.7 we obtain

$$\left.\begin{array}{rcl} \kappa & = & \frac{d\theta}{dt}\frac{dt}{\sqrt{1+t^2}dx} \\ \theta & = & \tan^{-1} t \\ \frac{d\theta}{dt} & = & \frac{1}{1+t^2} \end{array}\right\} \left.\begin{array}{rcl} \kappa & = & \frac{1}{1+t^2}\frac{1}{\sqrt{1+t^2}}\frac{dt}{dx} \\ \frac{dt}{dx} & = & \frac{d^2y}{dx^2} \end{array}\right\} \kappa = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{6.8}$$

14 Thus the slope θ, curvature κ, radius of curvature ρ are related to the y displacement at a point x along a flexural member by

$$\boxed{\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}} \tag{6.9}$$

15 If the displacements are very small, we will have $\frac{dy}{dx} << 1$, thus Eq. 6.9 reduces to

$$\boxed{\kappa = \frac{d^2y}{dx^2} = \frac{1}{\rho}} \tag{6.10}$$

6.1.2 Differential Equation of the Elastic Curve

16 Again with reference to Figure 6.1 a positive $d\theta$ at a positive y (upper fibers) will cause a *shortening* of the upper fibers

$$\Delta u = -y\Delta\theta \tag{6.11}$$

17 This equation can be rewritten as

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = -y \lim_{\Delta s \to 0} \frac{\Delta \theta}{\Delta s} \tag{6.12}$$

and since $\Delta s \approx \Delta x$

$$\underbrace{\frac{du}{dx}}_{\varepsilon} = -y\frac{d\theta}{dx} \tag{6.13}$$

Combining this with Eq. 6.10

$$\boxed{\frac{1}{\rho} = \kappa = -\frac{\varepsilon}{y}} \tag{6.14}$$

This is the fundamental relationship between curvature (κ), elastic curve (y), and linear strain (ε).

18 Note that so far we made no assumptions about material properties, i.e. it can be elastic or inelastic.

19 For the elastic case:

$$\left.\begin{array}{rcl} \varepsilon_x & = & \frac{\sigma}{E} \\ \sigma & = & -\frac{My}{I} \end{array}\right\} \varepsilon = -\frac{My}{EI} \tag{6.15}$$

Combining this last equation with Eq. 6.14 yields

$$\boxed{\frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI}} \tag{6.16}$$

This fundamental equation relates moment to curvature.

20 Combining this equation with the moment-shear-force relations determined in the previous chapter

$$\left.\begin{array}{rcl} \frac{dV}{dx} & = & w(x) \\ \frac{dM}{dx} & = & V(x) \end{array}\right\} \frac{d^2M}{dx^2} \tag{6.17-a}$$

we obtain

$$\boxed{\frac{w(x)}{EI} = \frac{d^4y}{dx^4}} \tag{6.18}$$

6.1.3 Moment Temperature Curvature Relation

21 Assuming linear variation in temperature

$$\Delta_T = \alpha\Delta T_T dx \text{ Top} \tag{6.19-a}$$

$$\Delta_B = \alpha\Delta T_B dx \text{ Bottom} \tag{6.19-b}$$

22 Next, considering

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{\varepsilon}{y} \tag{6.20}$$

In this case, we can take $\varepsilon = \alpha\Delta T_T$ at $y = \frac{h}{2}$ or $\varepsilon = \alpha(T_T - T_B)$ at $y = h$ thus

$$\boxed{\frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{\alpha(T_T - T_B)}{h}} \tag{6.21}$$

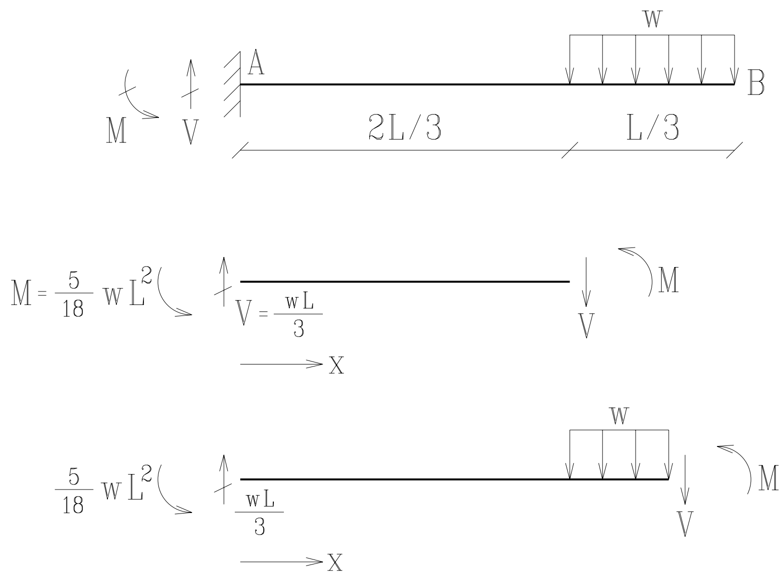
6.2 Flexural Deformations

6.2.1 Direct Integration Method

23 Equation 6.18 lends itself naturally to the method of double integration which was presented in *Strength of Materials*

Example 6-1: Double Integration

Determine the deflection at B for the following cantilevered beam



Solution:

At: $0 \le x \le \frac{2L}{3}$

1. Moment Equation
$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 \tag{6.22}$$
2. Integrate once
$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x + C_1 \tag{6.23}$$
However we have at $x = 0$, $\frac{dy}{dx} = 0$, $\Rightarrow C_1 = 0$
3. Integrate twice
$$EIy = \frac{wL}{18}x^3 - \frac{5wL^2}{36}x^2 + C_2 \tag{6.24}$$
Again we have at $x = 0$, $y = 0$, $\Rightarrow C_2 = 0$

At: $\frac{2L}{3} \leq x \leq L$

1. Moment equation

$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 - w(x - \frac{2L}{3})(\frac{x - \frac{2L}{3}}{2}) \tag{6.25}$$

2. Integrate once

$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x - \frac{w}{6}(x - \frac{2L}{3})^3 + C_3 \tag{6.26}$$

Applying the boundary condition at $x = \frac{2L}{3}$, we must have $\frac{dy}{dx}$ equal to the value coming from the left, $\Rightarrow C_3 = 0$

3. Integrating twice

$$EIy = \frac{wL}{18}x^3 - \frac{5}{36}wL^2x^2 - \frac{w}{24}(x - \frac{2L}{3})^4 + C_4 \tag{6.27}$$

Again following the same argument as above, $C_4 = 0$

Substituting for $x = L$ we obtain

$$y = \frac{163}{1944}\frac{wL^4}{EI} \tag{6.28}$$

■

6.2.2 Curvature Area Method (Moment Area)

6.2.2.1 First Moment Area Theorem

24 From equation 6.16 we have

$$\frac{d\theta}{dx} = \frac{M}{EI} \tag{6.29}$$

this can be rewritten as (note similarity with $\frac{dV}{dx} = w(x)$).

$$\boxed{\theta_{21} = \theta_2 - \theta_1 = \int_{x_1}^{x_2} d\theta = \int_{x_1}^{x_2} \frac{M}{EI}dx} \tag{6.30}$$

or with reference to Figure 6.2

First Area Moment Theorem: The change in slope from point 1 to point 2 on a beam is equal to the area under the M/EI diagram between those two points.

6.2.2.2 Second Moment Area Theorem

25 Similarly, with reference to Fig. 6.2, we define by t_{21} the distance between point 2 and the tangent at point 1. For an infinitesimal distance $ds = \rho d\theta$ and for small displacements

$$\left.\begin{array}{rcr} dt & = & d\theta(x_2 - x) \\ \frac{d\theta}{dx} & = & \frac{M}{EI} \end{array}\right\} dt = \frac{M}{EI}(x_2 - x)dx \tag{6.31}$$

To evaluate t_{21}

$$\boxed{t_{21} = \int_{x_1}^{x_2} dt = \int_{x_1}^{x_2} \frac{M}{EI}(x_2 - x)dx} \tag{6.32}$$

or

Second Moment Area Theorem: The tangent distance t_{21} between a point, 2, on the beam and the tangent of another point, 1, is equal to the moment of the M/EI diagram between points 1 and 2, with respect to point 2.

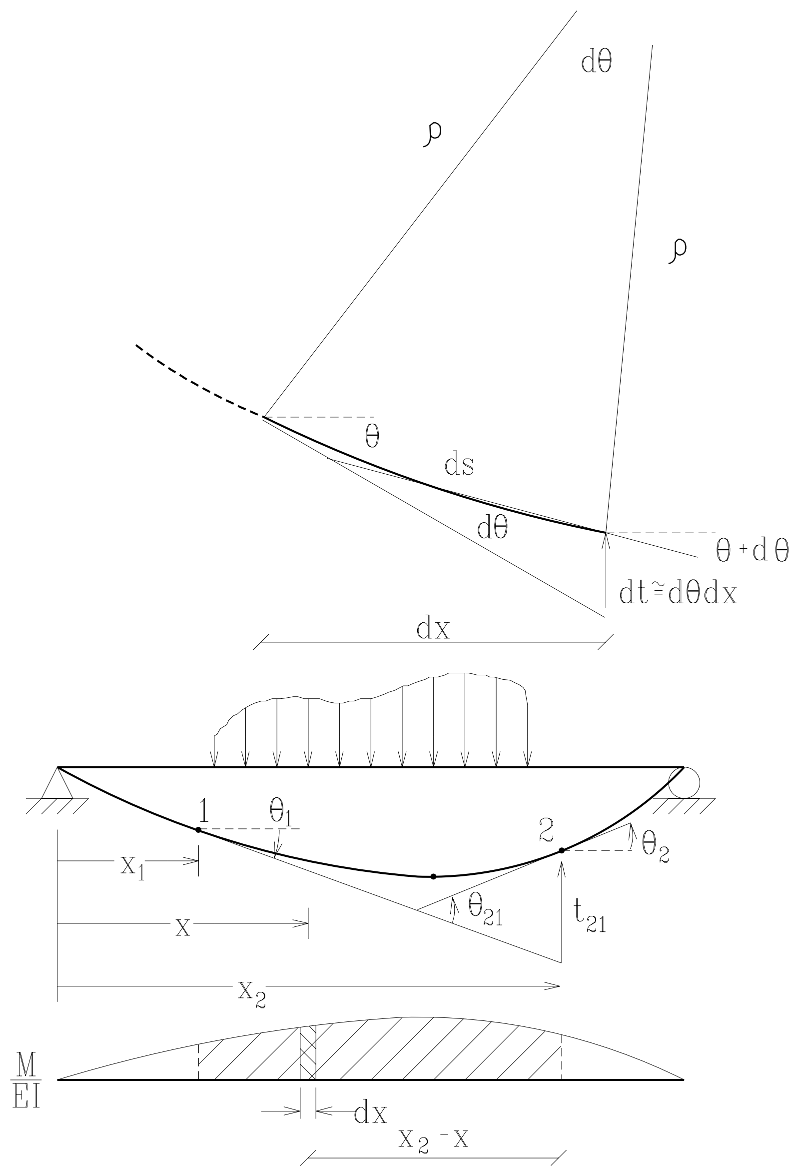


Figure 6.2: Moment Area Theorems

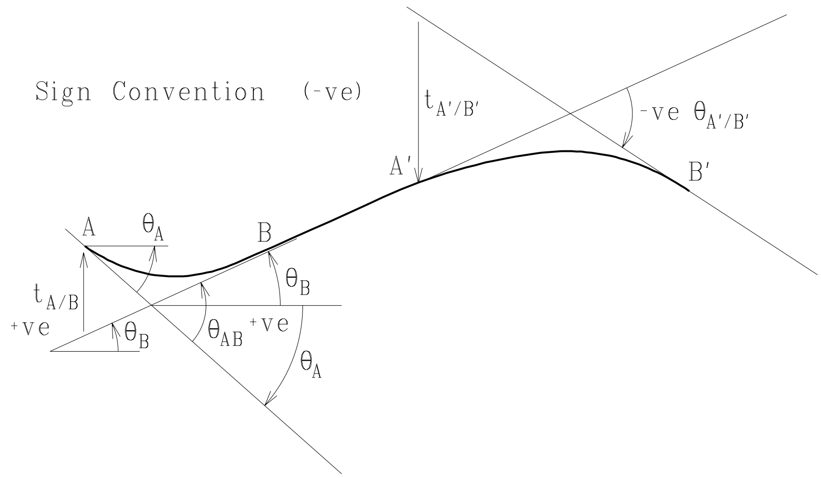
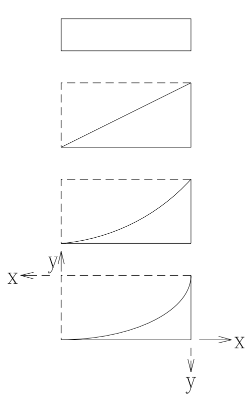


Figure 6.3: Sign Convention for the Moment Area Method



degree	AREA		CENTROID	
0	xy	—	$\frac{x}{2}$	—
1	$\frac{xy}{2}$	$\frac{xy}{2}$	$\frac{2}{3}x$	$\frac{x}{3}$
2	$\frac{xy}{3}$	$\frac{2}{3}xy$	$\frac{3}{4}x$	$\frac{3x}{8}$
3	$\frac{xy}{4}$	$\frac{3}{4}xy$	$\frac{4}{5}x$	$\frac{2x}{5}$
n	$\frac{xy}{n+1}$	$\frac{nxy}{n+1}$	$\frac{x}{n+2}$	$\frac{(n+1)x}{2(n+2)}$

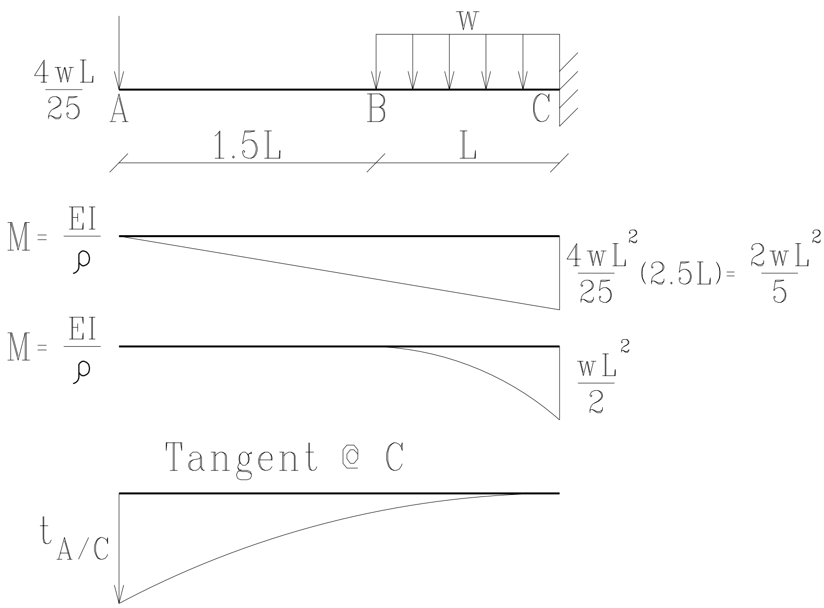
Figure 6.4: Areas and Centroid of Polynomial Curves

26 The sign convention is as shown in Fig. 6.3

27 Fig. 6.4 is a helpful tool to determine centroid and areas.

■ Example 6-2: Moment Area, Cantilevered Beam

Determine the deflection of point A



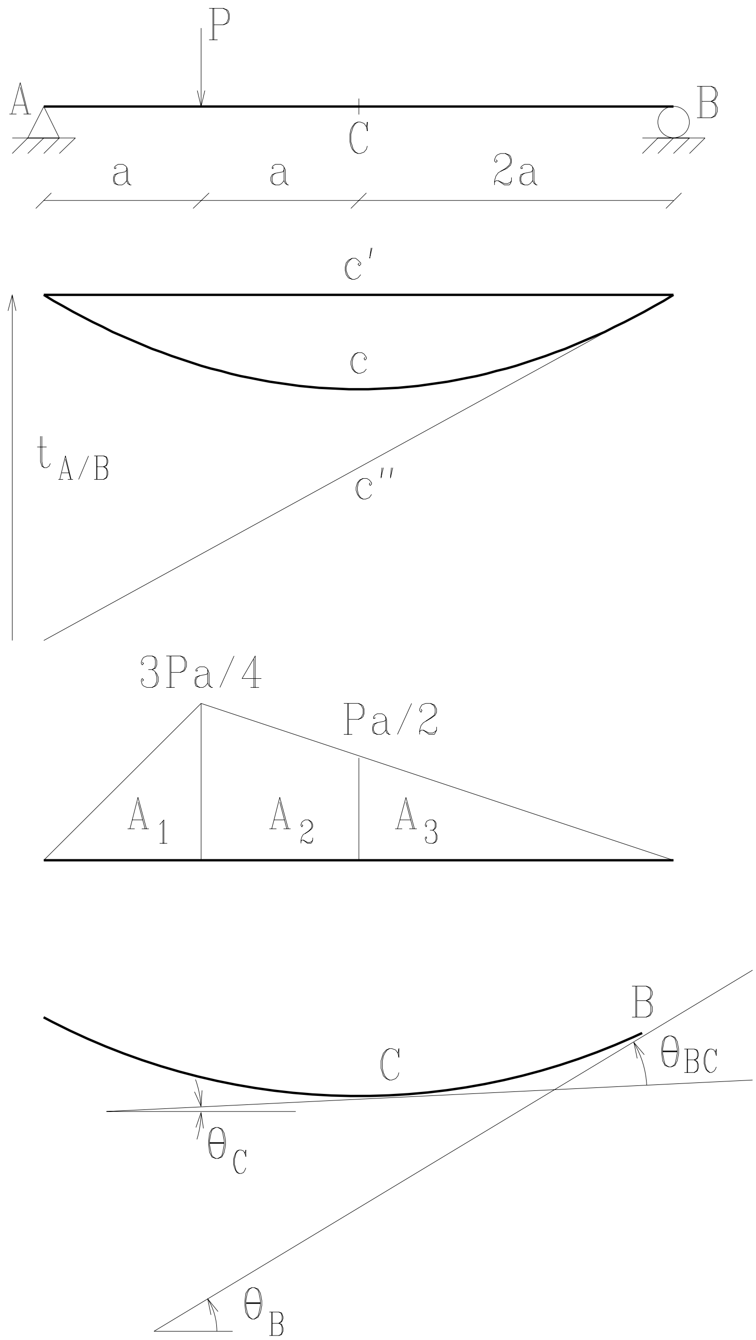
Solution:

$$EIt_{A/C} = \underbrace{\underbrace{\frac{1}{2}\left(\frac{-2wL^2}{5}\right)\left(\frac{5L}{2}\right)}_{\text{area}}\left(\frac{2}{3}\frac{5L}{2}\right)}_{\text{Moment wrt A}} + \underbrace{\underbrace{\frac{1}{3}\left(\frac{-wL^2}{2}\right)(L)}_{\text{area}}\left(\frac{9L}{4}\right)}_{\text{Moment wrt A}} = -\frac{29wL^4}{24} \qquad (6.33)$$

Thus, $\Delta_A = \frac{29wL^4}{24EI}$ ■

■ Example 6-3: Moment Area, Simply Supported Beam

Determine Δ_C and θ_C for the following example



Solution:

Deflection Δ_C is determined from $\Delta_C = c'c = c'c" - c"c$, $c"c = t_{C/B}$, and $c'c" = \frac{t_{A/B}}{2}$ from geometry

$$t_{A/B} = \frac{1}{EI}\left[\underbrace{\underbrace{\left(\frac{3Pa}{4}\right)\left(\frac{a}{2}\right)}_{A_1}\left(\frac{2a}{3}\right)}_{\text{Moment}} + \underbrace{\underbrace{\left(\frac{3Pa}{4}\right)\left(\frac{3a}{2}\right)}_{A_2+A_3}\left(a+\frac{3a}{3}\right)}_{\text{Moment}}\right] = \frac{5Pa^3}{2EI} \tag{6.34}$$

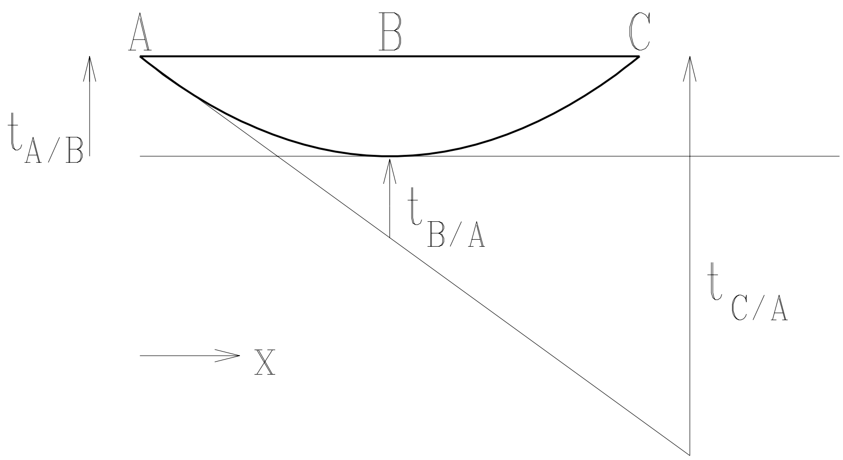


Figure 6.5: Maximum Deflection Using the Moment Area Method

This is positive, thus above tangent from B

$$t_{C/B} = \frac{1}{EI}\left(\frac{Pa}{2}\right)\left(\frac{2a}{2}\right)\left(\frac{2a}{3}\right) = \frac{Pa^3}{3EI} \tag{6.35}$$

Positive, thus above the tangent from B Finally,

$$\Delta_C = \frac{5Pa^3}{4EI} - \frac{Pa^3}{3EI} = \frac{11}{12}\frac{Pa^3}{EI} \tag{6.36}$$

Rotation θ_C is

$$\left.\begin{array}{rcl} \theta_{BC} & = & \theta_B - \theta_C \Rightarrow \theta_C = \theta_B - \theta_{BC} \\ \theta_{BC} & = & A_3 \\ \theta_B & = & \frac{t_{A/B}}{L} \end{array}\right\} \Rightarrow \theta_C = \frac{5Pa^3}{2EI(4a)} - \left(\frac{Pa}{2}\right)\left(\frac{2a}{2}\right) = \frac{Pa^2}{8EI} \tag{6.37}$$

■

6.2.2.3 Maximum Deflection

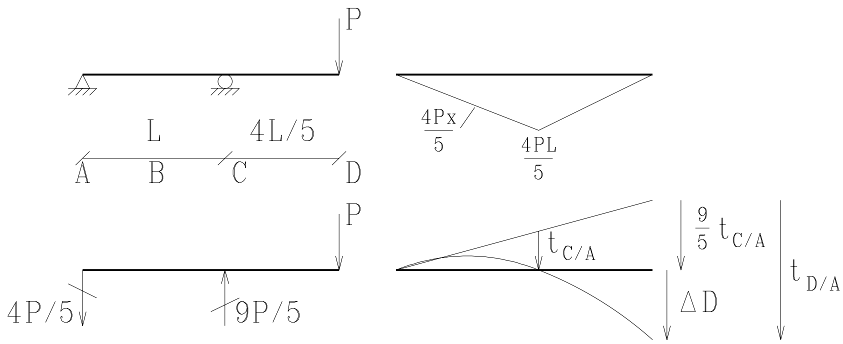
28 A joint along the beam will have the maximum (or minimum) relative deflection if $\frac{dy}{dx} = 0$ or $\theta = 0$ So we can determine Δ_{max} if x is known, Fig. 6.5.

To determine x:

1. Compute $t_{C/A}$
2. $\theta_A = \frac{t_{C/A}}{L}$
3. $\theta_B = 0$ and $\theta_{AB} = \theta_A - \theta_B$, thus $\theta_{AB} = \theta_A$. Hence, compute θ_{AB} in terms of x using First Theorem.
4. Equate items 2 and 3, then solve for x.

■ **Example 6-4: Maximum Deflection**

Determine the deflection at D, and the maximum deflection at B



Solution:

Deflection at D:

$$\Delta_D = t_{D/A} - \frac{9}{5}t_{C/A} \tag{6.38-a}$$

$$EIt_{C/A} = \frac{1}{2}\left(\frac{-4PL}{5}\right)(L)\left(\frac{L}{3}\right) \tag{6.38-b}$$

$$t_{C/A} = -\frac{2PL^3}{15EI} \tag{6.38-c}$$

$$EIt_{D/A} = \frac{1}{2}\left(\frac{-4PL}{5}\right)(L)\left(\frac{17L}{15}\right) + \frac{1}{2}\left(\frac{-4PL}{5}\right)\left(\frac{4L}{5}\right)\left(\frac{8L}{15}\right) \tag{6.38-d}$$

$$t_{D/A} = -\frac{234}{375}\frac{PL^3}{EI} \tag{6.38-e}$$

Substituting we obtain

$$\Delta_D = -\frac{48}{125}\frac{PL^3}{EI} \tag{6.39}$$

Maximum deflection at B:

$$t_{C/A} = -\frac{2PL^3}{15EI} \tag{6.40-a}$$

$$\theta_A = \frac{t_{C/A}}{L} = -\frac{2PL^2}{15EI} \tag{6.40-b}$$

$$\theta_{AB} = \frac{1}{EI}\left[\frac{1}{2}\left(\frac{4Px}{5}\right)(x)\right] = -\frac{2}{5}\frac{Px^2}{EI} \tag{6.40-c}$$

$$\theta_A = \theta_{AB} \Rightarrow -\frac{2}{15}\frac{PL^2}{EI} = -\frac{2}{5}\frac{Px^2}{EI} \Rightarrow x = \frac{L}{\sqrt{3}} \tag{6.40-d}$$

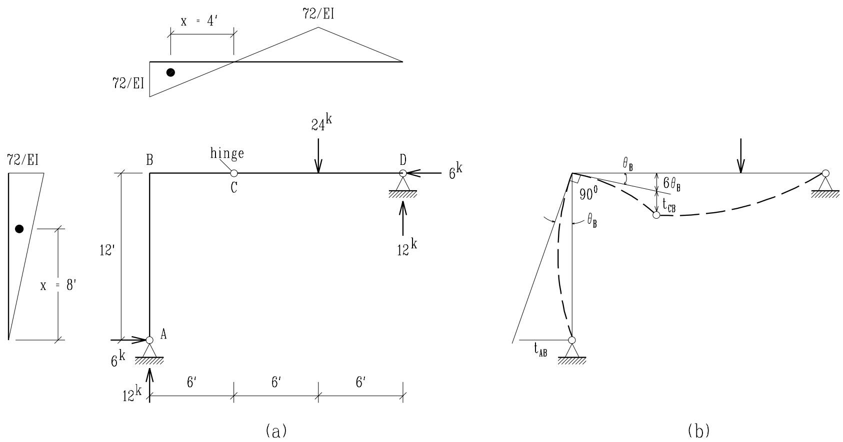
$$\Delta_{max} = \left(\frac{4Px}{5EI}\right)\left(\frac{x}{2}\right)\left(\frac{2}{3}x\right) \text{ at } x = \frac{L}{\sqrt{3}} \tag{6.40-e}$$

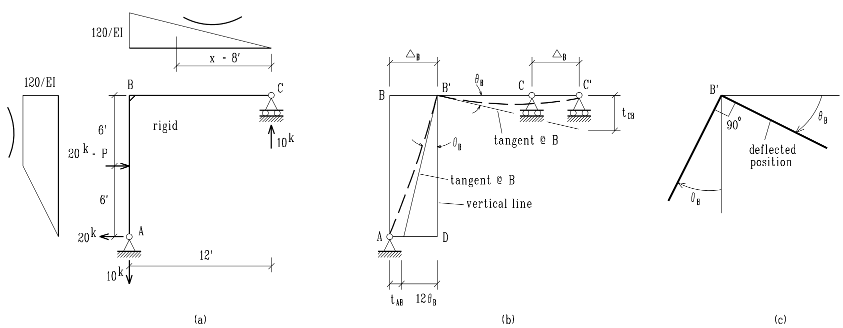
$$\Rightarrow \Delta_{max} = \frac{4PL^3}{45\sqrt{3}EI} \tag{6.40-f}$$

■

■ **Example 6-5: Frame Deflection**

Complete the following example problem

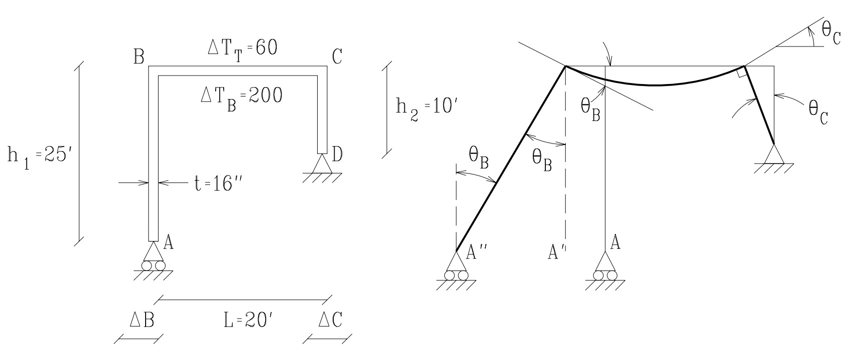




Solution:
xxxx ■

■ Example 6-6: Frame Subjected to Temperature Loading

Neglecting axial deformation, compute displacement at A for the following frame



Solution:

1. First let us sketch the deformed shape

2. BC flexes $\Rightarrow \theta_B = \theta_C \neq 0$

3. Rigid hinges at B and C with no load on AB and CD

4. Deflection at A

$$\begin{aligned}\Delta_A &= AA'' = AA' + A'A'' && (6.41\text{-a})\\ AA' &= \Delta_B = \Delta_C = |\theta_C|h_2 && (6.41\text{-b})\\ A''A' &= |\theta_B|h_1 && (6.41\text{-c})\end{aligned}$$

5. We need to compute θ_B & θ_C

$$\begin{aligned}\theta_B &= \frac{t_{C/B}}{L} && (6.42\text{-a})\\ \theta_C &= \theta_{CB} + \theta_B \text{ or } \theta_C = \frac{t_{B/C}}{L} && (6.42\text{-b})\end{aligned}$$

6. In order to apply the curvature are theorem, we need a curvature (or moment diagram).

$$\frac{1}{\rho} = \alpha(\frac{T_B - T_T}{h}) = \frac{M}{EI} \tag{6.43}$$

7.

$$t_{C/B} = A\left(\frac{L}{2}\right) \Rightarrow |\theta_B| = A\left(\frac{L}{2}\right)\left(\frac{1}{L}\right) = \frac{A}{2} \text{ or } \theta_B = -\frac{A}{2} \text{ (-ve)} \tag{6.44}$$

8.

$$\left.\begin{aligned}\theta_{CB} &= A\\ \theta_C &= \theta_{CB} + \theta_B\end{aligned}\right\}\theta_C = A - \frac{A}{2} = \frac{A}{2} \text{ (+ve)} \tag{6.45}$$

9. From above,

$$\left.\begin{aligned}\Delta_A &= |\theta_C|h_2 + |\theta_B|h_1 = \tfrac{A}{2}h_2 + \tfrac{A}{2}h_1\\ &= \tfrac{A}{2}(h_2 + h_1)\\ A &= \tfrac{\alpha(T_B - T_T)}{h}L\end{aligned}\right\}\Delta_A = \alpha(T_B - T_T)\frac{L}{h}\left(\frac{1}{2}\right)(h_2 + h_1) \tag{6.46}$$

10. Substitute

$$\begin{aligned}\Delta_A &= (6.5 \times 10^{-6})(200 - 60)\frac{(20)(12)}{(16)}(\frac{1}{2})(10 + 25)(12) && (6.47\text{-a})\\ &= 2.87 \text{ in} && (6.47\text{-b})\end{aligned}$$

11. Other numerical values:

$$\begin{aligned}\theta_B &= \theta_C = \frac{A}{2} = \frac{1}{2}\frac{\alpha(T_B - T_T)}{h}L && (6.48\text{-a})\\ &= \frac{1}{2}(6.5 \times 10^{-6})\frac{(200 - 60)}{(16)}(20)(12) && (6.48\text{-b})\\ &= .00683rad. \quad \text{(.39 degrees)} \quad = \frac{dy}{dx} << 1 && (6.48\text{-c})\end{aligned}$$

Note: $\sin(.00683) = .006829947$ and $\tan(.00683) = .006830106$

$$\begin{aligned}\frac{M}{EI} &= \frac{1}{\rho} = \alpha\frac{(T_B - T_T)}{h} && (6.49\text{-a})\\ &= (6.5 \times 10^{-6})\frac{(200 - 60)}{16} = 5.6875 \times 10^{-5} && (6.49\text{-b})\\ \Rightarrow \rho &= \frac{1}{5.6875 \times 10^{-5}} = 1.758 \times 10^4 \text{ in} = 1,465 \text{ ft} && (6.49\text{-c})\end{aligned}$$

12. In order to get M, we need E & I. Note the difference with other statically determinate structures; the stiffer the beam, the higher the moment; the higher the moment, the higher the stress? NO!!

13. $\sigma = \frac{My}{I} = \frac{EI}{\rho}\frac{y}{I} = \frac{Ey}{\rho}$

14. ρ is constant $\Rightarrow$ BC is on arc of circle M is constant & $\frac{M}{EI} = \frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{M}{EI} = C \Rightarrow y = C\frac{x^2}{2} + dx + e$

15. The slope is a parabola, (Why?)

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{\frac{d^2y}{dx^2}}{(}1 + (\frac{dy}{dx})^2)^{\frac{3}{2}} \approx \frac{d^2y}{dx^2} \tag{6.50}$$

29 Let us get curvature from the parabola slope and compare it with ρ

$$y = \frac{cx^2}{2} + dx + e \tag{6.51-a}$$
$$\frac{dy}{dx} = cx + d \tag{6.51-b}$$
$$\frac{d^2y}{dx^2} = c \tag{6.51-c}$$

at $x = 0$, $y = 0$, thus $e = 0$
at $x = 0$, $\frac{dy}{x} = \theta_B = -.00683$ thus $d = -.00683$
at $x = 20$ ft, $\frac{dy}{dx} = \theta_C = .00683$ thus $c(20) - .00683 = .00683$ thus $c = 6.83 \times 10^{-4}$ thus

$$y = 6.83 \times 10^{-4}\left(\frac{x^2}{2} - 10x\right) \tag{6.52-a}$$
$$\frac{dy}{dx} = 6.83 \times 10^{-4}(x - 10) \tag{6.52-b}$$
$$\frac{d^2y}{dx^2} = 6.83 \times 10^{-4} \tag{6.52-c}$$
$$\kappa = 6.83 \times 10^{-4} \tag{6.52-d}$$
$$\rho = \frac{1}{6.83 \times 10{-}4} = 1,464 ft \quad \text{as expected!} \tag{6.52-e}$$

If we were to use the exact curvature formula

$$\frac{\frac{d^2y}{dx^2}}{1 + (\frac{dy}{dx})^2})^{\frac{3}{2}} = \frac{6.83 \times 10^{-4}}{[1 + (.00683)^2]^{\frac{3}{2}}} \tag{6.53-a}$$
$$= 6.829522 \times 10^{-4} \tag{6.53-b}$$
$$\Rightarrow \rho = 1464.23 ft \quad \text{(compared with 1465ft)} \tag{6.53-c}$$

■

6.2.3 Elastic Weight/Conjugate Beams

V and M		θ and y	
$V_{12} = \int_{x_1}^{x_2} w dx$	$V = \int w dx + C_1$	$\theta_{12} = \int_{x_1}^{x_2} \frac{1}{\rho} dx$	$\theta = \int \frac{1}{\rho} dx + C_1$
$M_{12} = \int_{x_1}^{x_2} V dx$	$M = \int V dx + C_2$	$t_{21} = \int_{x_1}^{x_2} \theta dx$	$y = \int \theta dx + C_2$

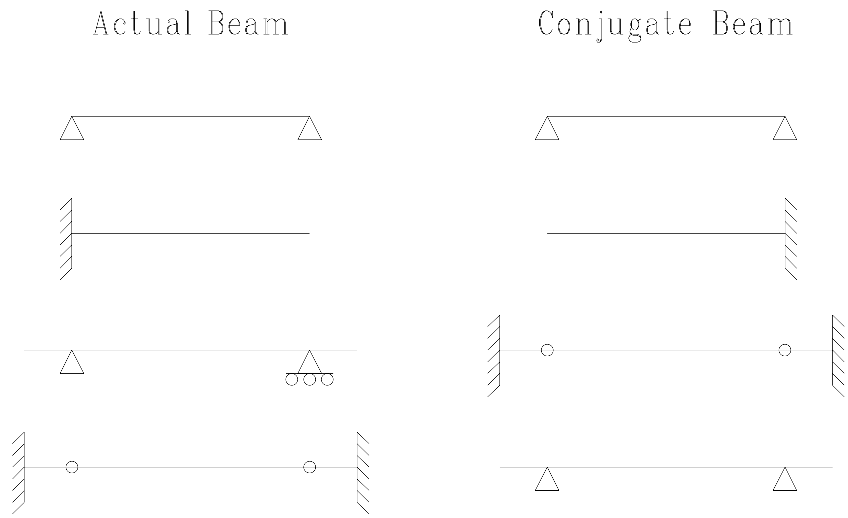


Figure 6.6: Conjugate Beams

29 This leads to the following

$$\boxed{\begin{array}{rcll} \text{Load } q & \equiv & \text{curvature } \dfrac{1}{\rho} = \kappa = \dfrac{M}{EI} & (6.54) \\ \text{Shear } V & \equiv & \text{slope } \theta & (6.55) \\ \text{Moment } M & \equiv & \text{deflection } y & (6.56) \end{array}}$$

30 Since V & M can be conjugated from statics, by analogy θ & y can be thought of as the V & M of a fictitious beam (or **conjugate beam**) loaded by $\frac{M}{EI}$ **elastic weight**.

31 What about Boundary Conditions? Table 6.1, and Fig. 6.6.

Actual Beam			Conjugate Beam		
Hinge	$\theta \neq 0$	$y = 0$	$V \neq 0$	$M = 0$	"Hinge"
Fixed End	$\theta = 0$	$y = 0$	$V = 0$	$M = 0$	Free end
Free End	$\theta \neq 0$	$y \neq 0$	$V \neq 0$	$M \neq 0$	Fixed end
Interior Hinge	$\theta \neq 0$	$y \neq 0$	$V \neq 0$	$M \neq 0$	Interior support
Interior Support	$\theta \neq 0$	$y = 0$	$V \neq 0$	$M = 0$	Interior hinge

Table 6.1: Conjugate Beam Boundary Conditions

32 Whereas the Moment area method has a well defined basis, its direct application can be sometimes confusing.

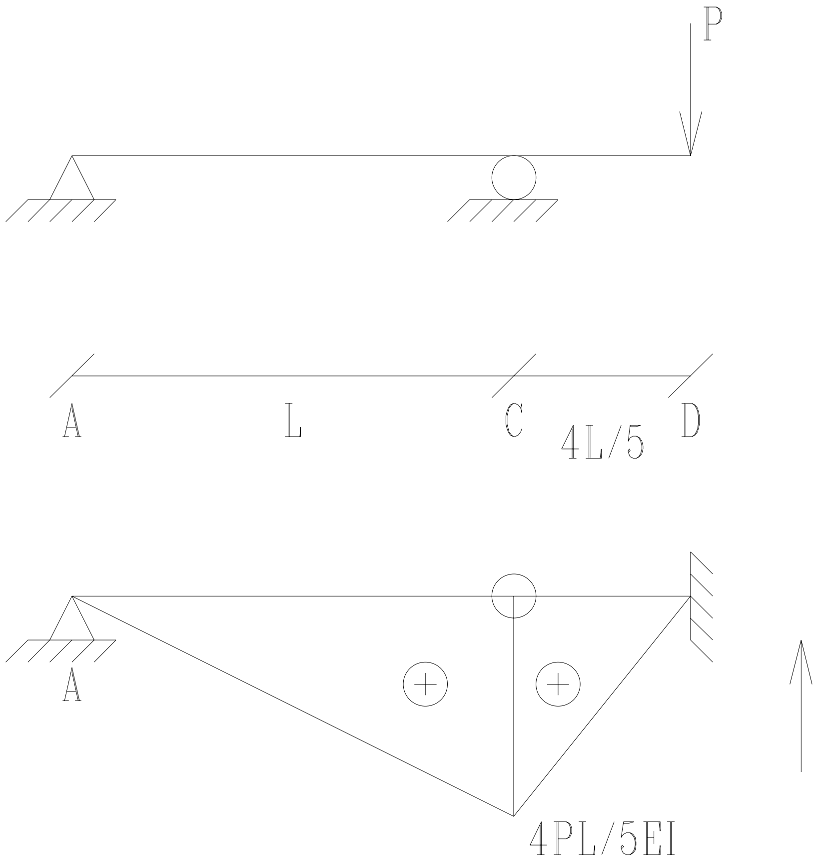
33 Alternatively, the moment area method was derived from the moment area method, and is a far simpler method to remember and use in practice when simple "back of the envelope" calculations are required.

34 Note that we can only have distributed load, and that the load the load is positive for a positive moment, and negative for a negative moment. "Shear" and "Moment" diagrams should be drawn accordingly.

35 **Units** of the "distributed load" w^* are $\frac{FL}{EI}$ (force time length divided by EI). Thus the "Shear" would have units of $w^* \times L$ or $\frac{FL^2}{EI}$ and the "moment" would have units of $(w^* \times L) \times L$ or $\frac{FL^3}{EI}$. Recalling that EI has units of $FL^{-2}L^4 = FL^2$, we observe that indeed the "shear" corresponds to a rotation in radians and the "moment" to a displacement.

■ **Example 6-7: Conjugate Beam**

Analyze the following beam.



Solution:

3 equations of equilibrium and 1 equation of condition = 4 = number of reactions.
Deflection at D = Shear at D of the corresponding conjugate beam (Reaction at D)
Take AC and ΣM with respect to C

$$R_A(L) - \left(\frac{4PL}{5EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) = 0 \quad (6.57\text{-a})$$

$$\Rightarrow R_A = \frac{2PL^2}{15EI} \quad (6.57\text{-b})$$

(Slope in real beam at A) As computed before!
Let us draw the Moment Diagram for the conjugate beam

$$M = \frac{P}{EI}\left[\frac{2}{15}L^2x - \left(\frac{4}{5}x\right)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right)\right] \quad (6.58\text{-a})$$

$$= \frac{P}{EI}\left(\frac{2}{15}L^2x - \frac{2}{15}x^3\right) \quad (6.58\text{-b})$$

$$= \frac{2P}{15EI}\left(L^2x - x^3\right) \quad (6.58\text{-c})$$

Point of Maximum Moment (Δ_{max}) occurs when $\frac{dM}{dx} = 0$

$$\frac{dM}{dx} = \frac{2P}{15EI}(L^2 - 3x^2) = 0 \Rightarrow 3x^2 = L^2 \Rightarrow x = \frac{L}{\sqrt{3}} \quad (6.59)$$

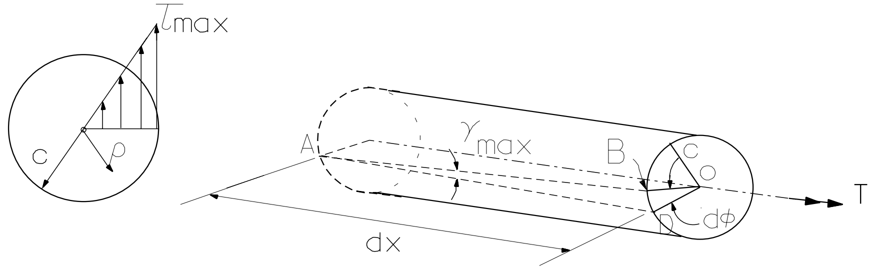


Figure 6.7: Torsion Rotation Relations

as previously determined

$$x = \frac{L}{\sqrt{3}} \tag{6.60-a}$$

$$\Rightarrow M = \frac{2P}{15EI}\left(\frac{L^2 L}{\sqrt{3}} - \frac{L^3}{3\sqrt{3}}\right) \tag{6.60-b}$$

$$= \frac{4PL^3}{45\sqrt{3}EI} \tag{6.60-c}$$

as before. ■

6.3 Axial Deformations

Statics: $\sigma = \frac{P}{A}$

Material: $\sigma = E\varepsilon$

Kinematics: $\Sigma = \frac{\Delta}{L}$

$$\boxed{\Delta = \frac{PL}{AE}} \tag{6.61}$$

6.4 Torsional Deformations

36 Since torsional effects are seldom covered in basic structural analysis, and students may have forgotten the derivation of the basic equations from the Strength of Material course, we shall briefly review the basic equations.

37 Assuming a linear elastic material, and a linear strain (and thus stress) distribution along the radius of a circular cross section subjected to torsional load, Fig. 6.7 we have:

$$T = \underbrace{\underbrace{\int_A \underbrace{\frac{\rho}{c}\tau_{max}}_{stress} \underbrace{dA}_{area}}_{Force} \underbrace{\rho}_{arm}}_{torque} \tag{6.62}$$

$$= \frac{\tau_{max}}{c}\underbrace{\int_A \rho^2 dA}_{J} \tag{6.63}$$

$$\tau_{max} = \frac{Tc}{J} \tag{6.64}$$

Note the analogy of this last equation with $\sigma = \frac{Mc}{I_y}$

38 $\displaystyle\int_A \rho^2 dA$ is the polar moment of inertia J for circular cross sections and is equal to:

$$\begin{aligned} J &= \int_A \rho^2 dA = \int_0^c \rho^2 \left(2\pi\rho d\rho\right) \\ &= \frac{\pi c^4}{2} = \frac{\pi d^4}{32} \end{aligned} \tag{6.65}$$

39 Having developed a relation between torsion and shear stress, we now seek a relation between torsion and torsional rotation. Considering Fig. 6.7-b, we look at the arc length BD

$$\left.\left.\begin{array}{l} \gamma_{max} dx = d\Phi c \Rightarrow \frac{d\Phi}{dx} = \frac{\gamma_{max}}{c} \\ \gamma_{max} = \frac{\tau_{max}}{G} \end{array}\right\} \begin{array}{l} \frac{d\Phi}{dx} = \frac{\tau_{max}}{Gc} \\ \tau_{max} = \frac{TC}{J} \end{array}\right\} \frac{d\Phi}{dx} = \frac{T}{GJ} \tag{6.66}$$

40 Finally, we can rewrite this last equation as $\displaystyle\int T dx = \int G j d\Phi$ and obtain:

$$\boxed{T = \frac{GJ}{L}\Phi} \tag{6.67}$$

Chapter 7

ENERGY METHODS; Part I

7.1 Introduction

1 Energy methods are powerful techniques for both formulation (of the stiffness matrix of an element[1]) and for the analysis (i.e. deflection) of structural problems.

2 We shall explore two techniques:

1. Real Work
2. Virtual Work (Virtual force)

7.2 Real Work

3 We start by revisiting the first law of thermodynamics:

> The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time.

$$\boxed{\frac{d}{dt}(K+U) = W_e + H} \tag{7.1}$$

where K is the kinetic energy, U the internal strain energy, W_e the external work, and H the heat input to the system.

4 For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \tag{7.2}$$

5 Simply stated, the first law stipulates that the external work must be equal to the internal strain energy due to the external load.

7.2.1 External Work

6 The external work is given by, Fig. 7.1

$$\boxed{\begin{aligned} W_e &= \int_0^{\Delta_f} P d\Delta \\ &= \int_0^{\theta_f} M d\theta \end{aligned}} \tag{7.3}$$

[1] More about this in *Matrix Structural Analysis*.

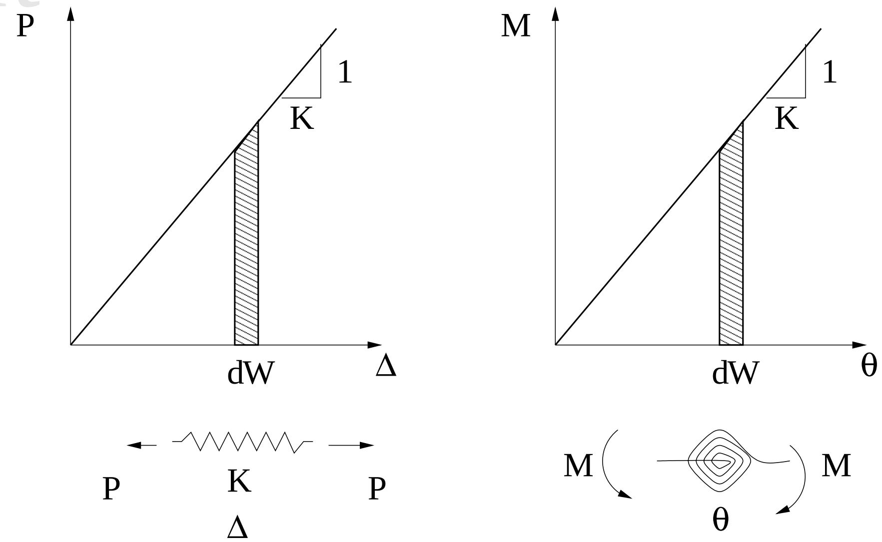


Figure 7.1: Load Deflection Curves

for point loads and concentrated moments respectively.

7 For **linear elastic systems**, we have for point loads

$$\left.\begin{array}{rcl} P & = & K\Delta \\ W_e & = & \int_0^{\Delta_f} P d\Delta \end{array}\right\} W_e = K\int_0^{\Delta_f}\Delta d\Delta = \frac{1}{2}K\Delta_f^2 \tag{7.4}$$

When this last equation is combined with $P_f = K\Delta_f$ we obtain

$$\boxed{W_e = \frac{1}{2}P_f\Delta_f} \tag{7.5}$$

where K is the **stiffness** of the structure.

8 Similarly for an applied moment we have

$$\boxed{W_e = \frac{1}{2}M_f\theta_f} \tag{7.6}$$

7.2.2 Internal Work

9 Considering an infinitesimal element from an arbitrary structure subjected to uniaxial state of stress, the strain energy can be determined with reference to Fig. 7.2. The net force acting on the element while deformation is taking place is $P = \sigma_x dydz$. The element will undergo a displacement $u = \varepsilon_x dx$. Thus, for a linear elastic system, the strain energy density is $dU = \frac{1}{2}\sigma\varepsilon$. And the total strain energy will thus be

$$\boxed{U = \frac{1}{2}\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol}} \tag{7.7}$$

10 When this relation is applied to various structural members it would yield:

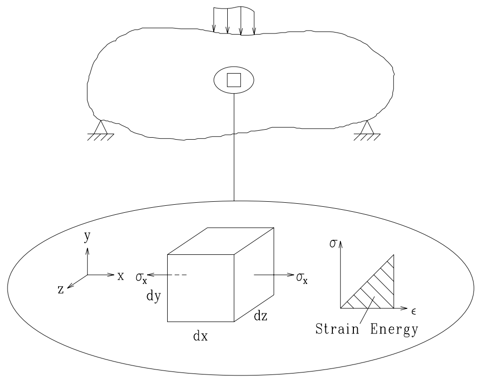


Figure 7.2: Strain Energy Definition

Axial Members:

$$\left.\begin{array}{rcl} U &=& \displaystyle\int_{\text{Vol}} \frac{\varepsilon\sigma}{2} d\text{Vol} \\ \sigma &=& \frac{P}{A} \\ \varepsilon &=& \frac{P}{AE} \\ dV &=& Adx \end{array}\right\} \boxed{U = \int_0^L \frac{P^2}{2AE} dx} \tag{7.8}$$

Torsional Members:

$$\left.\begin{array}{rcl} U &=& \frac{1}{2}\displaystyle\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} d\text{Vol} \\ U &=& \frac{1}{2}\displaystyle\int_{\text{Vol}} \gamma_{xy} \underbrace{G\gamma_{xy}}_{\tau_{xy}} d\text{Vol} \\ \tau_{xy} &=& \frac{Tr}{J} \\ \gamma_{xy} &=& \frac{\tau_{xy}}{G} \\ d\text{Vol} &=& rd\theta drdx \\ J &=& \displaystyle\int_o^r \int_0^{2\pi} r^2 d\theta\, dr \end{array}\right\} \boxed{U = \int_0^L \frac{T^2}{2GJ} dx} \tag{7.9}$$

Flexural Members:

$$\left.\begin{array}{rcl} U &=& \frac{1}{2}\displaystyle\int_{\text{Vol}} \varepsilon \underbrace{E\varepsilon}_{\sigma} \\ \sigma_x &=& \frac{M_z y}{I_z} \\ \varepsilon &=& \frac{M_z y}{EI_z} \\ d\text{Vol} &=& dA\, dx \\ I_z &=& \displaystyle\int_A y^2 dA \end{array}\right\} \boxed{U = \int_0^L \frac{M^2}{2EI_z} dx} \tag{7.10}$$

■ Example 7-1: Deflection of a Cantilever Beam, (Chajes 1983)

Determine the deflection of the cantilever beam, Fig. 7.3 with span L under a point load P applied at its free end. Assume constant EI.

Solution:

$$W_e = \frac{1}{2}P\Delta_f \tag{7.11-a}$$

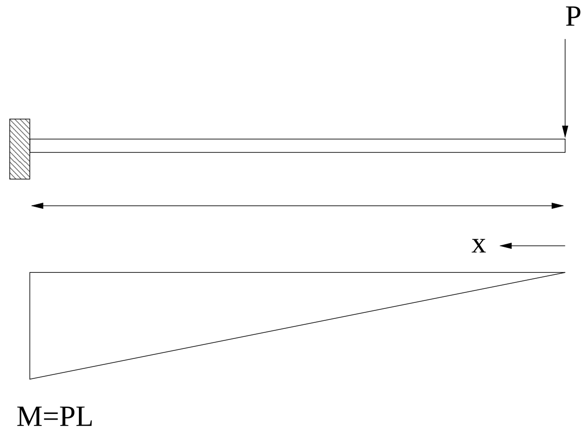


Figure 7.3: Deflection of Cantilever Beam

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{7.11-b}$$
$$M = -Px \tag{7.11-c}$$
$$U = \frac{P^2}{2EI}\int_0^L x^2 dx = \frac{P^2L^3}{6EI} \tag{7.11-d}$$
$$\frac{1}{2}P\Delta_f = \frac{P^2L^3}{6EI} \tag{7.11-e}$$
$$\Delta_f = \frac{PL^3}{3EI} \tag{7.11-f}$$

■

7.3 Virtual Work

11 A severe limitation of the method of real work is that only deflection along the externally applied load can be determined.

12 A more powerful method is the virtual work method.

13 The principle of Virtual Force (VF) relates *force* systems which satisfy the requirements of *equilibrium*, and *deformation* systems which satisfy the requirement of *compatibility*

14 In any application the force system could either be the actual set of **external** loads $d\mathbf{p}$ or some **virtual** force system which happens to satisfy the condition of *equilibrium* $\delta\overline{\mathbf{p}}$. This set of external forces will induce internal actual forces $d\boldsymbol{\sigma}$ or internal virtual forces $\delta\overline{\boldsymbol{\sigma}}$ compatible with the externally applied load.

15 Similarly the deformation could consist of either the actual joint deflections $d\mathbf{u}$ and compatible internal deformations $d\boldsymbol{\varepsilon}$ of the structure, or some **virtual** external and internal deformation $\delta\overline{\mathbf{u}}$ and $\delta\overline{\boldsymbol{\varepsilon}}$ which satisfy the conditions of *compatibility.*

16 It is often simplest to assume that the **virtual load is a unit load**.

17 Thus we may have 4 possible combinations, Table 7.1: where: d corresponds to the actual, and δ (with an overbar) to the hypothetical values.

This table calls for the following observations

1. The second approach is the same one on which the method of virtual or unit load is based. It is simpler to use than the third as a internal force distribution compatible with the assumed virtual

	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$		
2	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$	$\delta\overline{U}^*$	Flexibility
3	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$	$\delta\overline{U}$	Stiffness
4	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$		

Table 7.1: Possible Combinations of Real and Hypothetical Formulations

force can be easily obtained for statically determinate structures. This approach will yield exact solutions for statically determinate structures.

2. The third approach is favored for statically indeterminate problems or in conjunction with approximate solution. It requires a proper "guess" of a displacement shape and is the basis of the stiffness method.

18 Let us consider an arbitrary structure and load it with both real and virtual loads in the following sequence, Fig. 7.4. For the sake of simplicity, let us assume (or consider) that this structure develops only axial stresses.

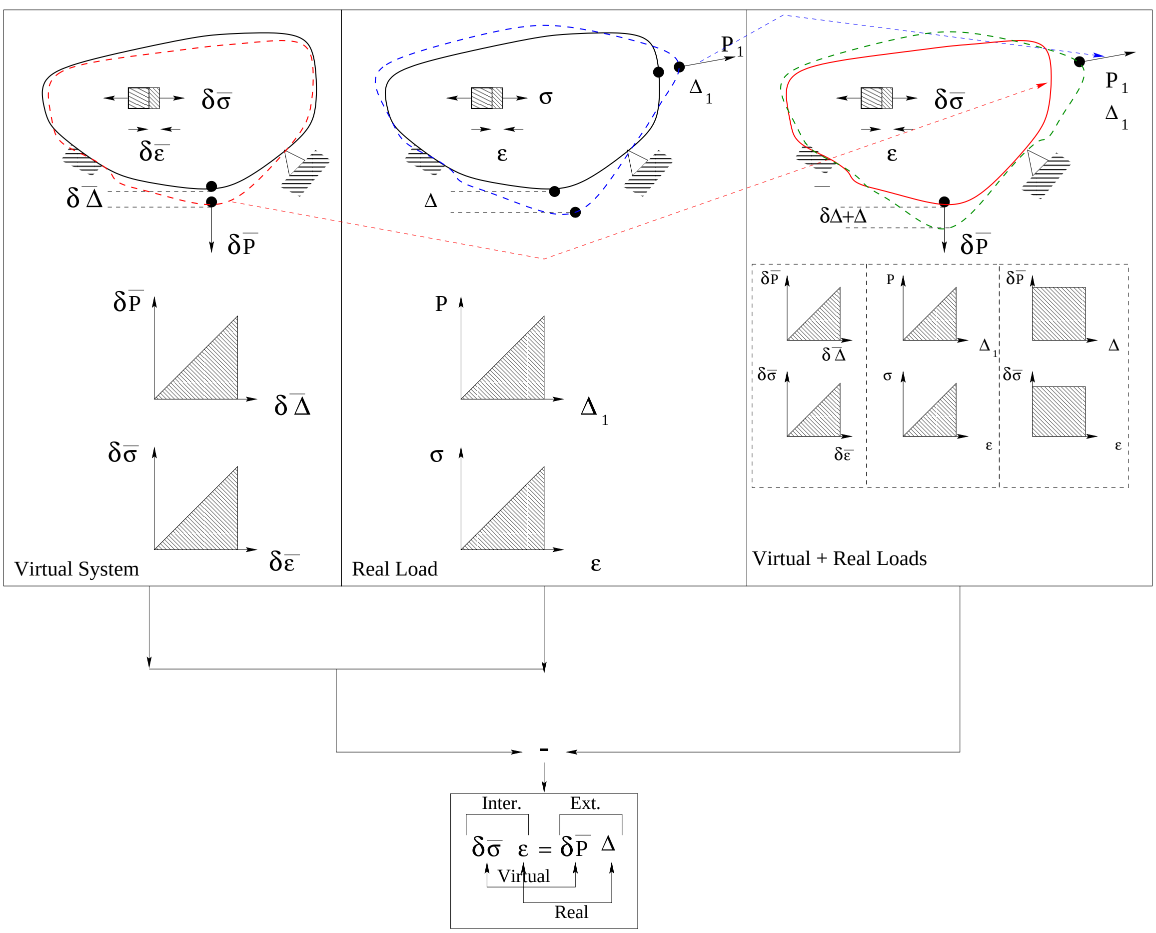


Figure 7.4: Real and Virtual Forces

1. If we apply the virtual load, then

$$\frac{1}{2}\delta\overline{P}\delta\overline{\Delta} = \frac{1}{2}\int_{d\mathrm{Vol}} \delta\overline{\sigma}\delta\overline{\varepsilon}d\mathrm{Vol} \tag{7.12}$$

2. Load with the real (applied) load, since the external work must be equal to the internal strain energy over the entire volume, then:

$$\frac{1}{2}P_1\Delta_1 = \frac{1}{2}\int_{d\mathrm{Vol}}\sigma\epsilon d\mathrm{Vol} \tag{7.13}$$

3. We now immagine that the virtual load was first applied, and we then apply the real (actual) load on top of it, then the total work done is

$$\underbrace{\frac{1}{2}\delta\overline{P}\delta\overline{\Delta} + \frac{1}{2}P_1\Delta_1}_{} + \underbrace{\delta\overline{P}\Delta}_{} = \underbrace{\frac{1}{2}\int_{\mathrm{Vol}}\delta\overline{\sigma}\delta\overline{\varepsilon}d\mathrm{Vol} + \frac{1}{2}\int_{\mathrm{Vol}}\sigma\varepsilon d\mathrm{Vol}}_{} + \underbrace{\int_{\mathrm{Vol}}\delta\overline{\sigma}\varepsilon d\mathrm{Vol}}_{} \tag{7.14}$$

4. Since the strain energy and work done must be the same whether the loads are applied together or separately, we obtain, from substracting the sum of Eqs. 7.13 and 7.12 from 7.14 and generalizing, we obtain

$$\boxed{\underbrace{\int\left(\delta\overline{\sigma}\varepsilon + \delta\overline{\tau}\gamma\right)d\mathrm{Vol}}_{\delta\overline{U}^*} = \underbrace{\delta\overline{P}\Delta}_{\delta\overline{W}^*}} \tag{7.15}$$

19 This last equation is the key to the method of virtual forces. The left hand side is the internal virtual strain energy $\delta\overline{U}^*$[2]. Similarly the right hand side is the external virtual work.

7.3.1 External Virtual Work $\delta\overline{W}^*$

20 The general expression for the external virtual work $\delta\overline{W}^*$ is

$$\boxed{\delta\overline{W}^* = \sum_{i=1}^{n}(\Delta_i)\delta\overline{P}_i + \sum_{i=1}^{n}(\theta_i)\delta\overline{M}_i} \tag{7.16}$$

for distributed load, point load, and concentrated moment respectively.

21 Recall that all overbar quantities are virtual and the other ones are the real.

7.3.2 Internal Virtual Work $\delta\overline{U}^*$

22 The general expression for the internal virtual work is

$$\boxed{\underbrace{\int\left(\delta\overline{\sigma}\varepsilon + \delta\overline{\tau}\gamma\right)d\mathrm{Vol}}_{\delta\overline{U}^*} = \underbrace{\delta\overline{P}\Delta}_{\delta\overline{W}^*}} \tag{7.17}$$

We will generalize it to different types of structural members.

23 We will first write the equations independently of the material stress strain relation, and then we will rewrite those same equations for a linear elastic system.

General :

Axial Members:

$$\left.\begin{array}{rcl} \delta\overline{U}^* & = & \int_0^L \delta\overline{\sigma}\varepsilon d\mathrm{Vol} \\ d\mathrm{Vol} & = & Adx \end{array}\right\} \boxed{\delta\overline{U}^* = A\int_0^L \delta\overline{\sigma}\varepsilon dx} \tag{7.18}$$

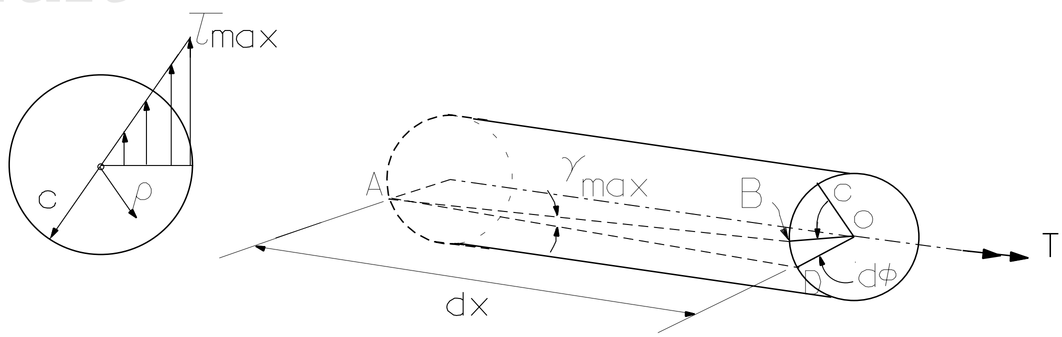


Figure 7.5: Torsion Rotation Relations

Torsional Members: With reference to Fig. 7.5 Note that in torsion the "strain" is γ the rate of change of rotation of the cross section about the longitudinal axis x.

$$\left.\begin{array}{rcl} \delta\overline{U}^* & = & \int_{\text{Vol}} \delta\overline{\tau}_{xy}\gamma_{xy}\, d\text{Vol} \\ \delta\overline{T} & = & \int_A \delta\overline{\tau}_{xy} r dA \\ \gamma_{xy} & = & r\frac{d\theta}{dx} \\ d\text{Vol} & = & dAdx \end{array}\right\} \delta\overline{U}^* = \int_0^L \underbrace{(\int_A \delta\overline{\tau}_{xy} r dA)}_{\delta\overline{T}} \theta dx \Rightarrow \boxed{\delta\overline{U}^* = \int_0^L \delta\overline{T}\frac{d\theta}{dx}dx} \tag{7.19}$$

Shear Members:

$$\left.\begin{array}{rcl} \delta\overline{U}^* & = & \int_{\text{Vol}} \delta\overline{\tau}_{xy}\gamma_{xy}\, d\text{Vol} \\ \delta\overline{V} & = & \int_A \delta\overline{\tau}_{xy} dA \\ d\text{Vol} & = & dAdx \end{array}\right\} \delta\overline{U}^* = \int_0^L \underbrace{(\int_A \delta\overline{\tau}_{xy} dA)}_{\delta\overline{V}} \gamma_{xy} dx \Rightarrow \boxed{\delta\overline{U}^* = \int_0^L \delta\overline{V}\gamma_{xy}dx} \tag{7.20}$$

Flexural Members:

$$\left.\begin{array}{rcl} \delta\overline{U}^* & = & \int \delta\overline{\sigma}_x\varepsilon_x d\text{Vol} \\ \delta\overline{M} & = & \int_A \delta\overline{\sigma}_x y dA \Rightarrow \frac{\delta\overline{M}}{y} = \int_A \delta\overline{\sigma}_x dA \\ \phi & = & \frac{\varepsilon}{y} \\ \phi y & = & \varepsilon \\ d\text{Vol} & = & \int_0^L \int_A dAdx \end{array}\right\} \boxed{\delta\overline{U}^* = \int_0^L \delta\overline{M}\phi dx} \tag{7.21}$$

Linear Elastic Systems Should we have a linear elastic material

$$\boxed{\sigma = E\varepsilon} \tag{7.22}$$

then the previous equations can be rewritten as:

[2] We use the * to distinguish it from the internal virtual strain energy obtained from the **virtual displacement** method $\delta\overline{U}$.

Axial Members:

$$\left.\begin{array}{rcl}\delta\overline{U}^* & = & \int_{\text{Vol}} \varepsilon\delta\overline{\sigma}d\text{Vol} \\ \delta\overline{\sigma} & = & \frac{\delta\overline{P}}{A} \\ \varepsilon & = & \frac{P}{AE} \\ dV & = & Adx\end{array}\right\} \boxed{\delta\overline{U}^* = \int_0^L \delta\overline{P}\underbrace{\frac{P}{AE}}_{\Delta}dx} \tag{7.23}$$

Note that for a **truss** where we have n members, the above expression becomes

$$\boxed{\delta\overline{U}^* = \Sigma_1^n\delta\overline{P}_i\frac{P_iL_i}{A_iE_i}} \tag{7.24}$$

Shear Members:

$$\left.\begin{array}{rcl}\delta\overline{U}^* & = & \int_{\text{Vol}} \delta\overline{\tau}_{xy}\gamma_{xy}d\text{Vol} \\ \tau_{xy} & = & k\frac{V}{A} \\ \gamma_{xy} & = & \frac{\tau_{xy}}{G} \\ d\text{Vol} & = & Adx \\ \lambda & \stackrel{\text{def}}{=} & \int_A k^2dydz\end{array}\right\} \boxed{\delta\overline{U}^* = \lambda\int_0^L \delta\overline{V}\underbrace{\frac{V}{GA}}_{\gamma_{xy}}dx} \tag{7.25}$$

24 Note that the exact expression for the shear stress is

$$\tau = \frac{VQ}{Ib} \tag{7.26}$$

where Q is the moment of the area from the external fibers to y with respect to the neutral axis; For a rectangular section, this yields

$$\tau = \frac{VQ}{Ib} \tag{7.27-a}$$

$$= \frac{V}{Ib}\int_y^{h/2} by'dy' = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right) \tag{7.27-b}$$

$$= \frac{6V}{bh^3}\left(\frac{h^2}{4} - 4\right) \tag{7.27-c}$$

and we observe that the shear stress is zero for $y = h/2$ and maximum at the neutral axis where it is equal to $1.5\frac{V}{bh}$.

25 To determine the form factor λ of a rectangular section

$$\left.\begin{array}{rcl}\tau & = & \frac{VQ}{Ib} \\ & = & k\frac{V}{A} \\ Q & = & \int_y^{h/2} by'dy' = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)\end{array}\right\} \left.\begin{array}{rcl} k & = & \frac{A}{2I}\left(\frac{h^2}{4} - y^2\right) \\ \lambda\underbrace{bh}_{A} & = & \int_A k^2dydz\end{array}\right\} \boxed{\lambda = 1.2} \tag{7.28}$$

26 Thus, the form factor λ may be taken as 1.2 for rectangular beams of ordinary proportions.

27 For I beams, k can be also approximated by 1.2, provided A is the area of the web.

Torsional Members:

$$\left.\begin{array}{rcl}\delta\overline{U}^* & = & \int_{\text{Vol}} \delta\overline{\tau}_{xy}\underbrace{\frac{\tau_{xy}}{G}}_{\gamma_{xy}}d\text{Vol} \\ \tau_{xy} & = & \frac{Tr}{J} \\ \gamma_{xy} & = & \frac{\tau_{xy}}{G} \\ d\text{Vol} & = & rd\theta drdx \\ J & = & \int_o^r\int_0^{2\pi} r^2d\theta\,dr\end{array}\right\} \boxed{\delta\overline{U}^* = \int_0^L \underbrace{\delta\overline{T}}_{\delta\overline{\sigma}}\underbrace{\frac{T}{GJ}}_{\varepsilon}dx} \tag{7.29}$$

28 Note the similarity with the corresponding equation for shear deformation.

29 The torsional stiffness of cylindrical sections is given by $J = \frac{\pi d^4}{32}$.

30 The torsional stiffness of solid rectangular sections is given by

$$\boxed{J = kb^3 d} \tag{7.30}$$

where b is the shorter side of the section, d the longer, and k a factor given by Table 7.2.

d/b	1.0	1.5	2.0	2.5	3.0	4.0	5.0	10	∞
k	0.141	0.196	0.229	0.249	0.263	0.281	0.291	0.312	0.333

Table 7.2: k Factors for Torsion

31 Recall from strength of materials that

$$\boxed{G = \frac{E}{2(1+\nu)}} \tag{7.31}$$

Flexural Members:

$$\left.\begin{array}{rcl} \delta\overline{U}^* & = & \displaystyle\int_{\text{Vol}} \varepsilon \underbrace{E\delta\overline{\varepsilon}}_{\delta\overline{\sigma}} dVol \\ \sigma_x & = & \frac{M_z y}{I_z} \\ \varepsilon & = & \frac{M_z y}{EI_z} \\ d\text{Vol} & = & dA\, dx \\ I_z & = & \displaystyle\int_A y^2 dA \end{array}\right\} \boxed{\delta\overline{U}^* = \int_0^L \delta\overline{M} \underbrace{\frac{M}{EI_z}}_{\Phi} dx} \tag{7.32}$$

7.3.3 Examples

■ **Example 7-2: Beam Deflection (Chajes 1983)**

Determine the deflection at point C in Fig. 7.6 $E = 29,000$ ksi, $I = 100$ in^4.
Solution:

For the virtual force method, we need to have two expressions for the moment, one due to the real load, and the other to the (unit) virtual one, Fig. 7.7.

Element	$x = 0$	M	$\delta\overline{M}$
AB	A	$15x - x^2$	$-0.5x$
BC	C	$-x^2$	$-x$

Applying Eq. 7.32 we obtain

$$\underbrace{\Delta_C \delta\overline{P}}_{\delta\overline{W}^*} = \underbrace{\int_0^L \delta\overline{M} \frac{M}{EI_z} dx}_{\delta\overline{U}^*} \tag{7.33-a}$$

$$(1)\Delta_C = \int_0^{20} (-0.5x)\frac{(15x - x^2)}{EI} dx + \int_0^{10} (-x)\frac{-x^2}{EI} dx \tag{7.33-b}$$

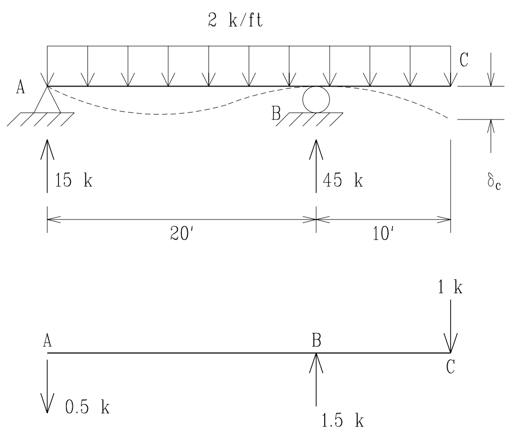


Figure 7.6:

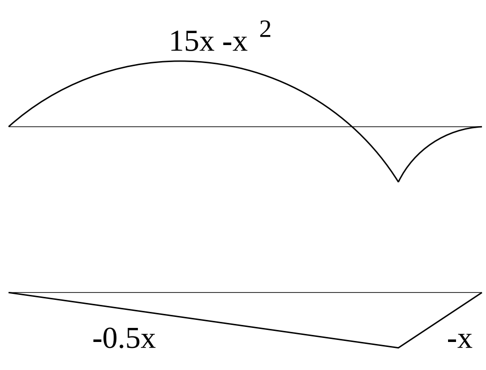


Figure 7.7:

.

$$= \frac{2,500}{EI} \tag{7.33-c}$$

$$\Delta_C = \frac{(2,500)\ \mathrm{k}^2 -\ \mathrm{ft}^3(1,728)\ \mathrm{in}^3/\ \mathrm{ft}^3}{(29,000)\ \mathrm{ksi}(100)\ \mathrm{in}^4} \tag{7.33-d}$$

$$= \boxed{1.49\ \mathrm{in}} \tag{7.33-e}$$

■

■ **Example 7-3: Deflection of a Frame (Chajes 1983)**

Determine both the vertical and horizontal deflection at A for the frame shown in Fig. 7.8. $E = 200 \times 10^6\ \mathrm{kN/\ m}^2$, $I = 200 \times 10^6\ \mathrm{mm}^4$.

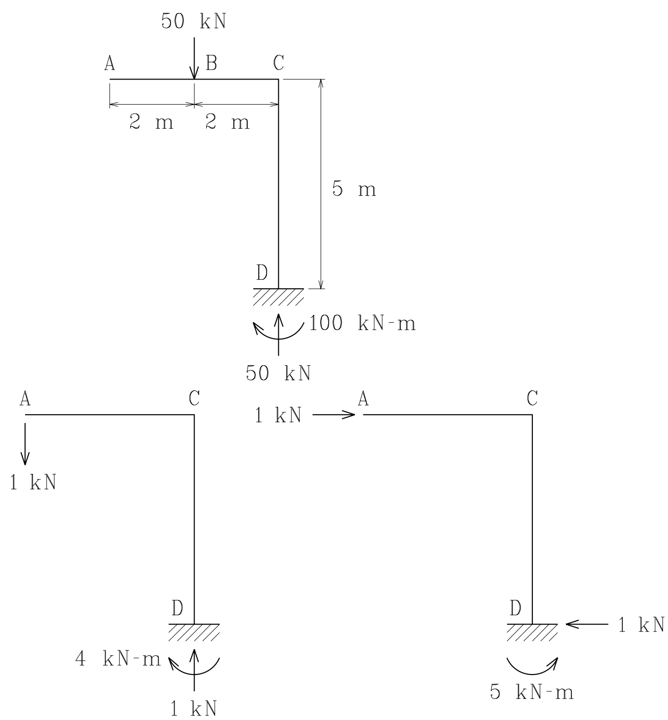


Figure 7.8:

.

Solution:

To analyse this frame we must determine analytical expressions for the moments along each member for the real load and the two virtual ones. One virtual load is a unit horizontal load at A, and the other a unit vertical one at A also, Fig. 7.9.

Element	$x = 0$	M	$\delta\overline{M}_v$	$\delta\overline{M}_h$
AB	A	0	x	0
BC	B	$50x$	$2 + x$	0
CD	C	100	4	$-x$

Note that moments are considered positive when they produce compression on the inside of the frame.

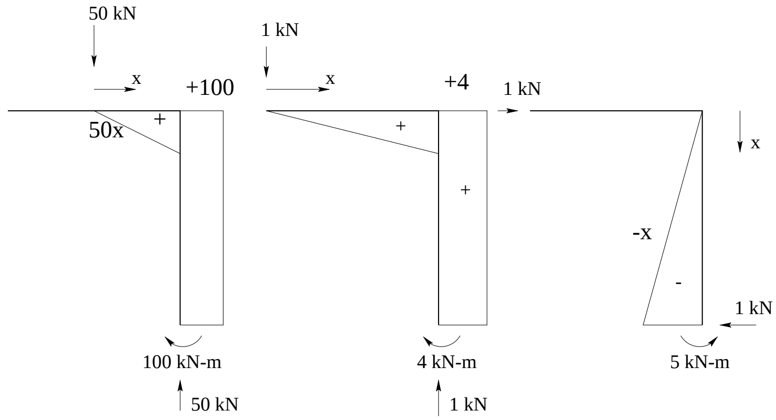


Figure 7.9:

Substitution yields:

$$\underbrace{\Delta_v \delta \overline{P}}_{\delta \overline{W}^*} = \underbrace{\int_0^L \delta \overline{M} \frac{M}{EI_z} dx}_{\delta \overline{U}^*} \tag{7.34-a}$$

$$(1)\Delta_v = \int_0^2 (x)\frac{(0)}{EI}dx + \int_0^2 (2+x)\frac{50x}{EI}dx + \int_0^5 (4)\frac{100}{EI}dx \tag{7.34-b}$$

$$= \frac{2,333 \text{ kN m}^3}{EI} \tag{7.34-c}$$

$$= \frac{(2,333) \text{ kN m}^3 (10^3)^4 \text{ mm}^4/\text{ m}^4}{(200 \times 10^6) \text{ kN/ m}^2 (200 \times 10^6) \text{ mm}^4} \tag{7.34-d}$$

$$= 0.058 \text{ m} = \boxed{5.8 \text{ cm}} \tag{7.34-e}$$

Similarly for the horizontal displacement:

$$\underbrace{\Delta_h \delta \overline{P}}_{\delta \overline{W}^*} = \underbrace{\int_0^L \delta \overline{M} \frac{M}{EI_z} dx}_{\delta \overline{U}^*} \tag{7.35-a}$$

$$(1)\Delta_h = \int_0^2 (0)\frac{(0)}{EI}dx + \int_0^2 (0)\frac{50x}{EI}dx + \int_0^5 (-x)\frac{100}{EI}dx \tag{7.35-b}$$

$$= \frac{-1,250 \text{ kN m}^3}{EI} \tag{7.35-c}$$

$$= \frac{(-1,250) \text{ kN m}^3 (10^3)^4 \text{ mm}^4/\text{ m}^4}{(200 \times 10^6) \text{ kN/ m}^2 (200 \times 10^6) \text{ mm}^4} \tag{7.35-d}$$

$$= -0.031 \text{ m} = \boxed{-3.1 \text{ cm}} \tag{7.35-e}$$

■

■ Example 7-4: Rotation of a Frame (Chajes 1983)

Determine the rotation of joint C for frame shown in Fig. 7.10. $E = 29,000$ ksi, $I = 240$ in^4.

Solution:

In this problem the virtual force is a unit moment applied at joint C, $\delta \overline{M}_e$. It will cause an internal moment $\delta \overline{M}_i$

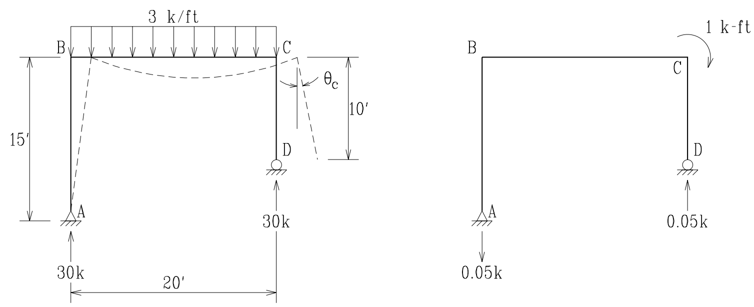


Figure 7.10:

Element	$x=0$	M	$\delta\overline{M}$
AB	A	0	0
BC	B	$30x - 1.5x^2$	$-0.05x$
CD	D	0	0

Note that moments are considered positive when they produce compression on the outside of the frame.

Substitution yields:

$$\underbrace{\theta_C \delta\overline{M}_e}_{\delta\overline{W}^*} = \underbrace{\int_0^L \delta\overline{M}\frac{M}{EI_z}dx}_{\delta\overline{U}^*} \tag{7.36-a}$$

$$(1)\theta_C = \int_0^{20} (-0.05x)\frac{(30x-1.5x^2)}{EI}dx \quad \text{k}^2\ \text{ft}^3 \tag{7.36-b}$$

$$= \frac{(1,000)(144)}{(29,000)(240)} \tag{7.36-c}$$

$$= \boxed{-0.021 \text{ radians}} \tag{7.36-d}$$

■

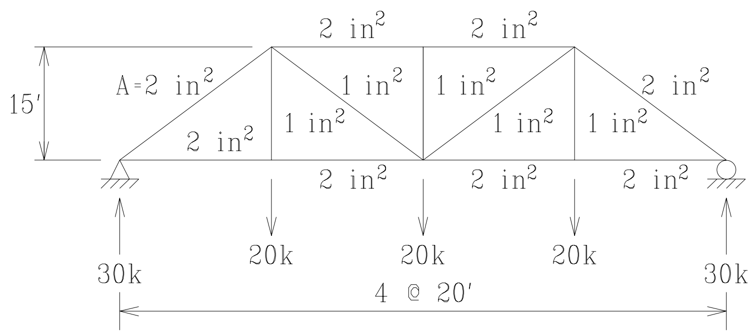
■ Example 7-5: Truss Deflection (Chajes 1983)

Determine the vertical deflection of joint 7 in the truss shown in Fig. 7.11. $E = 30,000$ ksi.

Solution:

Two analyses are required. One with the real load, and the other using a unit vertical load at joint 7. Results for those analysis are summarized below. Note that advantage was taken of the symmetric load and structure.

Member	A in^2	L ft	P k	$\delta\overline{P}$ k	$\frac{\delta\overline{P}PL}{A}$	n	$\frac{n\delta\overline{P}PL}{A}$
1 & 4	2	25	-50	-0.083	518.75	2	1,037.5
10 & 13	2	20	40	0.67	268.0	2	536.0
11 & 12	2	20	40	0.67	268.0	2	536.0
5 & 9	1	15	20	0	0	2	0
6 & 8	1	25	16.7	0.83	346.5	2	693.0
2& 3	2	20	-53.3	-1.33	708.9	2	1,417.8
7	1	15	0	0	0	1	0
Total							4,220.3



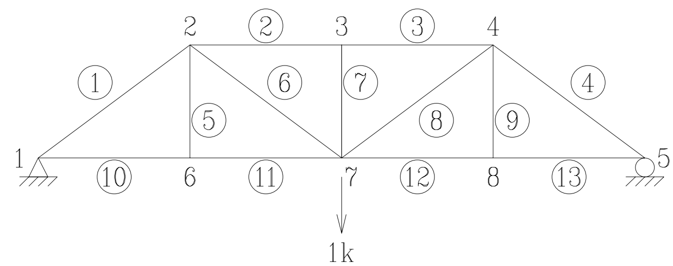


Figure 7.11:

Thus from Eq. 7.24 we have:

$$\underbrace{\Delta\delta\overline{P}}_{\delta\overline{W}^*} = \underbrace{\int_0^L \delta\overline{P}\frac{P}{AE}dx}_{\delta\overline{U}^*} \tag{7.37-a}$$

$$= \Sigma\delta\overline{P}\frac{PL}{AE} \tag{7.37-b}$$

$$1\Delta = \frac{(4,220.3)(12)}{30,000} \tag{7.37-c}$$

$$= \boxed{1.69 \text{ in}} \tag{7.37-d}$$

■

■ Example 7-6: Torsional and Flexural Deformation, (Chajes 1983)

Determine the vertical deflection at A in the structure shown in Fig. 7.12. $E = 30,000$ ksi, $I = 144$ in^4, $G = 12,000$ ksi, $J = 288$ in^4

Solution:

1. In this problem we have both flexural and torsional deformation. Hence we should determine the internal moment and torsion distribution for both the real and the unit virtual load.
2. Then we will use the following relation

$$\overbrace{\delta\overline{P}\Delta_A}^{\delta\overline{W}^*} = \overbrace{\underbrace{\int \delta\overline{M}\frac{M}{EI}dx}_{\text{flexure}} + \underbrace{\int \delta\overline{T}\frac{T}{GJ}dx}_{\text{Torsion}}}^{\delta\overline{U}^*}$$

3. The moment and torsion expressions are given by

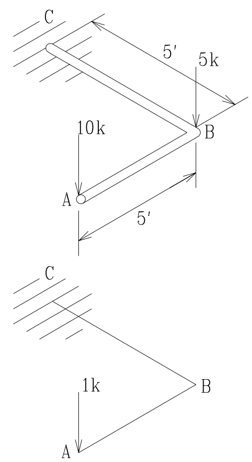


Figure 7.12:

Element	$x = 0$	M	$\delta\overline{M}$	T	$\delta\overline{T}$
AB	A	$10x$	x	0	0
BC	B	$15x$	x	50	5

4. Substituting,

$$\begin{aligned}
\delta\overline{P}\Delta_A &= \int \delta\overline{M}\frac{M}{EI}dx + \int \delta\overline{T}\frac{T}{GJ}dx \\
1\Delta_A &= \int_0^5 x\frac{10x}{EI}dx + \int_0^5 x\frac{15x}{EI}dx + \int_0^5 (5)\frac{50}{GJ}dx \\
&= \frac{(1{,}042)(1{,}728)}{(30{,}000)(144)} + \frac{(1{,}250)(1{,}728)}{(12{,}000)(288)} \\
&= 0.417 + 0.625 \\
&= \boxed{1.04 \text{ in}}
\end{aligned}$$

■

■ Example 7-7: Flexural and Shear Deformations in a Beam (White et al. 1976)

Determine the delection of a cantilevered beam, of length L subjected to an end force P due to both flexural and shear deformations. Assume $G = 0.4E$, and a square solid beam cross section.

Solution:

1. The virtual work equation is

$$\underbrace{\delta\overline{P}}_{1}\cdot\Delta = \int_0^L \delta\overline{M}d\phi + \int_0^L \delta\overline{V}\gamma_{xy}dx \tag{7.38-a}$$

$$= \int_0^L \delta\overline{M}\frac{M}{EI}dx + \lambda\int_0^L \delta\overline{V}\lambda\frac{V}{GA}dx \tag{7.38-b}$$

2. The first integral yields for $M = Px$, and $\delta\overline{M} = (1)(x)$

$$\int_0^L \delta\overline{M}\frac{M}{EI}dx = \frac{P}{EI}\int_0^L x^2dx \tag{7.39-a}$$

$$= PL^3/3EI \tag{7.39-b}$$

3. The second integral represents the contribution of the shearing action to the total internal virtual work and hence to the total displacement.
4. Both the real shear V and virtual shear $\delta\overline{V}$ are constant along the length of the member, hence

$$\int_0^L \delta\overline{V}\lambda\frac{V}{GA}dx = \frac{\lambda}{GA}\int_0^L 1(P)dx = \frac{\lambda PL}{GA} \tag{7.40}$$

5. Since λ= 1.2 for a square beam; hence

$$1\cdot\Delta = \frac{PL^3}{3EI} + \frac{1.2PL}{GA} \tag{7.41-a}$$

$$= \frac{PL}{3E}\left[\frac{L^2}{I} + \frac{3.6}{0.4A}\right] = \boxed{\frac{PL}{3E}\left[\frac{L^2}{I}+\frac{9}{A}\right]} \tag{7.41-b}$$

6. For a square beam of dimension h

$$I = \frac{h^4}{12} \quad \text{and} \quad A = h^2 \tag{7.42}$$

then

$$\Delta = \frac{PL}{3E}\left[\frac{12L^2}{h^4}+\frac{9}{h^2}\right] = \frac{3PL}{Eh^2}\left[\frac{1.33L^2}{h^2}+1\right] \tag{7.43}$$

7. Choosing $L = 20$ ft and $h = 1.5$ ft ($L/h = 13.3$)

$$\Delta = \frac{3PL}{Eh^2}[1.33\left(\frac{20}{1.5}\right)^2 + 1] = \boxed{\frac{3PL}{Eh^2}(237+1)} \tag{7.44}$$

Thus the flexural deformation is 237 times the shear displacement. This comparison reveals why we normally neglect shearing deformation in beams.

As the beam gets shorter or deeper, or as L/h decreases, the flexural deformation decreases relative to the shear displacement. At $L/h = 5$, the flexural deformation has reduced to $1.33(5)^2 = 33$ times the shear displacement. ■

■ Example 7-8: Thermal Effects in a Beam (White et al. 1976)

The cantilever beam of example 7-7 is subjected to a thermal environment that produces a temperature change of 70°C at the top surface and 230°C at the bottom surface, Fig. 7.13.

If the beam is a steel, wide flange section, 2 m long and 200 mm deep, what is the angle of rotation, θ_1, at the end of beam as caused by the temperature effect? The original uniform temperature of the beams was 40°C.

Solution:

1. The external virtual force conforming to the desired real displacement θ_1 is a moment $\delta\overline{M} = 1$ at the tip of the cantilever, producing an external work term of moment times rotation. The internal virtual force system for this cantilever beam is a uniform moment $\delta\overline{M}_{int} = 1$.
2. The real internal deformation results from: (a) the average beam temperature of 150°C, which is 110°C above that of the original temperature, and (b) the temperature gradient of 160°C across the depth of the beam.
3. The first part of the thermal effect produces only a lengthening of the beam and does not enter into the work equation since the virtual loading produces no axial force corresponding to an axial change in length of the beam.
4. The second effect (thermal gradient) produces rotation $d\phi$, and an internal virtual work term of $\int_0^L \delta\overline{M}d\phi$.

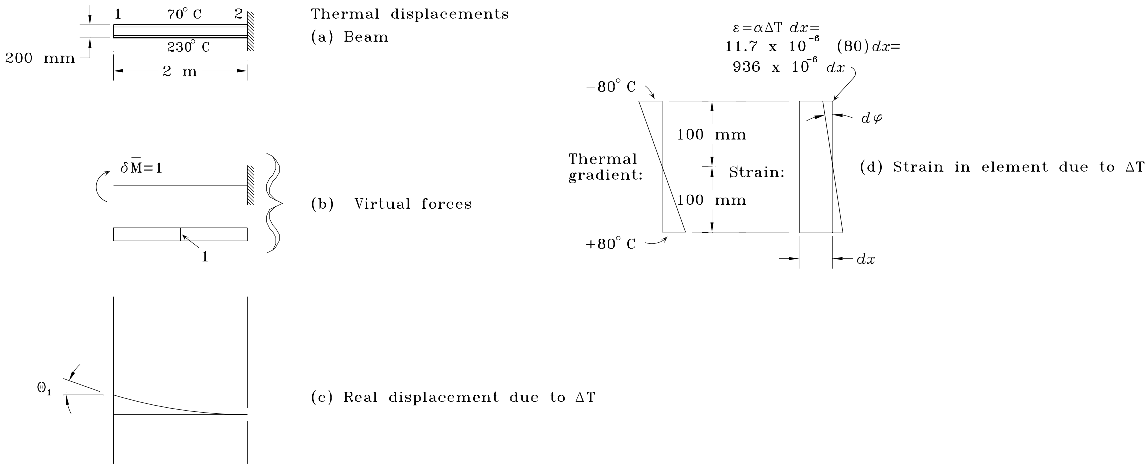


Figure 7.13:

5. We determine the value of $d\phi$ by considering the extreme fiber thermal strain as shown above. The angular rotation in the length dx is the extreme fiber thermal strain divided by half the beam depth, or

$$d\phi = \frac{\varepsilon}{h/2} = \frac{\alpha \Delta T dx}{h/2} = \frac{(11.7 \times 10^{-6})(80)dx}{100} = \frac{936 \times 10^{-6}}{100} = (9.36 \times 10^{-6})dx \tag{7.45}$$

6. Using the virtual work equation

$$1 \cdot \theta_1 = \int_0^L \delta\overline{M} d\phi \tag{7.46-a}$$

$$= \int_0^{2,000} 1(9.36 \times 10^{-6})dx \tag{7.46-b}$$

$$= +0.01872 \tag{7.46-c}$$

$$= \boxed{0.01872 \circlearrowright \text{ radians}} \tag{7.46-d}$$

7. This example raises the following points:

   1. The value of θ_1 would be the same for any shape of 200 mm deep steel beam that has its neutral axis of bending at middepth.

   2. Curvature is produced only by thermal gradient and is independent of absolute temperature values.

   3. The calculation of rotations by the method of virtual forces is simple and straightforward; the applied virtual force is a moment acting at the point where rotation is to be calculated.

   4. Internal angular deformation $d\phi$ has been calculated for an effect other than load-induced stresses. The extension of the method of virtual forces to treat inelastic displacements is obvious – all we need to know is a method for determining the inelastic internal deformations.

■

■ **Example 7-9: Deflection of a Truss (White et al. 1976)**

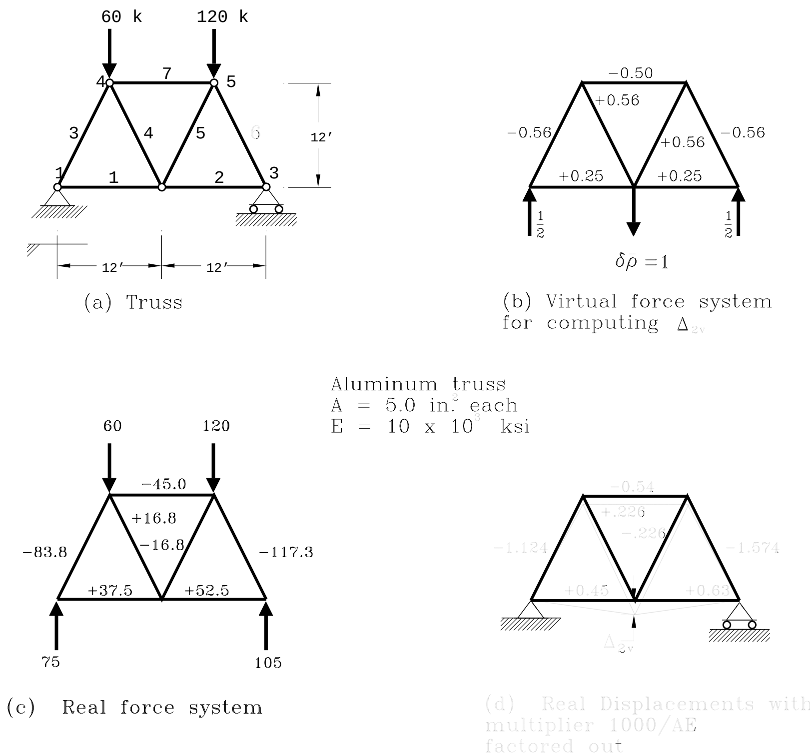


Figure 7.14:

Determine the deflection at node 2 for the truss shown in Fig. 7.14.
Solution:

Member	$\delta\overline{P}$ kips	P, kips	L, ft	A, in^2	E, ksi	$\delta\overline{P}\frac{PL}{AE}$
1	+0.25	+37.5	12	5.0	10×10^3	$+22.5\times10^{-4}$
2	+0.25	+52.5	12	5.0	10×10^3	$+31.5\times10^{-4}$
3	-0.56	-83.8	13.42	5.0	10×10^3	$+125.9\times10^{-4}$
4	+0.56	+16.8	13.42	5.0	10×10^3	$+25.3\times10^{-4}$
5	+0.56	-16.8	13.42	5.0	10×10^3	-25.3×10^{-4}
6	-0.56	-117.3	13.42	5.0	10×10^3	$+176.6\times10^{-4}$
7	-0.50	-45.0	12	5.0	10×10^3	$+54.0\times10^{-4}$
						$+410.5\times10^{-4}$

The deflection is thus given by

$$\delta\overline{P}\Delta = \sum_1^7 \delta\overline{P}\frac{PL}{AE} \tag{7.47-a}$$

$$\Delta = (410.5\times10^{-4})(12\text{ in/ ft}) = \boxed{0.493\text{ in}} \tag{7.47-b}$$

■

■ Example 7-10: Thermal Defelction of a Truss; I (White et al. 1976)

The truss in example 7-9 the preceding example is built such that the lower chords are shielded from the rays of the sun. On a hot summer day the lower chords are 30° F cooler than the rest of the truss members. What is the magnitude of the vertical displacement at joint 2 as a result of this temperature difference?
Solution:

1. The virtual force system remains identical to that in the previous example because the desired displacement component is the same.
2. The real internal displacements are made up of the shortening of those members of the truss that are shielded from the sun.
3. Both bottom chord members 1 and 2 thus shorten by

$$\Delta l = \alpha(\Delta T)(L) = (0.0000128)\text{ in/ in/}^{oF}(30)^o(12)\text{ ft}(12)\text{ in/ft} = 0.0553\text{ in} \tag{7.48}$$

4. Then,

$$1\cdot\Delta = \sum \delta\overline{P}(\Delta L) = .25(-0.0553) + .25(-0.0553) = -0.0276 \tag{7.49-a}$$

$$= \boxed{-0.0276\text{ in}\uparrow} \tag{7.49-b}$$

5. The negative sign on the displacement indicates that it is in opposite sense to the assumed direction of the displacement; the assumed direction is always identical to the direction of the applied virtual force.
6. Note that the same result would be obtained if we had considered the internal displacements to be made up of the lengthening of all truss members above the bottom chord. ■

■ Example 7-11: Thermal Deflections in a Truss; II (White et al. 1976)

A six-panel highway bridge truss, Fig. 7.15 is constructed with sidewalks outside the trusses so that

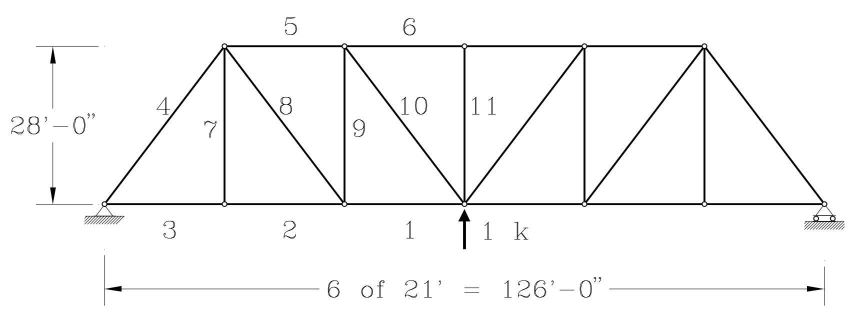


Figure 7.15:

the bottom chords are shaded. What will be the vertical deflection component of the bottom chord at the center of the bridge when the temperature of the bottom chord is 40°F (ΔT) below that of the top chord, endposts, and webs? (coefficient of steel thermal expansion is α=0.0000065 per degree F.)
Solution:

1. The deflection is given by

$$\underbrace{\Delta\delta\overline{P}}_{\delta\overline{W}^*} = \underbrace{\int_0^L \delta\overline{P}\frac{P}{AE}dx}_{\delta\overline{U}^*} = \Sigma\delta\overline{P}\frac{PL}{AE} = \Sigma\delta\overline{P}\Delta L = \Sigma\delta\overline{P}\alpha\Delta TL \tag{7.50}$$

where ΔL is the temperature change in the length of each member, and $\delta\overline{P}$ are the member virtual internal forces.

2. Taking advantage of symmetry:

Member	L (ft)	$(0.0000065)(40)L$ $\alpha \Delta T L$	$\delta\overline{P}$ (k)	$\delta\overline{P}\Delta L$
4	35	+0.00910	+ 0.625	+0.00568
5	21	+0.00546	+ 0.75	+0.00409
6	21	+0.00546	+1.13	+0.00616
7	0	0	0	0
8	35	+0.00910	-0.625	-0.00568
9	28	+0.00728	+0.5	+0.00364
10	35	+0.00910	-0.625	-0.00568
11	0	0	0	0
				+0.00821

3. Hence, the total deflection is

$$\Delta = (2)(0.00821)(12) \text{ in/ft} = \boxed{+0.20 \text{ in} \uparrow} \tag{7.51}$$

4. A more efficient solution would have consisted in considering members 1,2, and 3 only and apply a $\Delta T = -40$, we would obtain the same displacement.
5. Note that the forces in members 1, 2, and 3 (-0.75, -0.375, and -0.375 respectively) were not included in the table because the corresponding $\Delta T = 0$.
6. A simpler solution would have $\Delta T = -40$ in members 1, 2, and 3 thus,

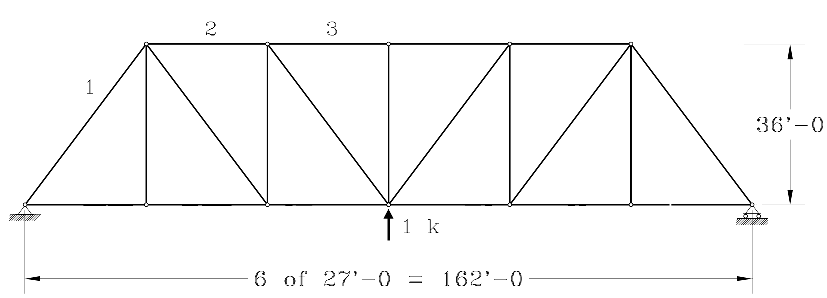
Member	L (ft)	$(0.0000065)(40)L$ $\alpha \Delta T L$	$\delta\overline{P}$ (k)	$\delta\overline{P}\Delta L$
1	21	-0.00546	-0.75	+0.004095
2	21	-0.00546	-0.375	+0.0020475
3	21	-0.00546	-0.375	+0.0020475
				+0.00819

$$\Delta = (2)(0.00819)(12) \text{ in/ft} = \boxed{+0.20 \text{ in} \uparrow} \tag{7.52}$$

■

■ Example 7-12: Truss with initial camber

It is desired to provide 3 in. of camber at the center of the truss shown below



by fabricating the endposts and top chord members additionally long. How much should the length of each endpost and each panel of the top chord be increased?

Solution:

1. Assume that each endpost and each section of top chord is increased 0.1 in.

Member	$\delta \overline{P}_{int}$	ΔL	$\delta \overline{P}_{int} \Delta L$
1	+0.625	+0.1	+0.0625
2	+0.750	+0.1	+0.0750
3	+1.125	+0.1	+0.1125
			+0.2500

Thus,

$$(2)(0.250) = 0.50 \text{ in} \tag{7.53}$$

2. Since the structure is linear and elastic, the required increase of length for each section will be

$$\left(\frac{3.0}{0.50}\right)(0.1) = 0.60 \text{ in} \tag{7.54}$$

3. If we use the practical value of 0.625 in., the theoretical camber will be

$$\frac{(6.25)(0.50)}{0.1} = \boxed{3.125 \text{ in}} \tag{7.55}$$

■

■ **Example 7-13: Prestressed Concrete Beam with Continously Variable I (White et al. 1976)**

A prestressed concrete beam is made of variable depth for proper location of the straight pretensioning tendon, Fig. 7.16. Determine the midspan displacement (point c) produced by dead weight of the girder. The concrete weights 23.6 kN/m^3 and has $E = 25,000$ MPa (N/mm^2). The beam is 0.25 m wide.
Solution:

1. We seek an expression for the real moment M, this is accomplished by first determining the reactions, and then considering the free body diagram.
2. We have the intermediary resultant forces

$$R_1 = (0.25)x(0.26)m^3(23.6)kN/m^3 = 3.54x \tag{7.56-a}$$
$$R_2 = frac12(0.25)x(0.04x)m^3(23.6)kN/m^3 = 0.118x^2 \tag{7.56-b}$$

Hence,

$$M(x) = 47.2x - 3.54x\left(\frac{x}{2}\right) - 0.118x^2\left(\frac{x}{3}\right) \tag{7.57-a}$$
$$= 47.2x - 1.76x^2 - 0.0393x^3 \tag{7.57-b}$$

3. The moment of inertia of the rectangular beam varies continously and is given, for the left half of the beam, by

$$I(x) = \frac{1}{12}bh^3 = \underbrace{\frac{1}{12}(0.25)}_{\frac{1}{48}}(0.6 + 0.04x)^3 \tag{7.58}$$

4. Thus, the real angle changes produced by dead load bending are

$$d\phi = \frac{M}{EI}dx = \frac{47.2x - 1.76x^2 - 0.0393x^3}{E\left(\frac{1}{48}\right)(0.6 + 0.04x)^3}dx \tag{7.59}$$

5. The virtual force system corresponding to the desired displacement is shown above with $\delta\overline{M} = (1/2)x$ for the left half of the span. Since the beam is symmetrical, the virtual work equations can be evaluated for only one half of the beam and the final answer is then obtained by multiplying the half-beam result by two.

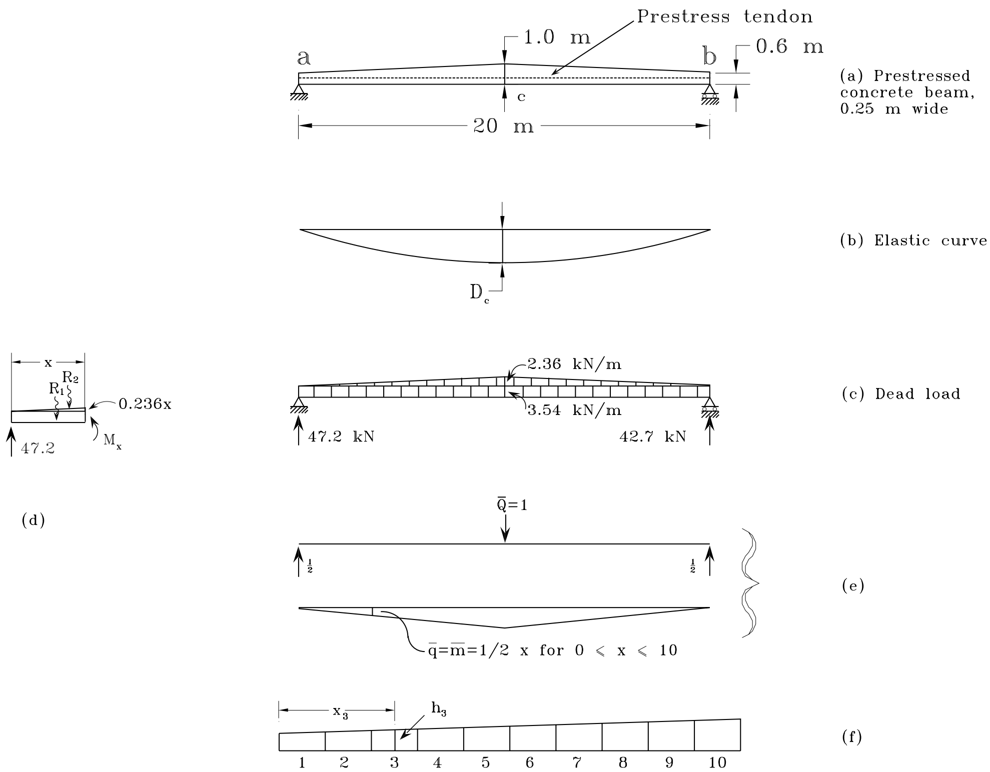


Figure 7.16: *(correct 42.7 to 47.2)

6. The direct evaluation of the integral $\int \delta\overline{M} d\phi$ is difficult because of the complexity of the expression for $d\phi$. Hence we shall use a numerical procedure, replacing the $\int \delta\overline{M}(M/EI)dx$ with the $\sum \delta\overline{M}(M/EI)\Delta x$, where each quantity in the summation is evaluated at the center of the interval Δx and held constant over the interval length. As Δx becomes very short, the solution approaches the exact answer.
7. An interval length of 1 meter, giving 10 elements in the half length of the beam, is chosen to establish an accurate result.
8. The internal virtual work quantity is then

$$\int_0^{L/2} \delta\overline{M}\frac{M}{EI}dx \simeq \sum \frac{\delta\overline{M}M}{EI}(\Delta x) \quad (7.60\text{-a})$$

$$= \sum \frac{\delta\overline{M}M}{E\left(\frac{0.25}{12}\right)h^3}(\Delta x) \quad (7.60\text{-b})$$

$$\simeq \frac{48}{E}\sum \frac{\delta\overline{M}M}{h^3}(\Delta x) \quad (7.60\text{-c})$$

9. The summation for the 10 elements in the left half of the beam gives

Segment	x	h	h^3	M	$\delta\overline{M}$	$\frac{\delta\overline{M}M}{h^3}$
1	0.5	0.62	0.238	23.2	0.25	24
2	1.5	0.66	0.288	66.7	0.75	174
3	2.5	0.70	0.343	106.4	1.25	388
4	3.5	0.74	0.405	150	1.75	648
5	4.5	0.78	0.475	173	2.25	820
6	5.5	0.82	0.551	200	2.75	998
7	6.5	0.86	0.636	222	3.25	1,134
8	7.5	0.90	0.729	238	3.75	1,224
9	8.5	0.94	0.831	250	4.25	1,279
10	9.5	0.98	0.941	256	4.75	1,292
						7,981

10. The SI units should be checked for consistency. Letting the virtual force carry the units of kN, the virtual moment $\delta\overline{M}$ has the units of m·kN, and the units of the equation

$$\Delta_c = \frac{1}{1\text{ kN}}\sum \frac{\delta\overline{M}M}{EI}\Delta x \quad (7.61)$$

are

$$\frac{1}{\text{kN}}\frac{(\text{ m}\cdot\text{ kN})(\text{ m}\cdot\text{ kN})}{(\text{ MN}/\text{ m}^2)\text{ m}^4}\text{ m} = \frac{\text{m}}{1,000} = \text{ mm} \quad (7.62)$$

11. Then

$$\int_0^L \frac{\delta\overline{M}M}{EI}dx \simeq 2\left[\frac{48}{25,000}(7,981)(1)\right] = 30.6mm \quad (7.63)$$

and the deflection at midspan is

$$\Delta_c = \boxed{30.6\text{ mm}} \quad (7.64)$$

12. Acceptably accurate results may be obtained with considerably fewer elements (longer intervals Δx). Using four elements with centers at 2, 5, 8, and 10, the $\sum(\delta\overline{M}M/h^3)\Delta x$ is

$$\sum = 3(174) + 3(820) + 3(1,224) + 1(1,292) = 7,946 \quad (7.65)$$

which is only 0.4% lower than the 10 element solution. If we go to two elements, 3 and 8, we obtain a summation of $5(388) + 5(1,224) = 8,060$, which is 1% high. A one element solution, with $x = 5$ m and $h = 0.8$ m, gives a summation of 9,136, which is 14.4% high and much less accurate than the 2 element solution.
13. Finally, it should be noted that the calculations involved in this example are essentially identical to those necessary in the moment area method. ■

7.4 *Maxwell Betti's Reciprocal Theorem

24 If we consider a beam with two points, A, and B, we seek to determine

1. The deflection at A due to a unit load at B, or f_{AB}
2. The deflection at B due to a unit load at A, or f_{BA}

25 Applying the theorem of vitual work

$$f_{AB} = \int_0^L \delta\overline{M}_A \frac{M_B}{EI} dx \quad (7.66\text{-a})$$

$$f_{BA} = \int_0^L \delta\overline{M}_B \frac{M_A}{EI} dx \quad (7.66\text{-b})$$

But since the *both* the real and the virtual internal moments are caused by a unit load, both moments are numerically equal

$$\delta\overline{M}_A = M_A \quad (7.67\text{-a})$$

$$\delta\overline{M}_B = M_B \quad (7.67\text{-b})$$

thus we conclude that

$$\boxed{f_{BA} = f_{BA}} \quad (7.68)$$

or *The displacement at a point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the unit load is acting at point B.*

26 Similarly *The rotation at a point B on a structure due to a unit couple moment acting at point A is equal to the rotation at point A when a unit couple moment is acting at point B.*

27 And *The rotation in radians at point B on a structure due to a unit load acting at point A is equal to the displacement of point A when a unit couple moment is acting at point B.*

28 These theorems will be used later on in justifying the symmetry of the stiffness matrix, and in construction of influence lines using the Müller-Breslau principle.

7.5 Summary of Equations

	U	Virtual Force $\delta\overline{U}^*$	
		General	Linear
Axial	$\int_0^L \frac{P^2}{2AE}dx$	$\int_0^L \delta\overline{\sigma}\varepsilon dx$	$\int_0^L \underbrace{\delta\overline{P}}_{\delta\overline{\sigma}} \underbrace{\frac{P}{AE}}_{\varepsilon} dx$
Shear	...	$\int_0^L \delta\overline{V}\gamma_{xy}dx$	...
Flexure	$\int_0^L \frac{M^2}{2EI_z}dx$	$\int_0^L \delta\overline{M}\phi dx$	$\int_0^L \underbrace{\delta\overline{M}}_{\delta\overline{\sigma}} \underbrace{\frac{M}{EI_z}}_{\varepsilon} dx$
Torsion	$\int_0^L \frac{T^2}{2GJ}dx$	$\int_0^L \delta\overline{T}\theta dx$	$\int_0^L \underbrace{\delta\overline{T}}_{\delta\overline{\sigma}} \underbrace{\frac{T}{GJ}}_{\varepsilon} dx$
	W	Virtual Force $\delta\overline{W}^*$	
P	$\Sigma_i \frac{1}{2} P_i \Delta_i$	$\Sigma_i \delta\overline{P}_i \Delta_i$	
M	$\Sigma_i \frac{1}{2} M_i \theta_i$	$\Sigma_i \delta\overline{M}_i \theta_i$	
w	$\int_0^L w(x)v(x)dx$	$\int_0^L \delta\overline{w}(x)v(x)dx$	

Table 7.3: Summary of Expressions for the Internal Strain Energy and External Work

Chapter 8

ARCHES and CURVED STRUCTURES

1 This chapter will concentrate on the analysis of arches.

2 The concepts used are identical to the ones previously seen, however the major (and only) difference is that equations will be written in polar coordinates.

3 Like cables, arches can be used to reduce the bending moment in long span structures. Essentially, an arch can be considered as an inverted cable, and is transmits the load primarily through axial compression, but can also resist flexure through its flexural rigidity.

4 A parabolic arch uniformly loaded will be loaded in compression only.

5 A semi-circular arch unifirmly loaded will have some flexural stresses in addition to the compressive ones.

8.1 Arches

6 In order to optimize dead-load efficiency, long span structures should have their shapes approximate the coresponding moment diagram, hence an arch, suspended cable, or tendon configuration in a prestressed concrete beam all are nearly parabolic, Fig. 8.1.

7 Long span structures can be built using flat construction such as girders or trusses. However, for spans in excess of 100 ft, it is often more economical to build a curved structure such as an arch, suspended cable or thin shells.

8 Since the dawn of history, mankind has tried to span distances using arch construction. Essentially this was because an arch required materials to resist compression only (such as stone, masonary, bricks), and labour was not an issue.

9 The basic issues of static in arch design are illustrated in Fig. 8.2 where the vertical load is per unit horizontal projection (such as an external load but not a self-weight). Due to symmetry, the vertical reaction is simply $V = \frac{wL}{2}$, and there is no shear across the midspan of the arch (nor a moment). Taking moment about the crown,

$$M = Hh - \frac{wL}{2}\left(\frac{L}{2} - \frac{L}{4}\right) = 0 \tag{8.1}$$

Solving for H

$$\boxed{H = \frac{wL^2}{8h}} \tag{8.2}$$

We recall that a similar equation was derived for arches., and H is analogous to the $C - T$ forces in a beam, and h is the overall height of the arch, Since h is much larger than d, H will be much smaller

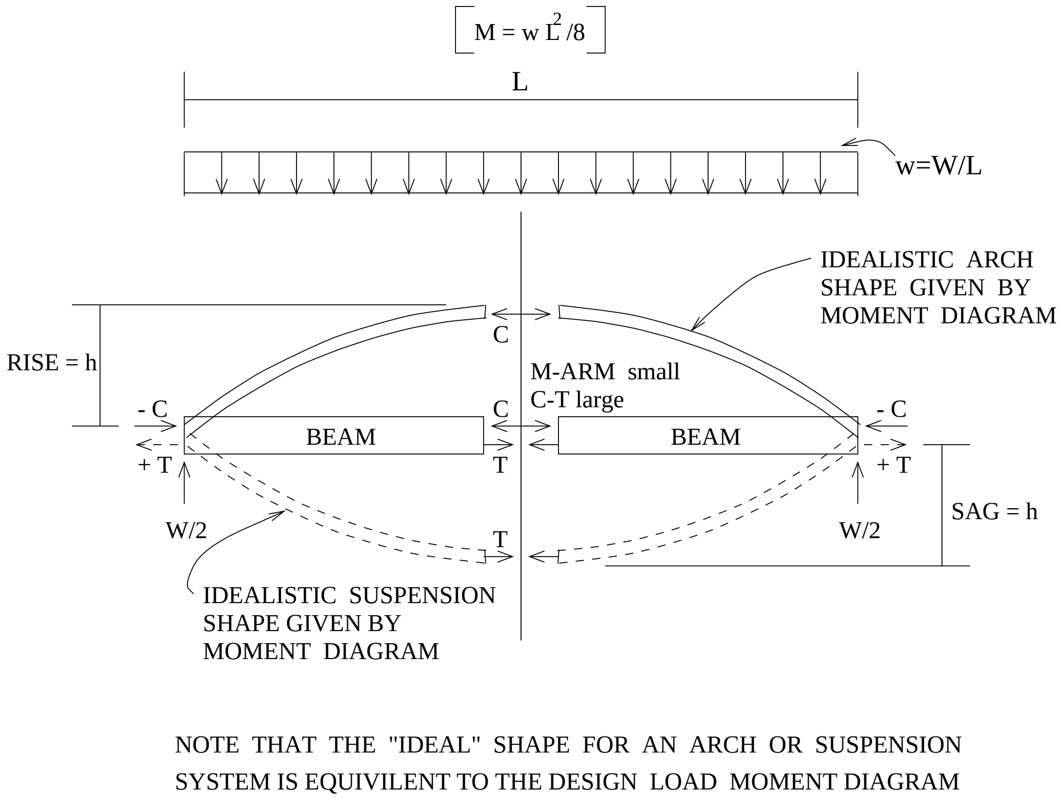


Figure 8.1: Moment Resisting Forces in an Arch or Suspension System as Compared to a Beam, (Lin and Stotesbury 1981)

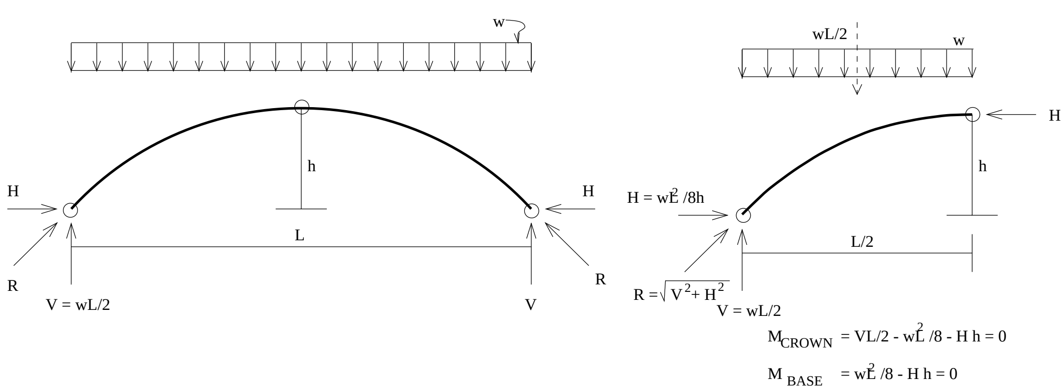


Figure 8.2: Statics of a Three-Hinged Arch, (Lin and Stotesbury 1981)

than $C - T$ in a beam.

10 Since equilibrium requires H to remain constant across thee arch, a parabolic curve would theoretically result in no moment on the arch section.

11 Three-hinged arches are statically determinate structures which shape can acomodate support settlements and thermal expansion without secondary internal stresses. They are also easy to analyse through statics.

12 An arch carries the vertical load across the span through a combination of axial forces and flexural ones. A well dimensioned arch will have a small to negligible moment, and relatively high normal compressive stresses.

13 An arch is far more efficient than a beam, and possibly more economical and aesthetic than a truss in carrying loads over long spans.

14 If the arch has only two hinges, Fig. 8.3, or if it has no hinges, then bending moments may exist either at the crown or at the supports or at both places.

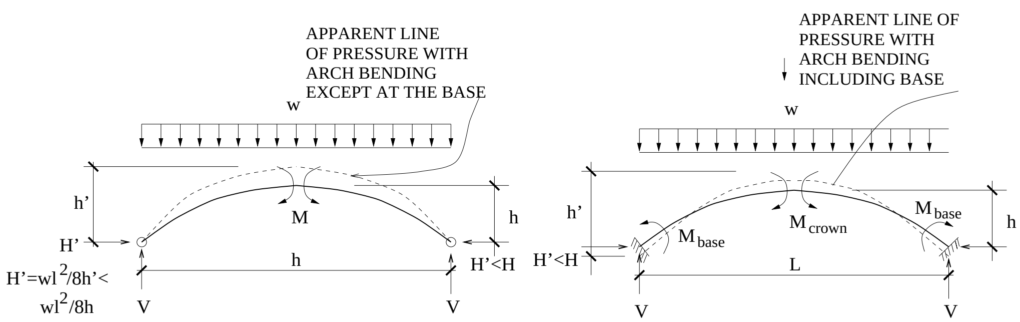


Figure 8.3: Two Hinged Arch, (Lin and Stotesbury 1981)

15 Since H varies inversely to the rise h, it is obvious that one should use as high a rise as possible. For a combination of aesthetic and practical considerations, a span/rise ratio ranging from 5 to 8 or perhaps as much as 12, is frequently used. However, as the ratio goes higher, we may have buckling problems, and the section would then have a higher section depth, and the arch advantage diminishes.

16 In a parabolic arch subjected to a uniform horizontal load there is no moment. However, in practice an arch is not subjected to uniform horizontal load. First, the depth (and thus the weight) of an arch is not usually constant, then due to the inclination of the arch the actual self weight is not constant. Finally, live loads may act on portion of the arch, thus the line of action will not necessarily follow the arch centroid. This last effect can be neglected if the live load is small in comparison with the dead load.

17 Since the greatest total force in the arch is at the support, ($R = \sqrt{V^2 + H^2}$), whereas at the crown we simply have H, the crown will require a smaller section than the support.

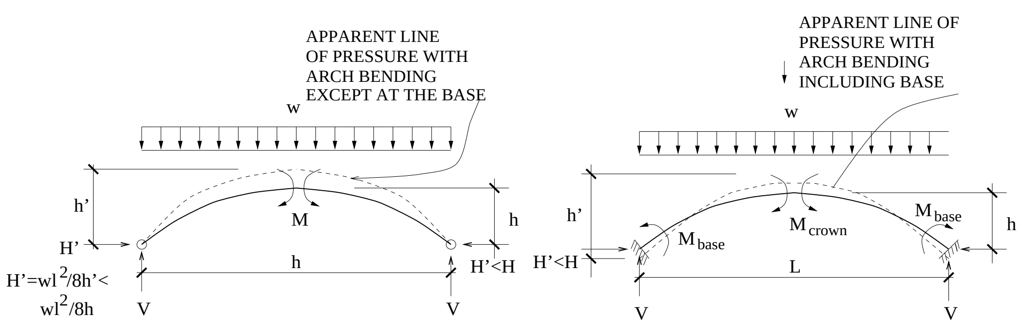


Figure 8.4: Arch Rib Stiffened with Girder or Truss, (Lin and Stotesbury 1981)

8.1.1 Statically Determinate

■ **Example 8-1: Three Hinged Arch, Point Loads. (Gerstle 1974)**

Determine the reactions of the three-hinged arch shown in Fig. 8.5

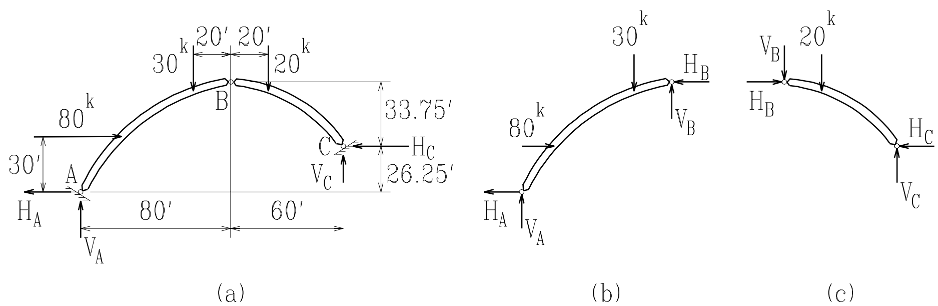


Figure 8.5:

Solution:

Four unknowns, three equations of equilibrium, one equation of condition ⇒ statically determinate.

$$
\begin{array}{rcll}
(+\circlearrowright)\ \Sigma M_z^C & = & 0; & (R_{Ay})(140) + (80)(3.75) - (30)(80) - (20)(40) + R_{Ax}(26.25) = 0 \\
 & & & \Rightarrow 140R_{Ay} + 26.25R_{Ax} = 2.900 \\
(+\rightarrow)\ \Sigma F_x & = & 0; & 80 - R_{Ax} - R_{Cx} = 0 \\
(+\uparrow)\ \Sigma F_y & = & 0; & R_{Ay} + R_{Cy} - 30 - 20 = 0 \\
(+\circlearrowright)\ \Sigma M_z^B & = & 0; & (R_{ax})(60) - (80)(30) - (30)(20) + (R_{Ay})(80) = 0 \\
 & & & \Rightarrow 80R_{Ay} + 60R_{Ax} = 3,000
\end{array}
\tag{8.3}
$$

Solving those four equations simultaneously we have:

$$
\begin{bmatrix} 140 & 26.25 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 80 & 60 & 0 & 0 \end{bmatrix}
\begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix} =
\begin{Bmatrix} 2,900 \\ 80 \\ 50 \\ 3,000 \end{Bmatrix} \Rightarrow
\begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix} =
\begin{Bmatrix} 15.1 \text{ k} \\ 29.8 \text{ k} \\ 34.9 \text{ k} \\ 50.2 \text{ k} \end{Bmatrix}
\tag{8.4}
$$

We can check our results by considering the summation with respect to b from the right:

$$
(+\circlearrowleft)\ \Sigma M_z^B = 0; -(20)(20) - (50.2)(33.75) + (34.9)(60) = 0\surd \tag{8.5}
$$

■

■ **Example 8-2: Semi-Circular Arch, (Gerstle 1974)**

Determine the reactions of the three hinged statically determined semi-circular arch under its own dead weight w (per unit arc length s, where $ds = rd\theta$). 8.6

Solution:

I Reactions The reactions can be determined by **integrating** the load over the entire structure

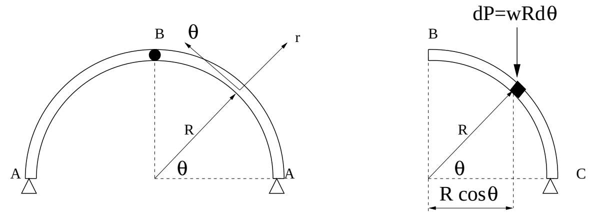


Figure 8.6: Semi-Circular three hinged arch

1. **Vertical Reaction** is determined first:

$$(+\curvearrowleft)\ \Sigma M_A = 0; -(C_y)(2R) + \int_{\theta=0}^{\theta=\pi} \underbrace{wRd\theta}_{dP} \underbrace{R(1+\cos\theta)}_{\text{moment arm}} = 0 \tag{8.6-a}$$

$$\begin{aligned} \Rightarrow C_y &= \frac{wR}{2}\int_{\theta=0}^{\theta=\pi}(1+\cos\theta)d\theta = \frac{wR}{2}[\theta-\sin\theta]\mid_{\theta=0}^{\theta=\pi} \\ &= \frac{wR}{2}[(\pi-\sin\pi)-(0-\sin 0)] \\ &= \boxed{\tfrac{\pi}{2}wR} \end{aligned} \tag{8.6-b}$$

2. **Horizontal Reactions** are determined next

$$(+\curvearrowright)\ \Sigma M_B = 0; -(C_x)(R) + (C_y)(R) - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \underbrace{wRd\theta}_{dP} \underbrace{R\cos\theta}_{\text{moment arm}} = 0 \tag{8.7-a}$$

$$\begin{aligned} \Rightarrow C_x &= \frac{\pi}{2}wR - \frac{wR}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\cos\theta d\theta = \frac{\pi}{2}wR - wR[\sin\theta]\mid_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{2}wR - wR(\frac{\pi}{2}-0) \\ &= \boxed{\left(\tfrac{\pi}{2}-1\right)wR} \end{aligned} \tag{8.7-b}$$

By symmetry the reactions at A are equal to those at C

II Internal Forces can now be determined, Fig. 8.7.

1. **Shear Forces:** Considering the free body diagram of the arch, and summing the forces in the radial direction ($\Sigma F_R = 0$):

$$\underbrace{-(\frac{\pi}{2}-1)wR}_{C_x}\cos\theta + \underbrace{\frac{\pi}{2}wR}_{C_y}\sin\theta - \int_{\alpha=0}^{\theta} wRd\alpha\sin\theta + V = 0 \tag{8.8}$$

$$\Rightarrow \boxed{V = wR\left[(\tfrac{\pi}{2}-1)\cos\theta + (\theta-\tfrac{\pi}{2})\sin\theta\right]} \tag{8.9}$$

2. **Axial Forces:** Similarly, if we consider the summation of forces in the axial direction ($\Sigma F_N = 0$):

$$(\tfrac{\pi}{2}-1)wR\sin\theta + \tfrac{\pi}{2}wR\cos\theta - \int_{\alpha=0}^{\theta} wRd\alpha\cos\theta + N = 0 \tag{8.10}$$

$$\Rightarrow \boxed{N = wR\left[(\theta-\tfrac{\pi}{2})\cos\theta - (\tfrac{\pi}{2}-1)\sin\theta\right]} \tag{8.11}$$

3. **Moment:** Now we can consider the third equation of equilibrium ($\Sigma M_z = 0$):

$$\begin{aligned} (+\curvearrowleft)\ \Sigma M \quad & (\tfrac{\pi}{2}-1)wR\cdot R\sin\theta - \tfrac{\pi}{2}wR^2(1-\cos\theta) + \\ & \int_{\alpha=0}^{\theta} wRd\alpha\cdot R(\cos\alpha-\cos\theta) + M = 0 \end{aligned} \tag{8.12}$$

$$\Rightarrow \boxed{M = wR^2\left[\tfrac{\pi}{2}(1-\sin\theta) + (\theta-\tfrac{\pi}{2})\cos\theta\right]} \tag{8.13}$$

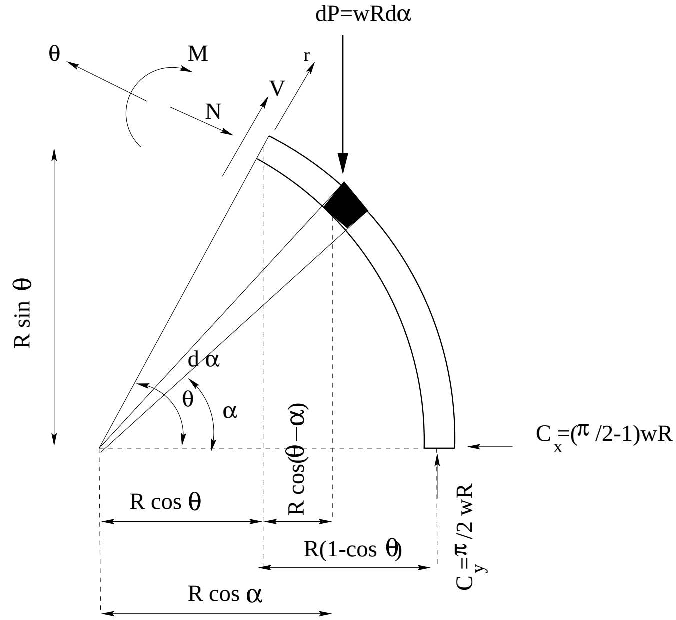


Figure 8.7: Semi-Circular three hinged arch; Free body diagram

III Deflection are determined last

1. The real curvature ϕ is obtained by dividing the moment by EI

$$\phi = \frac{M}{EI} = \frac{wR^2}{EI}\left[\frac{\pi}{2}(1-\sin\theta) + (\theta - \frac{\pi}{2})\cos\theta\right] \tag{8.14}$$

1. The virtual force $\delta\overline{P}$ will be a unit vertical point in the direction of the desired deflection, causing a virtual internal moment

$$\delta\overline{M} = \frac{R}{2}\left[1-\cos\theta-\sin\theta\right] \qquad 0 \le \theta \le \frac{\pi}{2} \tag{8.15}$$

p

2. Hence, application of the virtual work equation yields:

$$\begin{aligned}
\underbrace{1}_{\delta\overline{P}}\cdot\Delta &= 2\int_{\theta=0}^{\frac{\pi}{2}} \underbrace{\frac{wR^2}{EI}\left[\frac{\pi}{2}(1-\sin\theta)+(\theta-\frac{\pi}{2})\cos\theta\right]}_{\frac{M}{EI}=\phi}\cdot\underbrace{\frac{R}{2}\cdot\left[1-\cos\theta-\sin\theta\right]}_{\delta\overline{M}}\underbrace{Rd\theta}_{dx} \\
&= \frac{wR^4}{16EI}\left[7\pi^2-18\pi-12\right] \\
&= \boxed{.0337\frac{wR^4}{EI}} \qquad (8.16\text{-a})
\end{aligned}$$

■

8.1.2 Statically Indeterminate

■ **Example 8-3: Statically Indeterminate Arch, (Kinney 1957)**

Determine the value of the horizontal reaction component of the indicated two-hinged solid rib arch, Fig. 8.8 as caused by a concentrated vertical load of 10 k at the center line of the span. Consider shearing, axial, and flexural strains. Assume that the rib is a W24x130 with a total area of 38.21 in^2, that it has a web area of 13.70 in^2, a moment of inertia equal to 4,000 in^4, E of 30,000 k/in^2, and a shearing modulus G of 13,000 k/in^2.

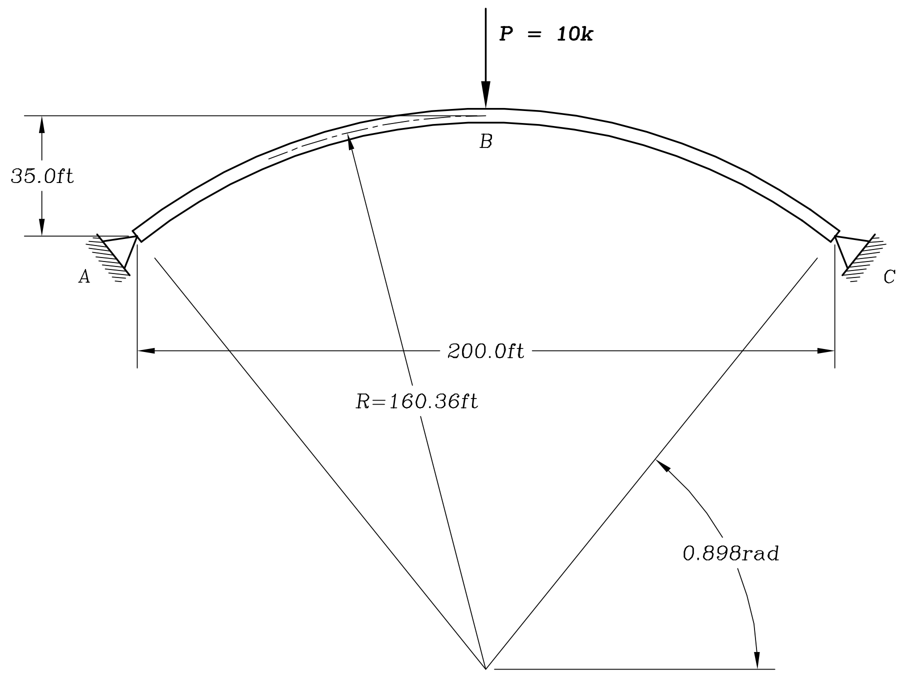


Figure 8.8: Statically Indeterminate Arch

Solution:

1. Consider that end C is placed on rollers, as shown in Fig. **??** A unit fictitious horizontal force is applied at C. The axial and shearing components of this fictitious force and of the vertical reaction at C, acting on any section θ in the right half of the rib, are shown at the right end of the rib in Fig. 13-7.

2. The expression for the horizontal displacement of C is

$$\underbrace{1}_{\delta\overline{P}}\Delta_{Ch} = 2\int_C^B \delta\overline{M}\frac{M}{EI}ds + 2\int_C^B \delta\overline{V}\frac{V}{A_wG}ds + 2\int_C^B \delta\overline{N}\frac{N}{AE}ds \tag{8.17}$$

3. From Fig. 8.9, for the rib from C to B,

$$M = \frac{P}{2}(100 - R\cos\theta) \tag{8.18-a}$$
$$\delta\overline{M} = 1(R\sin\theta - 125.36) \tag{8.18-b}$$
$$V = \frac{P}{2}\sin\theta \tag{8.18-c}$$
$$\delta\overline{V} = \cos\theta \tag{8.18-d}$$
$$N = \frac{P}{2}\cos\theta \tag{8.18-e}$$

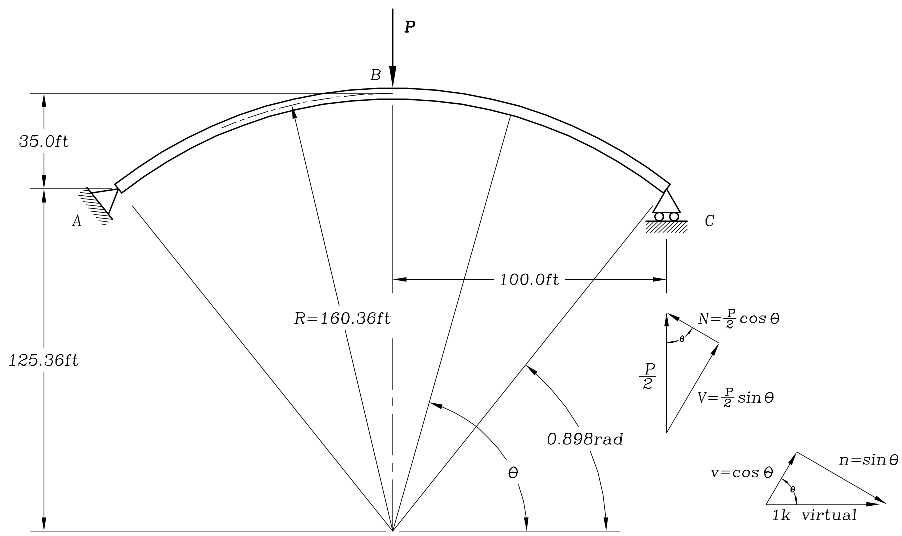


Figure 8.9: Statically Indeterminate Arch; ‘Horizontal Reaction Removed

$$\delta\overline{N} = -\sin\theta \tag{8.18-f}$$
$$ds = Rd\theta \tag{8.18-g}$$

4. If the above values are substituted in Eq. 8.17 and integrated between the limits of 0.898 and $\pi/2$, the result will be
$$\Delta_{Ch} = 22.55 + 0.023 - 0.003 = 22.57 \tag{8.19}$$

5. The load P is now assumed to be removed from the rib, and a real horizontal force of 1 k is assumed to act toward the right at C in conjunction with the fictitious horizontal force of 1 k acting to the right at the same point. The horizontal displacement of C will be given by
$$\delta_{ChCh} = 2\int_C^B \delta\overline{M}\frac{\overline{M}}{EI}ds + 2\int_C^B \delta\overline{V}\frac{\overline{V}}{A_wG}ds + 2\int_C^B \delta\overline{N}\frac{\overline{N}}{AE}ds \tag{8.20-a}$$
$$= 2.309 + 0.002 + 0.002 = 2.313 \text{ in} \tag{8.20-b}$$

6. The value of the horizontal reaction component will be
$$H_C = \frac{\Delta_{Ch}}{\delta_{ChCh}} = \frac{22.57}{2.313} = \boxed{9.75 \text{ k}} \tag{8.21}$$

7. If only flexural strains are considered, the result would be
$$H_C = \frac{22.55}{2.309} = \boxed{9.76 \text{ k}} \tag{8.22}$$

Comments

1. For the given rib and the single concentrated load at the center of the span it is obvious that the effects of shearing and axial strains are insignificant and can be disregarded.

2. Erroneous conclusions as to the relative importance of shearing and axial strains in the usual solid rib may be drawn, however, from the values shown in Eq. 8.19. These indicate that the effects of the shearing strains are much more significant than those of the axial strains. This is actually the case for the single concentrated load chosen for the demonstration, but only because the rib does not approximate the funicular polygon for the single load. As a result, the shearing components on most sections of the rib are more important than would otherwise be the case.

3. The usual arch encountered in practice, however, is subjected to a series of loads, and the axis of the rib will approximate the funicular polygon for these loads. In other words, the line of pressure is nearly perpendicular to the right section at all points along the rib. Consequently, the shearing components are so small that the shearing strains are insignificant and are neglected.

4. Axial strains, resulting in rib shortening, become increasingly important as the rise-to-span ratio of the arch decreases. It is advisable to determine the effects of rib shortening in the design of arches. The usual procedure is to first design the rib by considering flexural strains only, and then to check for the effects of rib shortening.

■

8.2 Curved Space Structures

■ **Example 8-4: Semi-Circular Box Girder, (Gerstle 1974)**

Determine the reactions of the semi-circular cantilevered box girder shown in Fig. 8.10 subjected to

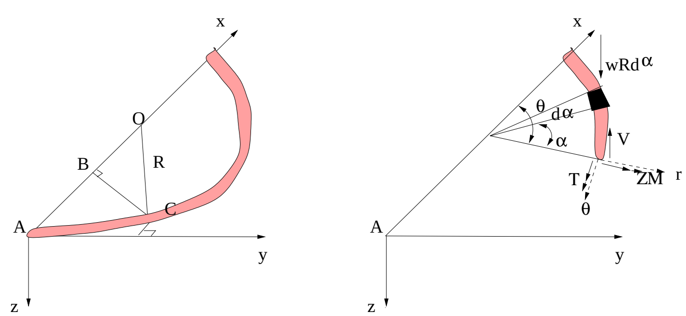


Figure 8.10: Semi-Circular Box Girder

its own weight w.
Solution:

I Reactions are again determined first From geometry we have $OA = R$, $OB = R\cos\theta$, $AB = OA - BO = R - R\cos\theta$, and $BP = R\sin\theta$. The moment arms for the moments with respect to the x and y axis are BP and AB respectively. Applying three equations of equilibrium we obtain

$$F_z^A - \int_{\theta=0}^{\theta=\pi} wRd\theta = 0 \quad \Rightarrow \quad \boxed{F_z^A = wR\pi} \tag{8.23-a}$$

$$M_x^A - \int_{\theta=0}^{\theta=\pi} (wRd\theta)(R\sin\theta) = 0 \quad \Rightarrow \quad \boxed{M_x^A = 2wR^2} \tag{8.23-b}$$

$$M_y^A - \int_{\theta=0}^{\theta=\pi} (wRd\theta)R(1-\cos\theta) = 0 \quad \Rightarrow \quad \boxed{M_y^A = -wR^2\pi} \tag{8.23-c}$$

II Internal Forces are determined next

1. **Shear Force:**

$$(+\uparrow)\, \Sigma F_z = 0 \Rightarrow V - \int_0^\theta wRd\alpha = 0 \Rightarrow \boxed{V = wr\theta} \tag{8.24}$$

2. Bending Moment:

$$\Sigma M_R = 0 \Rightarrow M - \int_0^\theta (wRd\alpha)(R\sin\alpha) = 0 \Rightarrow \boxed{M = wR^2(1-\cos\theta)} \tag{8.25}$$

3. Torsion:

$$\Sigma M_T = 0 \Rightarrow + \int_0^\theta (wRd\alpha)R(1-\cos\alpha) = 0 \Rightarrow \boxed{T = -wR^2(\theta - \sin\theta)} \tag{8.26}$$

III Deflection are determined last we assume a rectangular cross-section of width b and height $d = 2b$ and a Poisson's ratio $\nu = 0.3$.

1. Noting that the member will be subjected to both flexural and torsional deformations, we seek to determine the two stiffnesses.
2. The flexural stiffness EI is given by $EI = E\frac{bd^3}{12} = E\frac{b(2b)^3}{12} = \frac{2Eb^4}{3} = .667Eb^4$.
3. The torsional stiffness of solid rectangular sections $J = kb^3d$ where b is the shorter side of the section, d the longer, and k a factor equal to .229 for $\frac{d}{b} = 2$. Hence $G = \frac{E}{2(1+\nu)} = \frac{E}{2(1+.3)} = .385E$, and $GJ = (.385E)(.229b^4) = .176Eb^4$.
4. Considering both flexural and torsional deformations, and replacing dx by $rd\theta$:

$$\underbrace{\delta\overline{P}\Delta}_{\delta\overline{W}^*} = \underbrace{\underbrace{\int_0^\pi \delta\overline{M}\frac{M}{EI_z}Rd\theta}_{\text{Flexure}} + \underbrace{\int_0^\pi \delta\overline{T}\frac{T}{GJ}Rd\theta}_{\text{Torsion}}}_{\delta\overline{U}^*} \tag{8.27}$$

   where the real moments were given above.
5. Assuming a unit virtual downward force $\delta\overline{P} = 1$, we have

$$\delta\overline{M} = R\sin\theta \tag{8.28-a}$$
$$\delta\overline{T} = -R(1-\cos\theta) \tag{8.28-b}$$

6. Substituting these expression into Eq. 8.27

$$\begin{aligned}
\underbrace{1}_{\delta\overline{P}}\Delta &= \frac{wR^2}{EI}\int_0^\pi \underbrace{(R\sin\theta)}_{M}\underbrace{(1-\cos\theta)}_{\delta\overline{M}}Rd\theta + \frac{wR^2}{GJ}\int_0^\pi \underbrace{(\theta-\sin\theta)}_{\delta\overline{T}}\underbrace{R(1-\cos\theta)}_{T}Rd\theta \\
&= \frac{wR^4}{EI}\int_0^\pi \left[(\sin\theta - \sin\theta\cos\theta) + \frac{1}{.265}(\theta - \theta\cos\theta - \sin\theta + \sin\theta\cos\theta)\right]d\theta \\
&= \frac{wR^4}{EI}(\underbrace{2.}_{\text{Flexure}} + \underbrace{18.56}_{\text{Torsion}}) \\
&= \boxed{20.56\frac{wR^4}{EI}}
\end{aligned} \tag{8.29-a}$$

■

8.2.1 Theory

ßAdapted from (Gerstle 1974)

18 Because space structures may have complicated geometry, we must resort to vector analysis[1] to determine the internal forces.

19 In general we have six internal forces (forces and moments) acting at any section.

[1] To which you have already been exposed at an early stage, yet have very seldom used it so far in mechanics!

8.2.1.1 Geometry

20 In general, the geometry of the structure is most conveniently described by a parameteric set of equations

$$x = f_1(\theta); \qquad y = f_2(\theta); \qquad z = f_3(\theta) \tag{8.30}$$

as shown in Fig. 8.11. the global coordinate system is denoted by $X - Y - Z$, and its unit vectors are

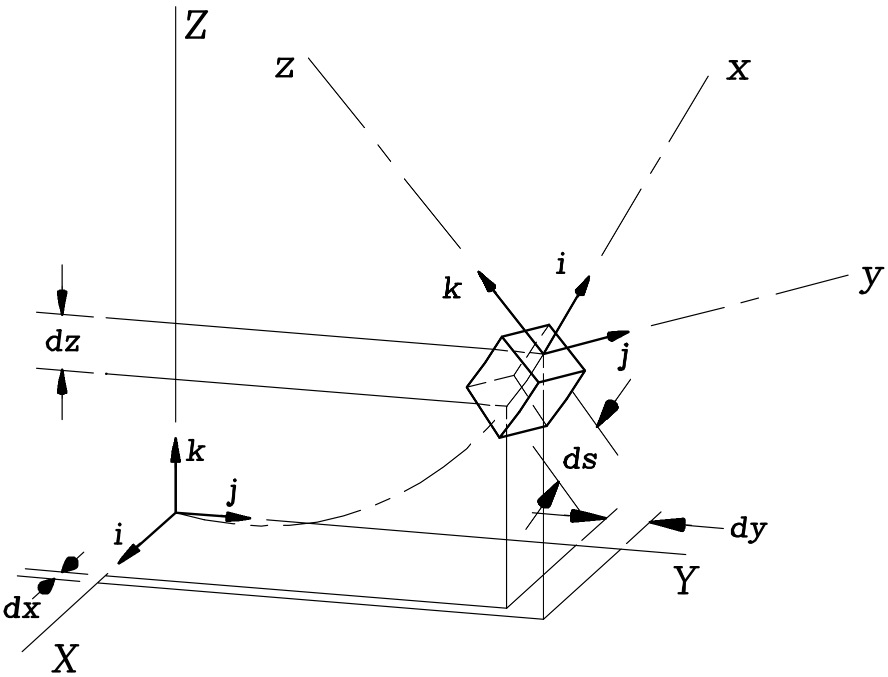


Figure 8.11: Geometry of Curved Structure in Space

denoted[2] $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

21 The section on which the internal forces are required is cut and the **principal** axes are identified as $N - S - W$ which correspond to the normal force, and bending axes with respect to the *Strong* and *Weak* axes. The corresponding unit vectors are $\mathbf{n}, \mathbf{s}, \mathbf{w}$.

22 The unit normal vector at any section is given by

$$\mathbf{n} = \frac{dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}}{ds} = \frac{dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}}{(dx^2 + dy^2 + dz^2)^{1/2}} \tag{8.31}$$

23 The principal bending axes must be defined, that is if the strong bending axis is parallel to the XY plane, or horizontal (as is generally the case for gravity load), then this axis is normal to both the N and Z axes, and its unit vector is

$$\mathbf{s} = \frac{\mathbf{n} \times \mathbf{k}}{|\mathbf{n} \times \mathbf{k}|} \tag{8.32}$$

24 The weak bending axis is normal to both N and S, and thus its unit vector is determined from

$$\mathbf{w} = \mathbf{n} \times \mathbf{s} \tag{8.33}$$

8.2.1.2 Equilibrium

25 For the equilibrium equations, we consider the free body diagram of Fig. 8.12 an applied load **P**

[2] All vectorial quantities are denoted by a bold faced character.

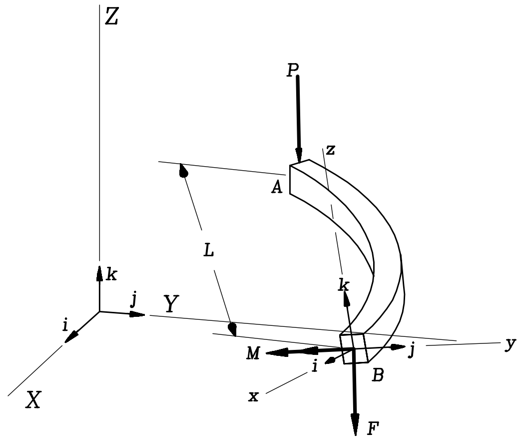


Figure 8.12: Free Body Diagram of a Curved Structure in Space

is acting at point A. The resultant force vector $\mathbf{F}$ and resultant moment vector $\mathbf{M}$ acting on the cut section B are determined from equilibrium

$$\Sigma \mathbf{F} = 0; \quad \mathbf{P} + \mathbf{F} = 0; \quad \mathbf{F} = -\mathbf{P} \tag{8.34-a}$$
$$\Sigma \mathbf{M}^B = 0; \quad \mathbf{L}\times\mathbf{P} + \mathbf{M} = 0; \quad \mathbf{M} = -\mathbf{L}\times\mathbf{P} \tag{8.34-b}$$

where $\mathbf{L}$ is the lever arm vector from B to A.

26 The axial and shear forces N, V_s and V_w are all three components of the force vector $\mathbf{F}$ along the N, S, and W axes and can be found by dot product with the appropriate unit vectors:

$$N = \mathbf{F}\cdot\mathbf{n} \tag{8.35-a}$$
$$V_s = \mathbf{F}\cdot\mathbf{s} \tag{8.35-b}$$
$$V_w = \mathbf{F}\cdot\mathbf{w} \tag{8.35-c}$$

27 Similarly the torsional and bending moments T, M_s and M_w are also components of the moment vector $\mathbf{M}$ and are determined from

$$T = \mathbf{M}\cdot\mathbf{n} \tag{8.36-a}$$
$$M_s = \mathbf{M}\cdot\mathbf{s} \tag{8.36-b}$$
$$M_w = \mathbf{M}\cdot\mathbf{w} \tag{8.36-c}$$

28 Hence, we do have a mean to determine the internal forces. In case of applied loads we summ, and for distributed load we integrate.

■ Example 8-5: Internal Forces in an Helicoidal Cantilevered Girder, (Gerstle 1974)

Determine the internal forces N, V_s, and V_w and the internal moments T, M_s and M_w along the helicoidal cantilevered girder shown in FIg. 8.13 due to a vertical load P at its free end.
Solution:

1. We first determine the geometry in terms of the angle θ

$$x = R\cos\theta; \quad y = R\sin\theta; \quad z = \frac{H}{\pi}\theta \tag{8.37}$$

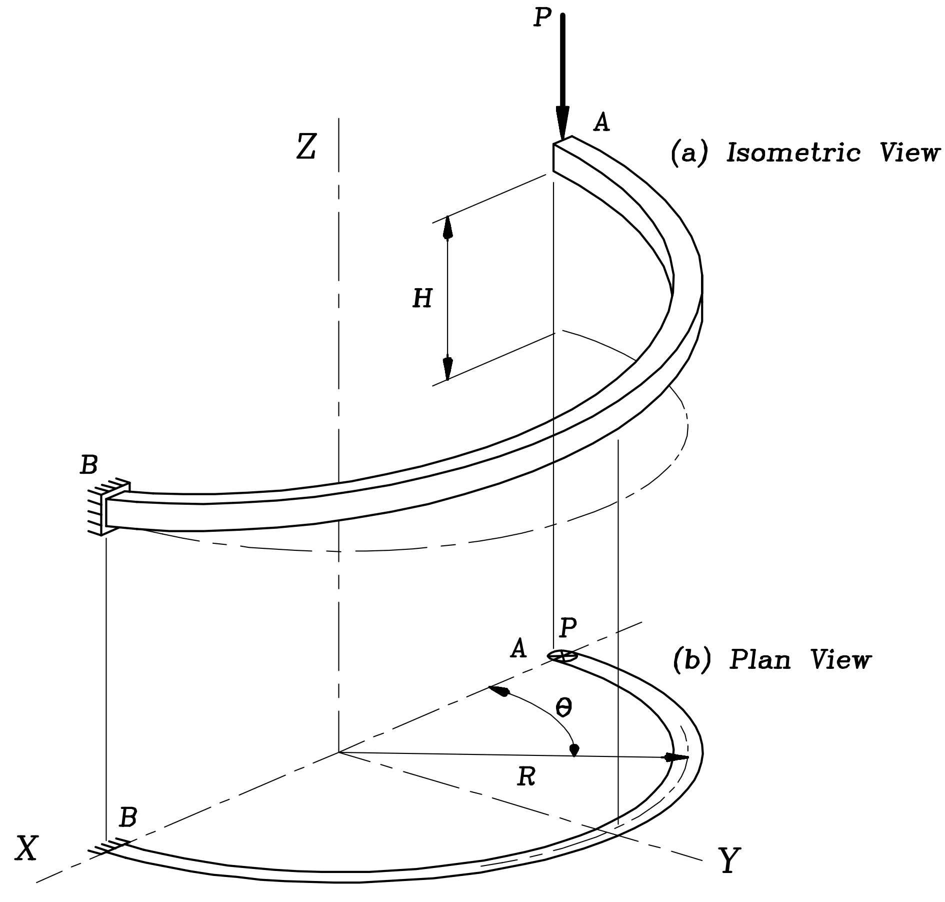


Figure 8.13: Helicoidal Cantilevered Girder

2. To determine the unit vector $\mathbf{n}$ at any point we need the derivatives:

$$dx = -R\sin\theta d\theta; \qquad dy = R\cos\theta d\theta; \qquad dz = \frac{H}{\pi}d\theta \tag{8.38}$$

and then insert into Eq. 8.31

$$\mathbf{n} = \frac{-R\sin\theta\mathbf{i} + R\cos\theta\mathbf{j} + H/\pi\mathbf{k}}{\left[R^2\sin^2\theta + R^2\cos^2\theta + (H/\pi)^2\right]^{1/2}} \tag{8.39-a}$$

$$= \underbrace{\frac{1}{\left[1 + (H/\pi R)^2\right]^{1/2}}}_{K}[\sin\theta\mathbf{i} + \cos\theta\mathbf{j} + (H/\pi R)\mathbf{k}] \tag{8.39-b}$$

Since the denominator depends only on the geometry, it will be designated by K.

3. The strong bending axis lies in a horizontal plane, and its unit vector can thus be determined from Eq. 8.32:

$$\mathbf{n}\times\mathbf{k} = \frac{1}{K}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & \frac{H}{\pi R} \\ 0 & 0 & 1 \end{vmatrix} \tag{8.40-a}$$

$$= \frac{1}{K}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \tag{8.40-b}$$

and the absolute magnitude of this vector $|\mathbf{k}\times\mathbf{n}| = \frac{1}{K}$, and thus

$$\mathbf{s} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \tag{8.41}$$

4. The unit vector along the weak axis is determined from Eq. 8.33

$$\mathbf{w} = \mathbf{s}\times\mathbf{n} = \frac{1}{K}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & \frac{H}{\pi R} \end{vmatrix} \tag{8.42-a}$$

$$= \frac{1}{K}\left(\frac{H}{\pi R}\sin\theta\mathbf{i} - \frac{H}{\pi R}\cos\theta\mathbf{j} + \mathbf{k}\right) \tag{8.42-b}$$

5. With the geometry definition completed, we now examine the equilibrium equations. Eq. 8.34-a and 8.34-b.

$$\Sigma F = 0; \quad \mathbf{F} = -P \tag{8.43-a}$$

$$\Sigma M_b = 0; \quad \mathbf{M} = -\mathbf{L}\times\mathbf{P} \tag{8.43-b}$$

where

$$\mathbf{L} = (R - R\cos\theta)\mathbf{i} + (0 - R\sin\theta)\mathbf{j} + \left(0 - \frac{\theta}{\pi}H\right)\mathbf{k} \tag{8.44}$$

and

$$\mathbf{L}\times\mathbf{P} = R\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (1-\cos\theta) & -\sin\theta & -\frac{\theta}{\pi}\frac{H}{R} \\ 0 & 0 & P \end{vmatrix} \tag{8.45-a}$$

$$= PR[-\sin\theta\mathbf{i} - (1-\cos\theta)\mathbf{j}] \tag{8.45-b}$$

and

$$\mathbf{M} = PR[\sin\theta\mathbf{i} + (1-\cos\theta)\mathbf{j}] \tag{8.46}$$

6. Finally, the components of the force $\mathbf{F} = -P\mathbf{k}$ and the moment $\mathbf{M}$ are obtained by appropriate dot products with the unit vectors

$$N = \mathbf{F}\cdot\mathbf{n} = \boxed{-\frac{1}{K}P\frac{H}{\pi R}} \tag{8.47-a}$$

$$V_s = \mathbf{F}\cdot\mathbf{s} = \boxed{0} \tag{8.47-b}$$

$$V_w = \mathbf{F}\cdot\mathbf{w} = \boxed{-\frac{1}{K}P} \tag{8.47-c}$$

$$T = \mathbf{M}\cdot\mathbf{n} = \boxed{-\frac{PR}{K}(1-\cos\theta)} \tag{8.47-d}$$

$$M_s = \mathbf{M}\cdot\mathbf{s} = \boxed{PR\sin\theta} \tag{8.47-e}$$

$$M_w = \mathbf{M}\cdot\mathbf{w} = \boxed{\frac{PH}{\pi K}(1-\cos\theta)} \tag{8.47-f}$$

■

Chapter 9

APPROXIMATE FRAME ANALYSIS

1 Despite the widespread availability of computers, approximate methods of analysis are justified by

1. Inherent assumption made regarding the validity of a linear elastic analysis *vis a vis* of an ultimate failure design.
2. Ability of structures to redistribute internal forces.
3. Uncertainties in load and material properties

2 Vertical loads are treated separately from the horizontal ones.

3 We use the design sign convention for moments (+ve tension below), and for shear (ccw +ve).

4 Assume girders to be numbered from left to right.

5 In all free body diagrams assume positivee forces/moments, and take algeebraic sums.

6 The key to the approximate analysis method is our ability to sketch the deflected shape of a structure and identify inflection points.

7 We begin by considering a uniformly loaded beam and frame. In each case we consider an extreme end of the restraint: a) free or b) restrained. For the frame a relativly flexible or stiff column would be analogous to a free or fixed restrain on the beam, Fig. 9.1.

9.1 Vertical Loads

8 With reference to Fig. 9.1, we now consider an *intermediary* case as shown in Fig. 9.2.

9 With the location of the inflection points identified, we may now determine all the reactions and internal forces from statics.

10 If we now consider a multi-bay/multi-storey frame, the girders at each floor are assumed to be continuous beams, and columns are assumed to resist the resulting unbalanced moments from the girders, we may make the following assumptions

1. Girders at each floor act as continous beams supporting a uniform load.
2. Inflection points are assumed to be at
   (a) One tenth the span from both ends of each girder.
   (b) Mid-height of the columns
3. Axial forces and deformation in the girder are negligibly small.

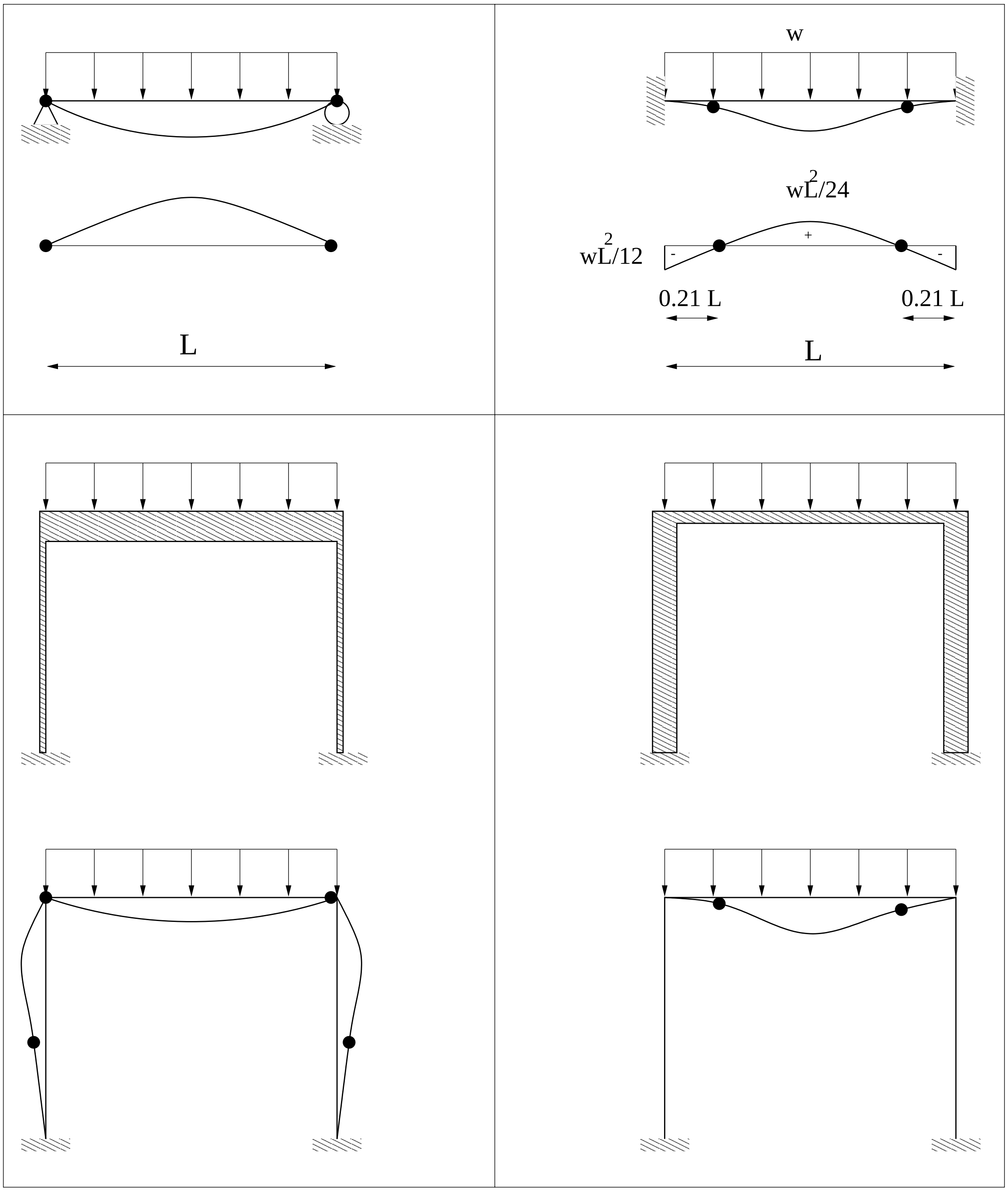


Figure 9.1: Uniformly Loaded Beam and Frame with Free or Fixed Beam Restraint

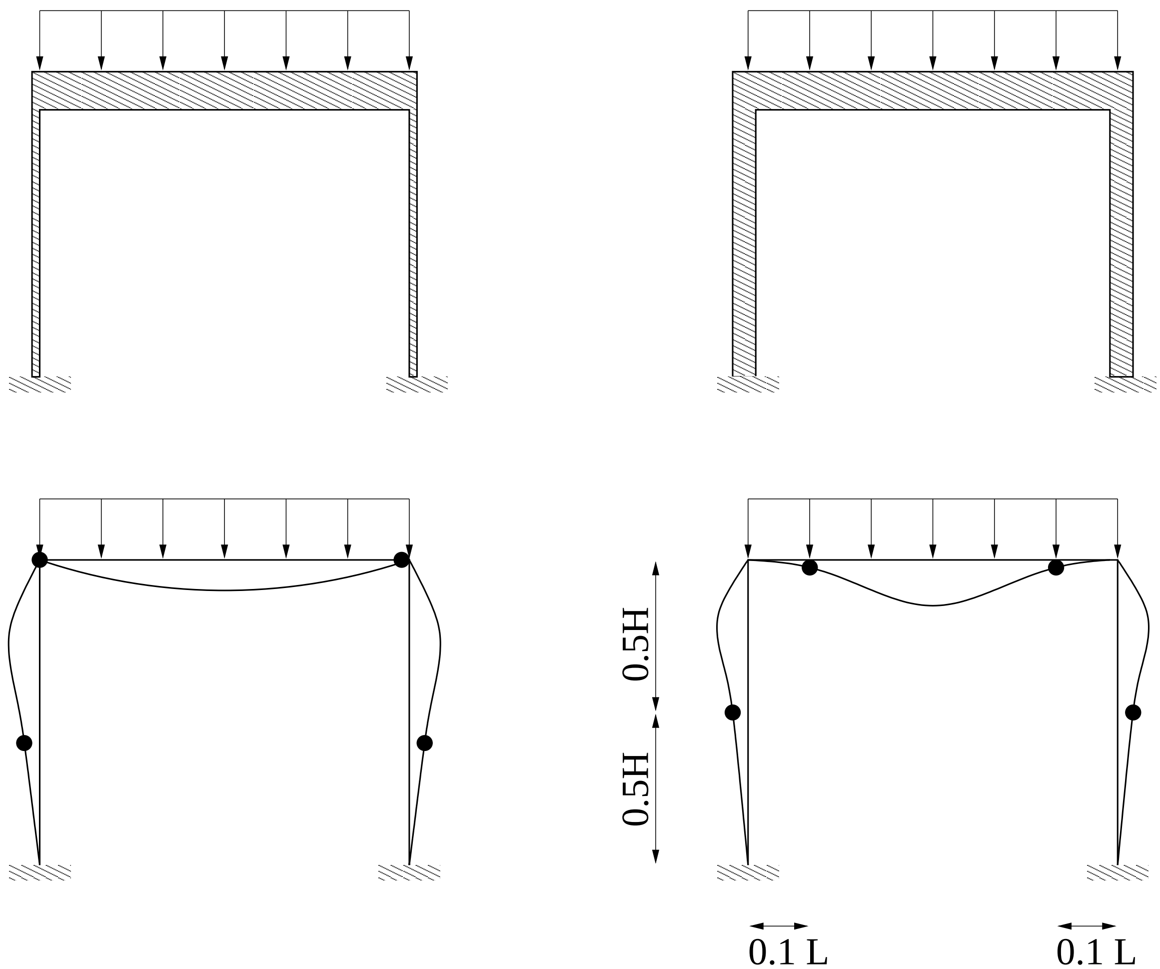


Figure 9.2: Uniformly Loaded Frame, Approximate Location of Inflection Points

4. Unbalanced end moments from the girders at each joint is distributed to the columns above and below the floor.

11 Based on the first assumption, all beams are statically determinate and have a span, L_s equal to 0.8 the original length of the girder, L. (Note that for a rigidly connected member, the inflection point is at 0.211 L, and at the support for a simply supported beam; hence, depending on the nature of the connection one could consider those values as upper and lower bounds for the approximate location of the hinge).

12 End forces are given by

Maximum positive moment at the center of each beam is, Fig. 9.3

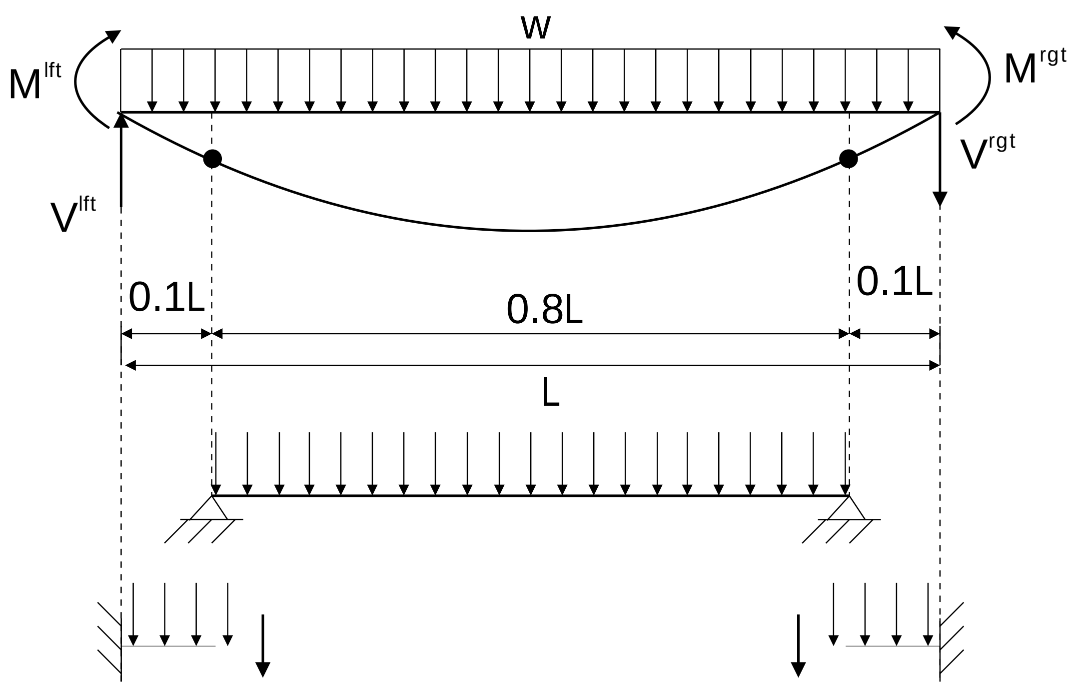


Figure 9.3: Approximate Analysis of Frames Subjected to Vertical Loads; Girder Moments

$$\boxed{M^+ = \frac{1}{8}wL_s^2 = w\frac{1}{8}(0.8)^2L^2 = 0.08wL^2} \tag{9.1}$$

Maximum negative moment at each end of the girder is given by, Fig. 9.3

$$\boxed{M^{left} = M^{rgt} = -\frac{w}{2}(0.1L)^2 - \frac{w}{2}(0.8L)(0.1L) = -0.045wL^2} \tag{9.2}$$

Girder Shear are obtained from the free body diagram, Fig. 9.4

$$\boxed{V^{lft} = \frac{wL}{2} \qquad V^{rgt} = -\frac{wL}{2}} \tag{9.3}$$

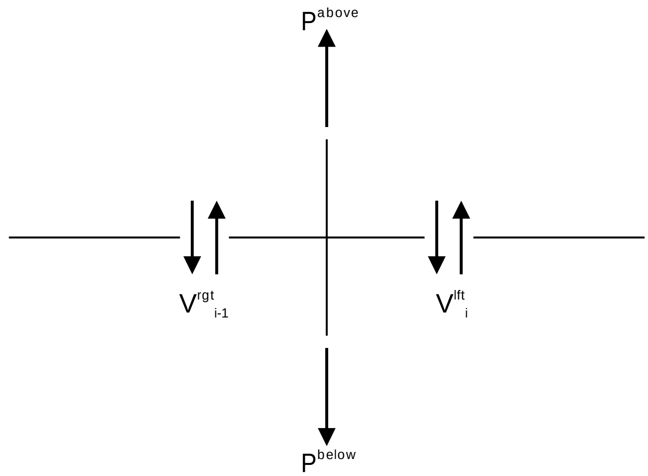


Figure 9.4: Approximate Analysis of Frames Subjected to Vertical Loads; Column Axial Forces

Column axial force is obtained by summing all the girder shears to the axial force transmitted by the column above it. Fig. 9.4

$$\boxed{P^{dwn} = P^{up} + V_{i-1}^{rgt} - V_i^{lft}} \tag{9.4}$$

Column Moment are obtained by considering the free body diagram of columns Fig. 9.5

$$\boxed{M^{top} = M_{above}^{bot} - M_{i-1}^{rgt} + M_i^{lft} \qquad M^{bot} = -M^{top}} \tag{9.5}$$

Column Shear Points of inflection are at mid-height, with possible exception when the columns on the first floor are hinged at the base, Fig. 9.5

$$\boxed{V = \frac{M^{top}}{\frac{h}{2}}} \tag{9.6}$$

Girder axial forces are assumed to be negligible eventhough the unbalanced column shears above and below a floor will be resisted by girders at the floor.

9.2 Horizontal Loads

13 Again, we begin by considering a simple frame subjected to a horizontal force, Fig. 9.6. depending on the boundary conditions, we will have different locations for the inflection points.

14 For the analysis of a multi-bays/multi-storeys frame, we must differentiate between low and high rise buildings.

Low rise buidlings, where the height is at least samller than the hrizontal dimension, the deflected shape is characterized by shear deformations.

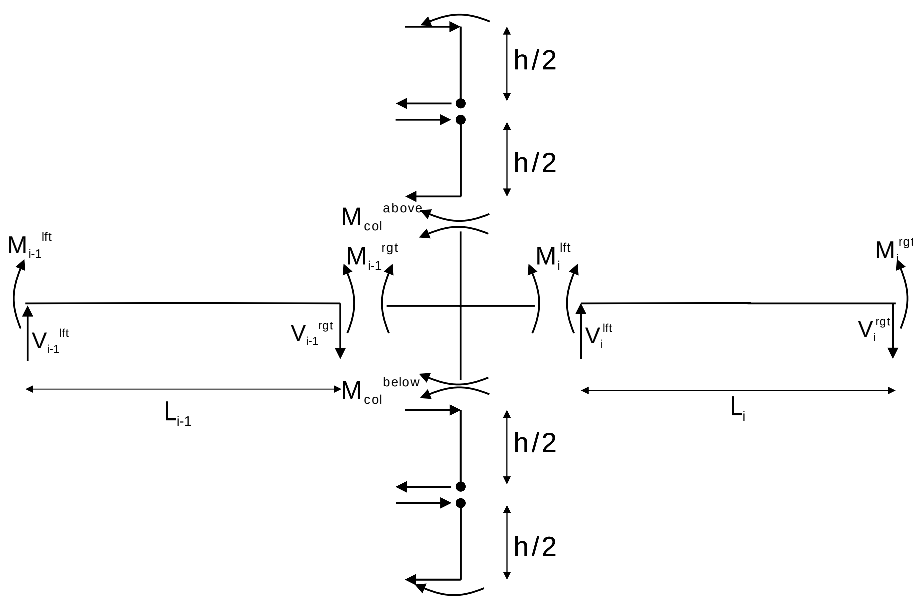


Figure 9.5: Approximate Analysis of Frames Subjected to Vertical Loads; Column Moments

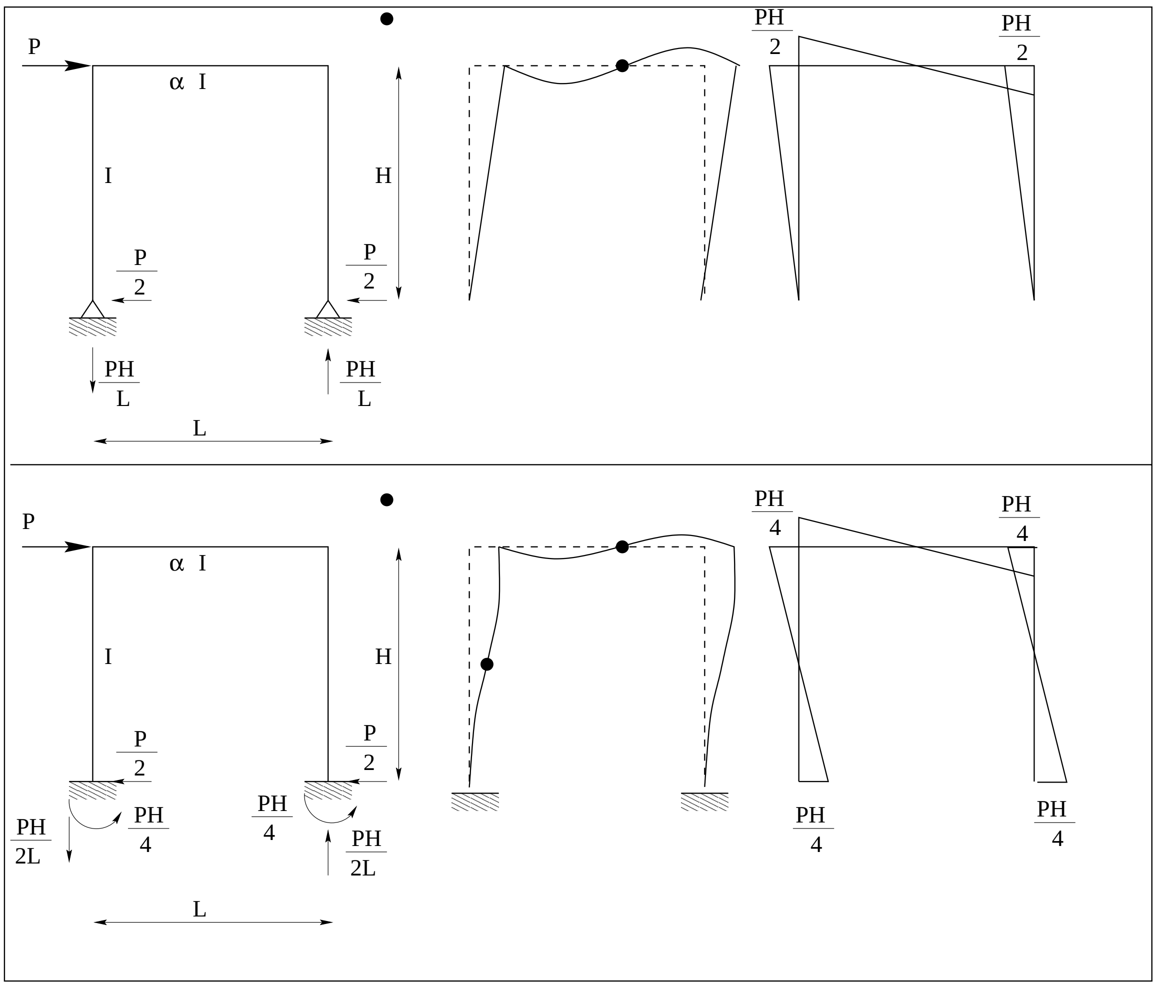


Figure 9.6: Horizontal Force Acting on a Frame, Approximate Location of Inflection Points

High rise buildings, where the height is several times greater than its least horizontal dimension, the deflected shape is dominated by overall flexural deformation.

9.2.1 Portal Method

15 Low rise buildings under lateral loads, have predominantly shear deformations. Thus, the approximate analysis of this type of structure is based on

1. Distribution of horizontal shear forces.
2. Location of inflection points.

16 The *portal method* is based on the following assumptions

1. Inflection points are located at
   (a) Mid-height of all columns above the second floor.
   (b) Mid-height of floor columns if rigid support, or at the base if hinged.
   (c) At the center of each girder.
2. Total horizontal shear at the mid-height of all columns at any floor level will be distributed among these columns so that each of the two exterior columns carry half as much horizontal shear as each interior columns of the frame.

17 Forces are obtained from

Column Shear is obtained by passing a horizontal section through the mid-height of the columns at each floor and summing the lateral forces above it, then Fig. 9.7

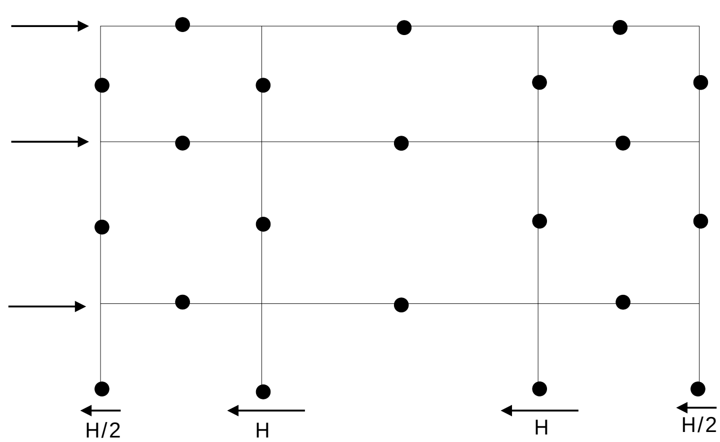


Figure 9.7: Approximate Analysis of Frames Subjected to Lateral Loads; Column Shear

$$\boxed{V^{ext} = \frac{\sum F^{lateral}}{2\text{No. of bays}} \qquad V^{int} = 2V^{ext}} \tag{9.7}$$

Column Moments at the end of each column is equal to the shear at the column times half the height of the corresponding column, Fig. 9.7

$$\boxed{M^{top} = V\frac{h}{2} \qquad M^{bot} = -M^{top}} \tag{9.8}$$

Girder Moments is obtained from the columns connected to the girder, Fig. 9.8

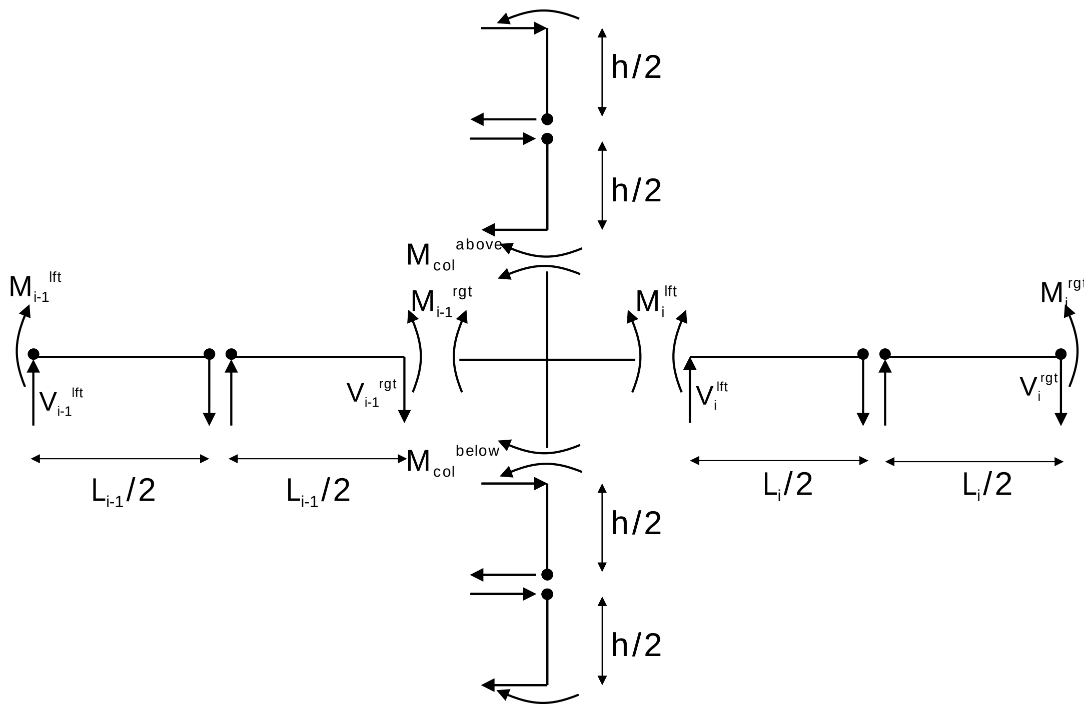


Figure 9.8: ***Approximate Analysis of Frames Subjected to Lateral Loads; Girder Moment

$$\boxed{M_i^{lft} = M_{col}^{above} - M_{col}^{below} + M_{i-1}^{rgt} \qquad M_i^{rgt} = -M_i^{lft}} \tag{9.9}$$

Girder Shears Since there is an inflection point at the center of the girder, the girder shear is obtained by considering the sum of moments about that point, Fig. 9.8

$$\boxed{V^{lft} = -\frac{2M}{L} \qquad V^{rgt} = V^{lft}} \tag{9.10}$$

Column Axial Forces are obtained by summing girder shears and the axial force from the column above, Fig. **??**

$$\boxed{P = P^{above} + P^{rgt} + P^{lft}} \tag{9.11}$$

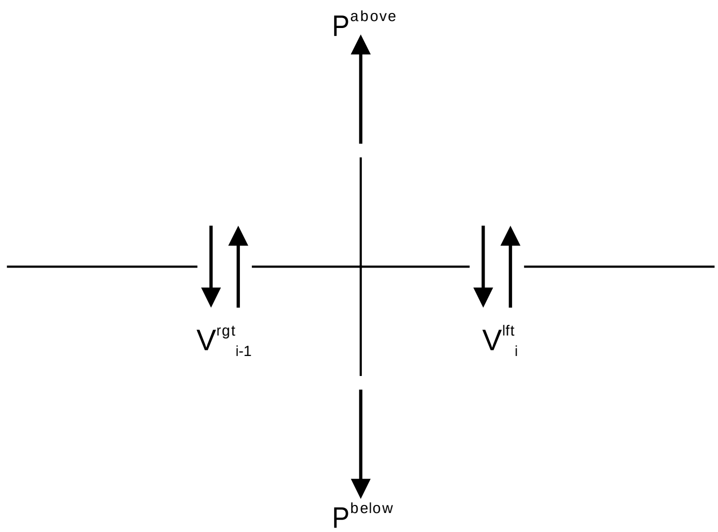


Figure 9.9: Approximate Analysis of Frames Subjected to Lateral Loads; Column Axial Force

18 In either case, you should **always** use a free body diagram in conjunction with this method, and **never** rely on a blind application of the formulae.

■ Example 9-1: Approximate Analysis of a Frame subjected to Vertical and Horizontal Loads

Draw the shear, and moment diagram for the following frame. **Solution:**

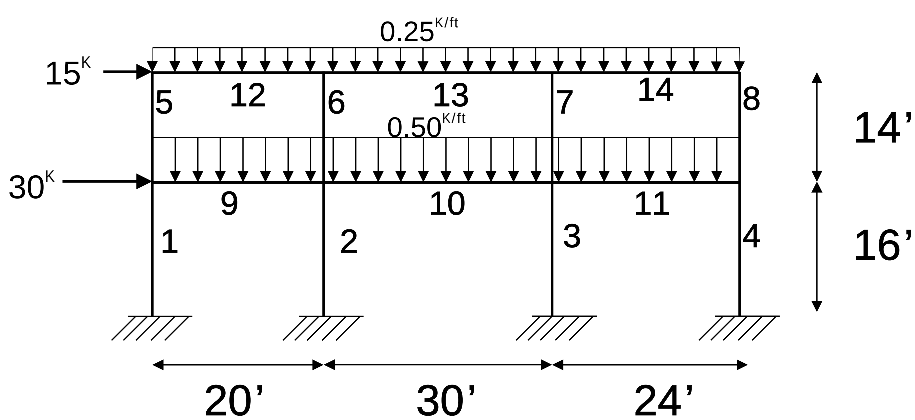


Figure 9.10: Example; Approximate Analysis of a Building

Vertical Loads

The analysis should be conducted in conjunction with the free body diagram shown in Fig. 9.11.

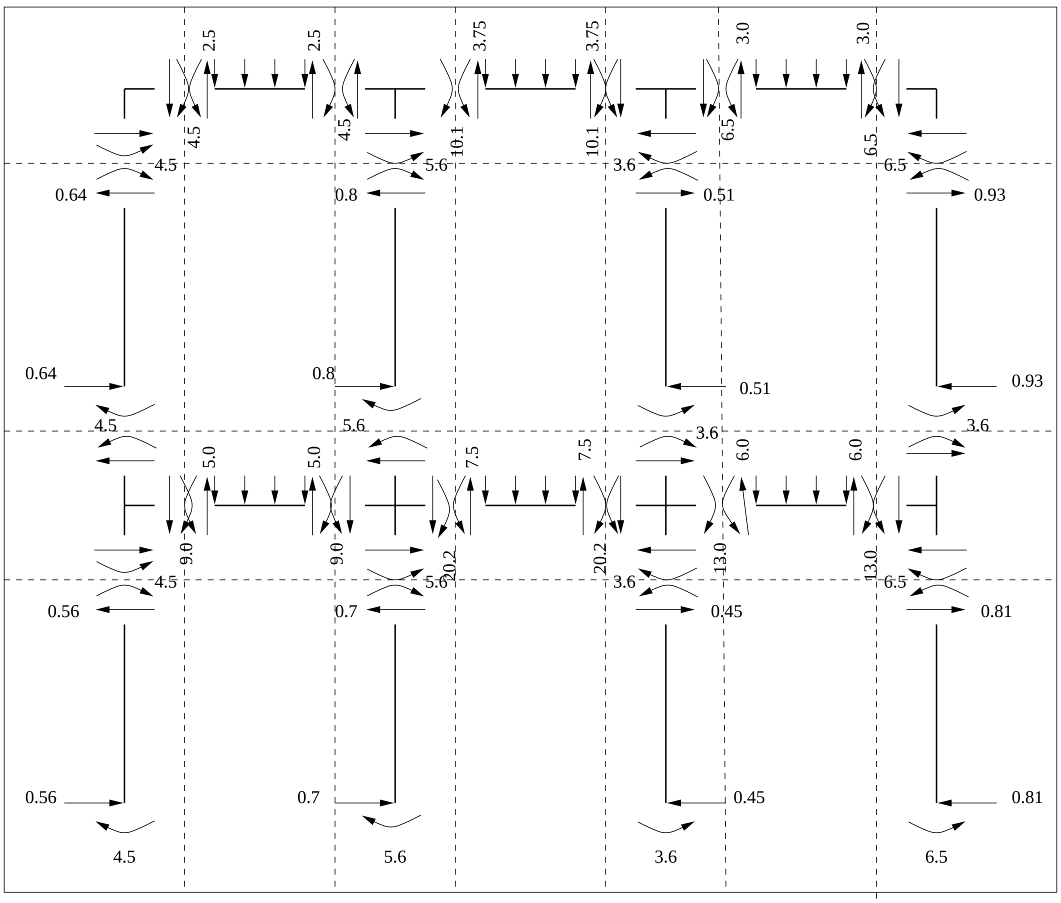


Figure 9.11: Free Body Diagram for the Approximate Analysis of a Frame Subjected to Vertical Loads

1. Top Girder Moments

$$\begin{array}{lcll}
M_{12}^{lft} & = & -0.045w_{12}L_{12}^2 = -(0.045)(0.25)(20)^2 & = - \quad 4.5 \text{ k.ft} \\
M_{12}^{cnt} & = & 0.08w_{12}L_{12}^2 = (0.08)(0.25)(20)^2 & = \quad 8.0 \text{ k.ft} \\
M_{12}^{rgt} & = & M_{12}^{lft} & = - \quad 4.5 \text{ k.ft} \\
M_{13}^{lft} & = & -0.045w_{13}L_{13}^2 = -(0.045)(0.25)(30)^2 & = - \quad 10.1 \text{ k.ft} \\
M_{13}^{cnt} & = & 0.08w_{13}L_{13}^2 = (0.08)(0.25)(30)^2 & = \quad 18.0 \text{ k.ft} \\
M_{13}^{rgt} & = & M_{13}^{lft} & = - \quad 10.1 \text{ k.ft} \\
M_{14}^{lft} & = & -0.045w_{14}L_{14}^2 = -(0.045)(0.25)(24)^2 & = - \quad 6.5 \text{ k.ft} \\
M_{14}^{cnt} & = & 0.08w_{14}L_{14}^2 = (0.08)(0.25)(24)^2 & = \quad 11.5 \text{ k.ft} \\
M_{14}^{rgt} & = & M_{14}^{lft} & = - \quad 6.5 \text{ k.ft}
\end{array}$$

2. Bottom Girder Moments

$$\begin{array}{lcll}
M_{9}^{lft} & = & -0.045w_{9}L_{9}^2 = -(0.045)(0.5)(20)^2 & = - \quad 9.0 \text{ k.ft} \\
M_{9}^{cnt} & = & 0.08w_{9}L_{9}^2 = (0.08)(0.5)(20)^2 & = \quad 16.0 \text{ k.ft} \\
M_{9}^{rgt} & = & M_{9}^{lft} & = - \quad 9.0 \text{ k.ft} \\
M_{10}^{lft} & = & -0.045w_{10}L_{10}^2 = -(0.045)(0.5)(30)^2 & = - \quad 20.3 \text{ k.ft} \\
M_{10}^{cnt} & = & 0.08w_{10}L_{10}^2 = (0.08)(0.5)(30)^2 & = \quad 36.0 \text{ k.ft} \\
M_{10}^{rgt} & = & M_{11}^{lft} & = - \quad 20.3 \text{ k.ft} \\
M_{11}^{lft} & = & -0.045w_{12}L_{12}^2 = -(0.045)(0.5)(24)^2 & = - \quad 13.0 \text{ k.ft} \\
M_{11}^{cnt} & = & 0.08w_{12}L_{12}^2 = (0.08)(0.5)(24)^2 & = \quad 23.0 \text{ k.ft} \\
M_{11}^{rgt} & = & M_{12}^{lft} & = - \quad 13.0 \text{ k.ft}
\end{array}$$

3. Top Column Moments

$$\begin{array}{lcll}
M_{5}^{top} & = & +M_{12}^{lft} & = - \quad 4.5 \text{ k.ft} \\
M_{5}^{bot} & = & -M_{5}^{top} & = \quad 4.5 \text{ k.ft} \\
M_{6}^{top} & = & -M_{12}^{rgt} + M_{13}^{lft} = -(-4.5) + (-10.1) & = - \quad 5.6 \text{ k.ft} \\
M_{6}^{bot} & = & -M_{6}^{top} & = \quad 5.6 \text{ k.ft} \\
M_{7}^{top} & = & -M_{13}^{rgt} + M_{14}^{lft} = -(-10.1) + (-6.5) & = - \quad 3.6 \text{ k.ft} \\
M_{7}^{bot} & = & -M_{7}^{top} & = \quad 3.6 \text{ k.ft} \\
M_{8}^{top} & = & -M_{14}^{rgt} = -(-6.5) & = \quad 6.5 \text{ k.ft} \\
M_{8}^{bot} & = & -M_{8}^{top} & = - \quad 6.5 \text{ k.ft}
\end{array}$$

4. Bottom Column Moments

$$\begin{array}{lcll}
M_{1}^{top} & = & +M_{5}^{bot} + M_{9}^{lft} = 4.5 - 9.0 & = - \quad 4.5 \text{ k.ft} \\
M_{1}^{bot} & = & -M_{1}^{top} & = \quad 4.5 \text{ k.ft} \\
M_{2}^{top} & = & +M_{6}^{bot} - M_{9}^{rgt} + M_{10}^{lft} = 5.6 - (-9.0) + (-20.3) & = - \quad 5.6 \text{ k.ft} \\
M_{2}^{bot} & = & -M_{2}^{top} & = \quad 5.6 \text{ k.ft} \\
M_{3}^{top} & = & +M_{7}^{bot} - M_{10}^{rgt} + M_{11}^{lft} = -3.6 - (-20.3) + (-13.0) & = \quad 3.6 \text{ k.ft} \\
M_{3}^{bot} & = & -M_{3}^{top} & = - \quad 3.6 \text{ k.ft} \\
M_{4}^{top} & = & +M_{8}^{bot} - M_{11}^{rgt} = -6.5 - (-13.0) & = \quad 6.5 \text{ k.ft} \\
M_{4}^{bot} & = & -M_{4}^{top} & = - \quad 6.5 \text{ k.ft}
\end{array}$$

5. Top Girder Shear

$$\begin{array}{lcll}
V_{12}^{lft} & = & \frac{w_{12}L_{12}}{2} = \frac{(0.25)(20)}{2} & = \quad 2.5 \text{ k} \\
V_{12}^{rgt} & = & -V_{12}^{lft} & = - \quad 2.5 \text{ k} \\
V_{13}^{lft} & = & \frac{w_{13}L_{13}}{2} = \frac{(0.25)(30)}{2} & = \quad 3.75 \text{ k} \\
V_{13}^{rgt} & = & -V_{13}^{lft} & = - \quad 3.75 \text{ k} \\
V_{14}^{lft} & = & \frac{w_{14}L_{14}}{2} = \frac{(0.25)(24)}{2} & = \quad 3.0 \text{ k} \\
V_{14}^{rgt} & = & -V_{14}^{lft} & = - \quad 3.0 \text{ k}
\end{array}$$

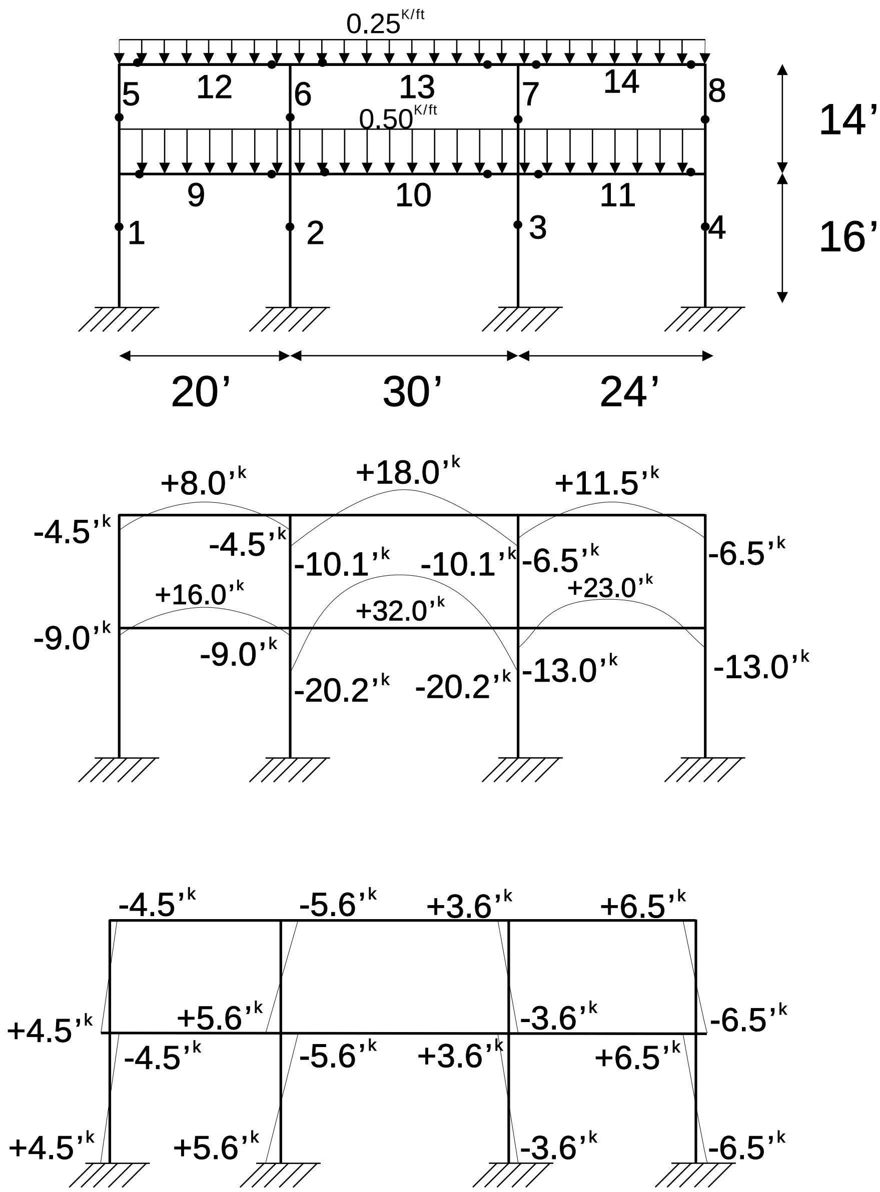


Figure 9.12: Approximate Analysis of a Building; Moments Due to Vertical Loads

6. Bottom Girder Shear

$$
\begin{array}{rcll}
V_9^{lft} & = & \frac{w_9L_9}{2} = \frac{(0.5)(20)}{2} & = 5.00 \text{ k} \\
V_9^{rgt} & = & -V_9^{lft} & = - 5.00 \text{ k} \\
V_{10}^{lft} & = & \frac{w_{10}L_{10}}{2} = \frac{(0.5)(30)}{2} & = 7.50 \text{ k} \\
V_{10}^{rgt} & = & -V_{10}^{lft} & = - 7.50 \text{ k} \\
V_{11}^{lft} & = & \frac{w_{11}L_{11}}{2} = \frac{(0.5)(24)}{2} & = 6.00 \text{ k} \\
V_{11}^{rgt} & = & -V_{11}^{lft} & = - 6.00 \text{ k}
\end{array}
$$

7. Column Shears

$$
\begin{array}{rcll}
V_5 & = & \frac{M_5^{top}}{\frac{H_5}{2}} = \frac{-4.5}{\frac{14}{2}} & = - 0.64 \text{ k} \\
V_6 & = & \frac{M_6^{top}}{\frac{H_6}{2}} = \frac{-5.6}{\frac{14}{2}} & = - 0.80 \text{ k} \\
V_7 & = & \frac{M_7^{top}}{\frac{H_7}{2}} = \frac{3.6}{\frac{14}{2}} & = 0.52 \text{ k} \\
V_8 & = & \frac{M_8^{top}}{\frac{H_8}{2}} = \frac{6.5}{\frac{14}{2}} & = 0.93 \text{ k} \\
V_1 & = & \frac{M_1^{top}}{\frac{H_1}{2}} = \frac{-4.5}{\frac{16}{2}} & = - 0.56 \text{ k} \\
V_2 & = & \frac{M_2^{top}}{\frac{H_2}{2}} = \frac{-5.6}{\frac{16}{2}} & = - 0.70 \text{ k} \\
V_3 & = & \frac{M_3^{top}}{\frac{H_3}{2}} = \frac{3.6}{\frac{16}{2}} & = 0.46 \text{ k} \\
V_4 & = & \frac{M_4^{top}}{\frac{H_4}{2}} = \frac{6.5}{\frac{16}{2}} & = 0.81 \text{ k}
\end{array}
$$

8. Top Column Axial Forces

$$
\begin{array}{rcll}
P_5 & = & V_{12}^{lft} & = 2.50 \text{ k} \\
P_6 & = & -V_{12}^{rgt} + V_{13}^{lft} = -(-2.50) + 3.75 & = 6.25 \text{ k} \\
P_7 & = & -V_{13}^{rgt} + V_{14}^{lft} = -(-3.75) + 3.00 & = 6.75 \text{ k} \\
P_8 & = & -V_{14}^{rgt} & = 3.00 \text{ k}
\end{array}
$$

9. Bottom Column Axial Forces

$$
\begin{array}{rcll}
P_1 & = & P_5 + V_9^{lft} = 2.50 + 5.0 & = 7.5 \text{ k} \\
P_2 & = & P_6 - V_{10}^{rgt} + V_9^{lft} = 6.25 - (-5.00) + 7.50 & = 18.75 \text{ k} \\
P_3 & = & P_7 - V_{11}^{rgt} + V_{10}^{lft} = 6.75 - (-7.50) + 6.0 & = 20.25 \text{ k} \\
P_4 & = & P_8 - V_{11}^{rgt} = 3.00 - (-6.00) & = 9.00 \text{ k}
\end{array}
$$

Horizontal Loads, Portal Method

1. Column Shears

$$
\begin{array}{rcll}
V_5 & = & \frac{15}{(2)(3)} & = 2.5 \text{ k} \\
V_6 & = & 2(V_5) = (2)(2.5) & = 5 \text{ k} \\
V_7 & = & 2(V_5) = (2)(2.5) & = 5 \text{ k} \\
V_8 & = & V_5 & = 2.5 \text{ k} \\
V_1 & = & \frac{15+30}{(2)(3)} & = 7.5 \text{ k} \\
V_2 & = & 2(V_1) = (2)(7.5) & = 15 \text{ k} \\
V_3 & = & 2(V_1) = (2)(2.5) & = 15 \text{ k} \\
V_4 & = & V_1 & = 7.5 \text{ k}
\end{array}
$$

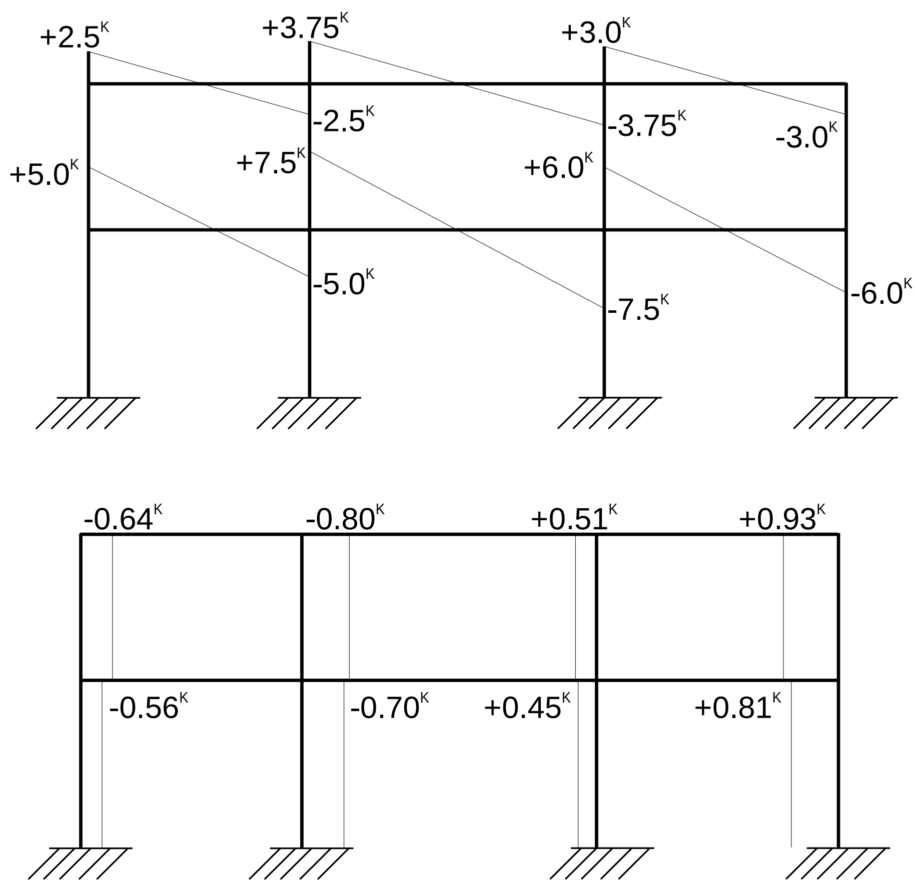


Figure 9.13: Approximate Analysis of a Building; Shears Due to Vertical Loads

Approximate Analysis Vertical Loads APROXVER.XLS Victor E. Saouma

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1																	
2					L1					L2					L3		
3	**Height**	**Span**			**20**					**30**					**24**		
4	**14**	**Load**			**0.25**					**0.25**					**0.25**		
5	**16**	**Load**			**0.5**					**0.5**					**0.5**		
6										**MOMENTS**							
7					**Bay 1**					**Bay 2**					**Bay 3**		
8			Col		Beam		Column			Beam		Column			Beam		Col
9				Lft	Cnt	Rgt			Lft	Cnr	Rgt			Lft	Cnt	Rgt	
10				-4.5	8.0	-4.5			-10.1	18.0	-10.1			-6.5	11.5	-6.5	
11			-4.5				-5.6					3.6					6.5
12			4.5				5.6					-3.6					-6.5
13				-9.0	16.0	-9.0			-20.3	36.0	-20.3			-13.0	23.0	-13.0	
14			-4.5				-5.6					3.6					6.5
15			4.5				5.6					-3.6					-6.5
16										**SHEAR**							
17					**Bay 1**					**Bay 2**					**Bay 3**		
18			Col		Beam		Column			Beam		Column			Beam		Col
19				Lft		Rgt			Lft		Rgt			Lft		Rgt	
20				2.50		-2.50			3.75		-3.75			3.00		-3.00	
21			-0.64				-0.80					0.52					0.93
22				5.00		-5.00			7.50		-7.50			6.00		-6.00	
23			-0.56				-0.70					0.46					0.81
24										**AXIAL FORCE**							
25					**Bay 1**					**Bay 2**					**Bay 3**		
26			Col		Beam		Column			Beam		Column			Beam		Col
27					0.00					0.00					0.00		
28			2.50				6.25					6.75					3.00
29					0.00					0.00					0.00		
30			7.50				18.75					20.25					9.00

Figure 9.14: Approximate Analysis for Vertical Loads; Spread-Sheet Format

Approximate Analysis Vertical Loads APROXVER.XLS Victor E. Saouma

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1																	
2				L1					L2					L3			
3	Height	Span		20					30					24			
4	14	Load		0.25					0.25					0.25			
5	16	Load		0.5					0.5					0.5			
6			MOMENTS														
7			Bay 1					Bay 2					Bay 3				
8			Col	Beam			Column		Beam			Column		Beam			Col
9				Lft	Cnt	Rgt			Lft	Cnr	Rgt			Lft	Cnt	Rgt	
10				=-0.045*D4*D3^2	=0.08*D4*D3*D3	=+D10			=-0.045*I4*I3^2	=0.08*I4*I3*I3	=+I10			=-0.045*N4*N3^2	=0.08*N4*N3*N3	=N10	
11			=+D10				=-F10+I10					=-K10+N10					=-P10
12			=-C11				=-G11					=-L11					=-Q11
13				=-0.045*D5*D3^2	=0.08*D5*D3*D3	=+D13			=-0.045*I5*I3^2	=0.08*I5*I3*I3	=+I13			=-0.045*N5*N3^2	=0.08*N5*N3*N3	=+N13	
14			=+D13+C12				=-F13+I13+G12					=-K13+N13+L12					=-P13+Q12
15			=-C14				=-G14					=-L14					=-Q14
16			SHEAR														
17			Bay 1					Bay 2					Bay 3				
18			Col	Beam			Column		Beam			Column		Beam			Col
19				Lft		Rgt			Lft		Rgt			Lft		Rgt	
20				=+D3*D4/2		=-D20			=+I3*I4/2		=-I20			=+N3*N4/2		=-N20	
21			=2*C11/A4				=2*G11/A4					=2*L11/A4					=2*Q11/A4
22				=+D3*D5/2		=-D22			=+I3*I5/2		=-I22			=+N3*N5/2		=-N22	
23			=2*C14/A5				=2*G14/A5					=2*L14/A5					=2*Q14/A5
24			AXIAL FORCE														
25			Bay 1					Bay 2					Bay 3				
26			Col	Beam			Column		Beam			Column		Beam			Col
27				0					0					0			
28			=+D20				=-F20+I20					=-K20+N20					=-P20
29				0					0					0			
30			=+C28+D22				=+G28-F22+I22					=+L28-K22+N22					=+Q28-P22

Figure 9.15: Approximate Analysis for Vertical Loads; Equations in Spread-Sheet

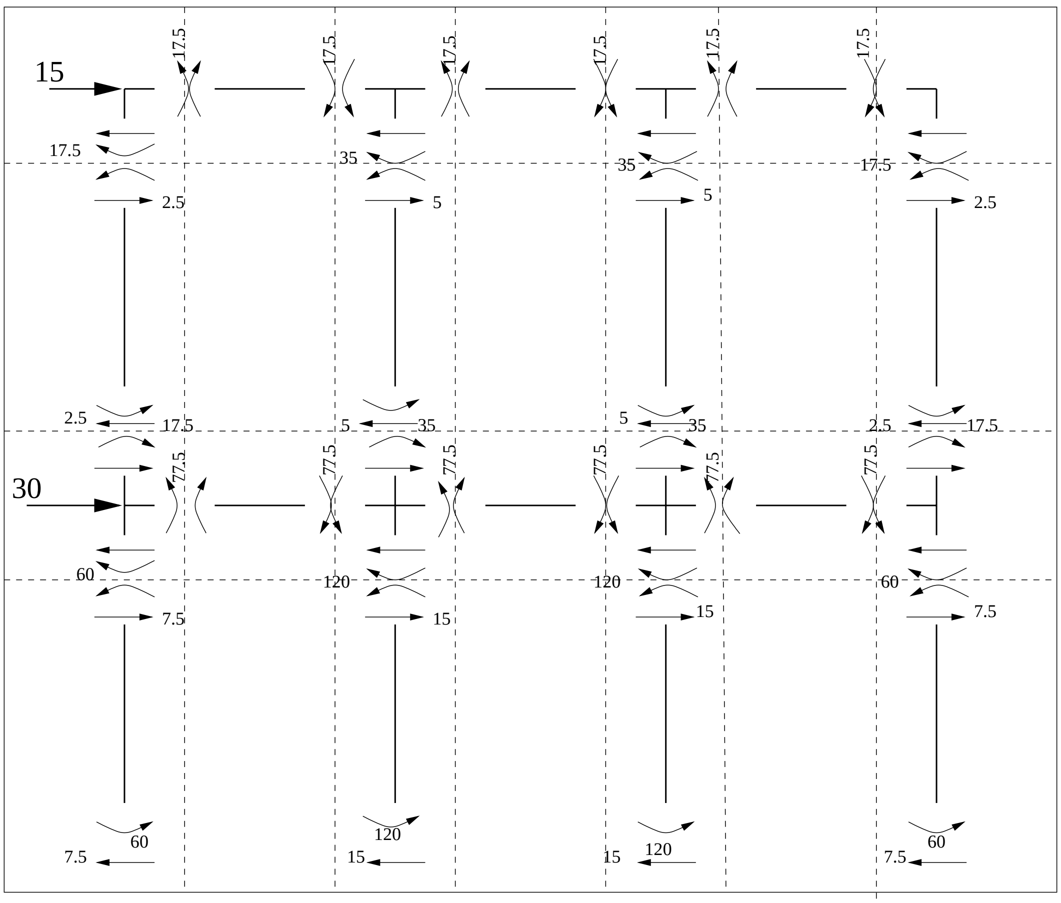


Figure 9.16: Free Body Diagram for the Approximate Analysis of a Frame Subjected to Lateral Loads

2. Top Column Moments

$$\begin{aligned}
M_5^{top} &= \frac{V_1H_5}{2} = \frac{(2.5)(14)}{2} &=& \quad 17.5 \text{ k.ft}\\
M_5^{bot} &= -M_5^{top} &=-& \quad 17.5 \text{ k.ft}\\
M_6^{top} &= \frac{V_6H_6}{2} = \frac{(5)(14)}{2} &=& \quad 35.0 \text{ k.ft}\\
M_6^{bot} &= -M_6^{top} &=-& \quad 35.0 \text{ k.ft}\\
M_7^{top} &= \frac{V_7^{up}H_7}{2} = \frac{(5)(14)}{2} &=& \quad 35.0 \text{ k.ft}\\
M_7^{bot} &= -M_7^{top} &=-& \quad 35.0 \text{ k.ft}\\
M_8^{top} &= \frac{V_8^{up}H_8}{2} = \frac{(2.5)(14)}{2} &=& \quad 17.5 \text{ k.ft}\\
M_8^{bot} &= -M_8^{top} &=-& \quad 17.5 \text{ k.ft}
\end{aligned}$$

3. Bottom Column Moments

$$\begin{aligned}
M_1^{top} &= \frac{V_1^{dwn}H_1}{2} = \frac{(7.5)(16)}{2} &=& \quad 60 \text{ k.ft}\\
M_1^{bot} &= -M_1^{top} &=-& \quad 60 \text{ k.ft}\\
M_2^{top} &= \frac{V_2^{dwn}H_2}{2} = \frac{(15)(16)}{2} &=& \quad 120 \text{ k.ft}\\
M_2^{bot} &= -M_2^{top} &=-& \quad 120 \text{ k.ft}\\
M_3^{top} &= \frac{V_3^{dwn}H_3}{2} = \frac{(15)(16)}{2} &=& \quad 120 \text{ k.ft}\\
M_3^{bot} &= -M_3^{top} &=-& \quad 120 \text{ k.ft}\\
M_4^{top} &= \frac{V_4^{dwn}H_4}{2} = \frac{(7.5)(16)}{2} &=& \quad 60 \text{ k.ft}\\
M_4^{bot} &= -M_4^{top} &=-& \quad 60 \text{ k.ft}
\end{aligned}$$

4. Top Girder Moments

$$\begin{aligned}
M_{12}^{lft} &= M_5^{top} &=& \quad 17.5 \text{ k.ft}\\
M_{12}^{rgt} &= -M_{12}^{lft} &=-& \quad 17.5 \text{ k.ft}\\
M_{13}^{lft} &= M_{12}^{rgt} + M_6^{top} = -17.5 + 35 &=& \quad 17.5 \text{ k.ft}\\
M_{13}^{rgt} &= -M_{13}^{lft} &=-& \quad 17.5 \text{ k.ft}\\
M_{14}^{lft} &= M_{13}^{rgt} + M_7^{top} = -17.5 + 35 &=& \quad 17.5 \text{ k.ft}\\
M_{14}^{rgt} &= -M_{14}^{lft} &=-& \quad 17.5 \text{ k.ft}
\end{aligned}$$

5. Bottom Girder Moments

$$\begin{aligned}
M_9^{lft} &= M_1^{top} - M_5^{bot} = 60 - (-17.5) &=& \quad 77.5 \text{ k.ft}\\
M_9^{rgt} &= -M_9^{lft} &=-& \quad 77.5 \text{ k.ft}\\
M_{10}^{lft} &= M_9^{rgt} + M_2^{top} - M_6^{bot} = -77.5 + 120 - (-35) &=& \quad 77.5 \text{ k.ft}\\
M_{10}^{rgt} &= -M_{10}^{lft} &=-& \quad 77.5 \text{ k.ft}\\
M_{11}^{lft} &= M_{10}^{rgt} + M_3^{top} - M_7^{bot} = -77.5 + 120 - (-35) &=& \quad 77.5 \text{ k.ft}\\
M_{11}^{rgt} &= -M_{11}^{lft} &=-& \quad 77.5 \text{ k.ft}
\end{aligned}$$

6. Top Girder Shear

$$\begin{aligned}
V_{12}^{lft} &= -\frac{2M_{12}^{lft}}{L_{12}} = -\frac{(2)(17.5)}{20} &=& \quad -1.75 \text{ k}\\
V_{12}^{rgt} &= +V_{12}^{lft} &=& \quad -1.75 \text{ k}\\
V_{13}^{lft} &= -\frac{2M_{13}^{lft}}{L_{13}} = -\frac{(2)(17.5)}{30} &=& \quad -1.17 \text{ k}\\
V_{13}^{rgt} &= +V_{13}^{lft} &=& \quad -1.17 \text{ k}\\
V_{14}^{lft} &= -\frac{2M_{14}^{lft}}{L_{14}} = -\frac{(2)(17.5)}{24} &=& \quad -1.46 \text{ k}\\
V_{14}^{rgt} &= +V_{14}^{lft} &=& \quad -1.46 \text{ k}
\end{aligned}$$

7. Bottom Girder Shear

$$\begin{aligned}
V_9^{lft} &= -\frac{2M_{12}^{lft}}{L_9} = -\frac{(2)(77.5)}{20} &=& \quad -7.75 \text{ k}\\
V_9^{rgt} &= +V_9^{lft} &=& \quad -7.75 \text{ k}\\
V_{10}^{lft} &= -\frac{2M_{10}^{lft}}{L_{10}} = -\frac{(2)(77.5)}{30} &=& \quad -5.17 \text{ k}\\
V_{10}^{rgt} &= +V_{10}^{lft} &=& \quad -5.17 \text{ k}\\
V_{11}^{lft} &= -\frac{2M_{11}^{lft}}{L_{11}} = -\frac{(2)(77.5)}{24} &=& \quad -6.46 \text{ k}\\
V_{11}^{rgt} &= +V_{11}^{lft} &=& \quad -6.46 \text{ k}
\end{aligned}$$

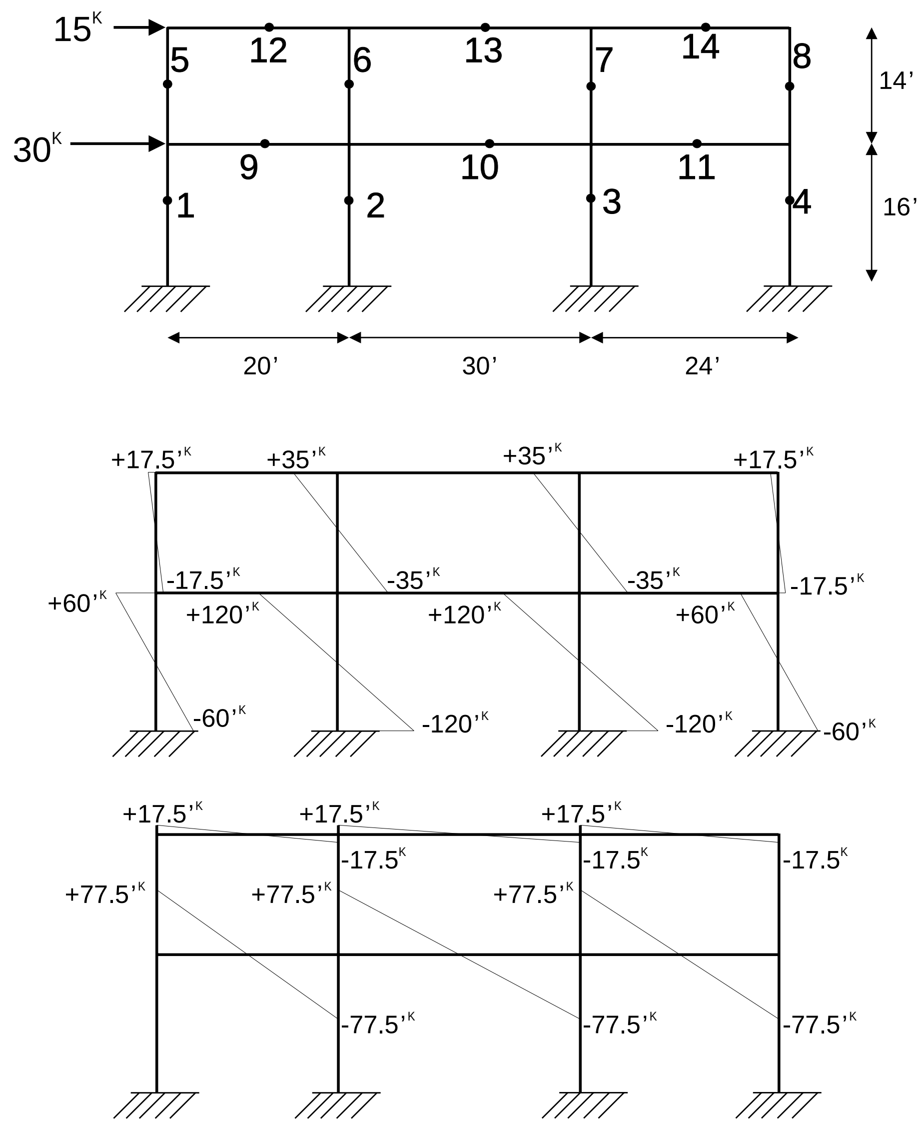


Figure 9.17: Approximate Analysis of a Building; Moments Due to Lateral Loads

8. Top Column Axial Forces (+ve tension, -ve compression)

$$\begin{array}{rcll} P_5 & = & -V_{12}^{lft} & = & -(-1.75)\text{ k} \\ P_6 & = & +V_{12}^{rgt} - V_{13}^{lft} = -1.75 - (-1.17) & = & -0.58\text{ k} \\ P_7 & = & +V_{13}^{rgt} - V_{14}^{lft} = -1.17 - (-1.46) & = & 0.29\text{ k} \\ P_8 & = & V_{14}^{rgt} = -1.46\text{ k} & & \end{array}$$

9. Bottom Column Axial Forces (+ve tension, -ve compression)

$$\begin{array}{rcll} P_1 & = & P_5 + V_9^{lft} = 1.75 - (-7.75) & = & 9.5\text{ k} \\ P_2 & = & P_6 + V_{10}^{rgt} + V_9^{lft} = -0.58 - 7.75 - (-5.17) & = & -3.16\text{ k} \\ P_3 & = & P_7 + V_{11}^{rgt} + V_{10}^{lft} = 0.29 - 5.17 - (-6.46) & = & 1.58\text{ k} \\ P_4 & = & P_8 + V_{11}^{rgt} = -1.46 - 6.46 & = & -7.66\text{ k} \end{array}$$

Design Parameters On the basis of the two approximate analyses, vertical and lateral load, we now seek the design parameters for the frame, Table 9.2.

Mem.		Vert.	Hor.	**Design Values**
1	Moment	4.50	60.00	**64.50**
	Axial	7.50	9.50	**17.00**
	Shear	0.56	7.50	**8.06**
2	Moment	5.60	120.00	**125.60**
	Axial	18.75	15.83	**34.58**
	Shear	0.70	15.00	**15.70**
3	Moment	3.60	120.00	**123.60**
	Axial	20.25	14.25	**34.50**
	Shear	0.45	15.00	**15.45**
4	Moment	6.50	60.00	**66.50**
	Axial	9.00	7.92	**16.92**
	Shear	0.81	7.50	**8.31**
5	Moment	4.50	17.50	**22.00**
	Axial	2.50	1.75	**4.25**
	Shear	0.64	2.50	**3.14**
6	Moment	5.60	35.00	**40.60**
	Axial	6.25	2.92	**9.17**
	Shear	0.80	5.00	**5.80**
7	Moment	3.60	35.00	**38.60**
	Axial	6.75	2.63	**9.38**
	Shear	0.51	5.00	**5.51**
8	Moment	6.50	17.50	**24.00**
	Axial	3.00	1.46	**4.46**
	Shear	0.93	2.50	**3.43**

Table 9.1: Columns Combined Approximate Vertical and Horizontal Loads

■

Portal Method PORTAL.XLS Victor E. Saouma

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	PORTAL METHOD																		
2	# of Bays					3			L1				L2				L3		
3									20				30				24		
4								MOMENTS											
5	# of Storeys					2		Bay 1					Bay 2				Bay 3		
6				Force	Shear			Col	Beam		Column		Beam		Column		Beam		Col
7		H	Lat.	Tot	Ext	Int			Lft	Rgt			Lft	Rgt			Lft	Rgt	
8									17.5	-17.5			17.5	-17.5			17.5	-17.5	
9	H1	14	15	15	2.5	5		17.5			35.0				35.0				17.5
10								-17.5			-35.0				-35.0				-17.5
11									77.5	-77.5			77.5	-77.5			77.5	-77.5	
12	H2	16	30	45	7.5	15		60.0			120.0				120.0				60.0
13								-60.0			-120.0				-120.0				-60.0
14								SHEAR											
15								Bay 1					Bay 2				Bay 3		
16								Col	Beam		Column		Beam		Column		Beam		Col
17									Lft	Rgt			Lft	Rgt			Lft	Rgt	
18									-1.75	-1.75			-1.17	-1.17			-1.46	-1.46	
19								2.50			5.00				5.00				2.50
20								2.50			5.00				5.00				2.50
21									-7.75	-7.75			-5.17	-5.17			-6.46	-6.46	
22								7.50			15.00				15.00				7.50
23								7.50			15.00				15.00				7.50
24								AXIAL FORCE											
25								Bay 1					Bay 2				Bay 3		
26								Col	Beam		Column		Beam		Column		Beam		Col
27									0.00				0.00				0.00		
28								1.75			-0.58				0.29				-1.46
29									0.00				0.00				0.00		
30								9.50			-3.17				1.58				-7.92

Figure 9.18: Portal Method; Spread-Sheet Format

Portal Method PORTAL.XLS Victor E. Saouma

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	**PORTAL METHOD**																		
2	**# of Bays**					**3**			**L1**				**L2**				**L3**		
3									**20**				**30**				**24**		
4								**MOMENTS**											
5	**# of Storeys**					**2**		**Bay 1**				**Bay 2**				**Bay 3**			
6			Force		Shear			Col	Beam		Column		Beam		Column		Beam		Col
7		**H**	**Lat.**	Tot	Ext	Int			Lft	Rgt			Lft	Rgt			Lft	Rgt	
8									=+H9	=-I8			=+J8+K9	=-M8			=+N8+O9	=-Q8	
9	**H1**	**14**	**15**	=+C9	=+D9/(2*F2)	=2*E9		=+E9*B9/2			=+F9*B9/2				=+K9				=+H9
10								=-H9			=-K9				=+K10				=+H10
11									=+H12-H10	=-I11			=+K12-K10+J11	=-M11			=+O12-O10+N11	=-Q11	
12	**H2**	**16**	**30**	=SUM(C9:C12)	=+D12/(2*F2)	=2*E12		=+E12*B12/2			=+F12*B12/2				=+K12				=+H12
13								=-H12			=-K12				=+K13				=+H13
14								**SHEAR**											
15								**Bay 1**				**Bay 2**				**Bay 3**			
16								Col	Beam		Column		Beam		Column		Beam		Col
17									Lft	Rgt			Lft	Rgt			Lft	Rgt	
18									=-2*I8/I$3	=+I18			=-2*M8/M$3	=+M18			=-2*Q8/Q$3	=+Q18	
19								=+E9			=+F9				=+F9				=+E9
20								=+H19			=+K19				=+O19				=+S19
21									=-2*I11/I$3	=+I21			=-2*M11/M$3	=+M21			=-2*Q11/Q$3	=+Q21	
22								=+E12			=+F12				=+F12				=+E12
23								=+H22			=+K22				=+O22				=+S22
24								**AXIAL FORCE**											
25								**Bay 1**				**Bay 2**				**Bay 3**			
26								Col	Beam		Column		Beam		Column		Beam		Col
27									0				0				0		
28								=-I18			=+J18-M18				=+N18-Q18				=+R18
29									0				0				0		
30								=+H28-I21			=+K28+J21-M21				=+O28+N21-Q21				=+S28+R21

Figure 9.19: Portal Method; Equations in Spread-Sheet

Mem.		Vert.	Hor.	**Design Values**
9	-ve Moment	9.00	77.50	**86.50**
	+ve Moment	16.00	0.00	**16.00**
	Shear	5.00	7.75	**12.75**
10	-ve Moment	20.20	77.50	**97.70**
	+ve Moment	36.00	0.00	**36.00**
	Shear	7.50	5.17	**12.67**
11	-ve Moment	13.0	77.50	**90.50**
	+ve Moment	23.00	0.00	**23.00**
	Shear	6.00	6.46	**12.46**
12	-ve Moment	4.50	17.50	**22.00**
	+ve Moment	8.00	0.00	**8.00**
	Shear	2.50	1.75	**4.25**
13	-ve Moment	10.10	17.50	**27.60**
	+ve Moment	18.00	0.00	**18.00**
	Shear	3.75	1.17	**4.92**
14	-ve Moment	6.50	17.50	**24.00**
	+ve Moment	11.50	0.00	**11.50**
	Shear	3.00	1.46	**4.46**

Table 9.2: Girders Combined Approximate Vertical and Horizontal Loads

Chapter 10

STATIC INDETERMINANCY; FLEXIBILITY METHOD

`All the examples in this chapter are taken verbatim from White, Gergely and Sexmith`

10.1 Introduction

1 A statically indeterminate structure has more unknowns than equations of equilibrium (and equations of conditions if applicable).

2 The advantages of a statically indeterminate structures are:

1. Lower internal forces
2. Safety in redundancy, i.e. if a support or members fails, the structure can *redistribute* its internal forces to accomodate the changing B.C. without resulting in a sudden failure.

3 Only disadvantage is that it is more complicated to analyse.

4 Analysis mehtods of statically indeterminate structures *must satisfy* three requirements

Equilibrium

Force-displacement (or stress-strain) relations (linear elastic in this course).

Compatibility of displacements (i.e. no discontinuity)

5 This can be achieved through two classes of solution

Force or Flexibility method;

Displacement or Stiffness method

6 The flexibility method is first illustrated by the following problem of a statically indeterminate cable structure in which a rigid plate is supported by two aluminum cables and a steel one. We seek to determine the force in each cable, Fig. 10.1

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.
2. Applying the equations of **equilibrium**

$$\begin{aligned} \Sigma M_z = 0; \quad &\Rightarrow \quad P_{Al}^{\text{left}} = P_{Al}^{\text{right}} \\ \Sigma F_y = 0; \quad &\Rightarrow \quad 2P_{Al} + P_{St} = P \end{aligned} \tag{10.1-a}$$

   Thus we effectively have two unknowns and one equation.

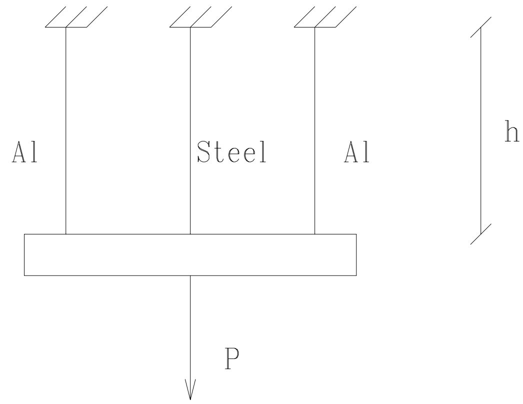


Figure 10.1: Statically Indeterminate 3 Cable Structure

3. We need to have a third equation to solve for the three unknowns. This will be derived from the **compatibility of the displacements** in all three cables, i.e. all three displacements must be equal:

$$D_{Al} = D_{St} \tag{10.2}$$

4. Finally, those isplacements are obtained from the **Force-Displacement** relations:

$$\left.\begin{array}{rcl} \sigma & = & \frac{P}{A} \\ \varepsilon & = & \frac{\Delta L}{L} \\ \varepsilon & = & \frac{\sigma}{E} \end{array}\right\} \Rightarrow \Delta L = \frac{PL}{AE} \tag{10.3}$$

$$\underbrace{\frac{P_{Al}L}{E_{Al}A_{Al}}}_{D_{Al}} = \underbrace{\frac{P_{St}L}{E_{St}A_{St}}}_{D_{St}} \Rightarrow \frac{P_{Al}}{P_{St}} = \frac{(EA)_{Al}}{(EA)_{St}} \tag{10.4}$$

or

$$-(EA)_{St}P_{Al} + (EA)_{Al}P_{St} = 0 \tag{10.5}$$

5. Solution of Eq. 10.1-a and 10.5 yields

$$\begin{array}{l} \left[\begin{array}{cc} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{array}\right] \left\{\begin{array}{c} P_{Al} \\ P_{St} \end{array}\right\} = \left\{\begin{array}{c} P \\ 0 \end{array}\right\} \\ \Rightarrow \left\{\begin{array}{c} P_{Al} \\ P_{St} \end{array}\right\} = \left[\begin{array}{cc} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{array}\right]^{-1} \left\{\begin{array}{c} P \\ 0 \end{array}\right\} \\ = \underbrace{\frac{1}{2(EA)_{Al} + (EA)_{St}}}_{\text{Determinant}} \left[\begin{array}{cc} (EA)_{Al} & -1 \\ (EA)_{St} & 2 \end{array}\right] \left\{\begin{array}{c} P \\ 0 \end{array}\right\} \end{array}$$

6. We observe that the solution of this sproblem, contrarily to statically determinate ones, depends on the elastic properties.

7 Another example is the propped cantiliver beam of length L, Fig. 10.2

1. First we remove the roller support, and are left with the cantilever as a primary structure.

2. We then determine the deflection at point B due to the applied load P using the virtual force method

$$1.D = \int \delta\overline{M}\frac{M}{EI}dx \tag{10.6-a}$$

$$= \int_0^{L/2} 0\frac{-px}{EI}dx + \int_0^{L/2} -\left(\frac{PL}{2} + Px\right)(-x)dx \tag{10.6-b}$$

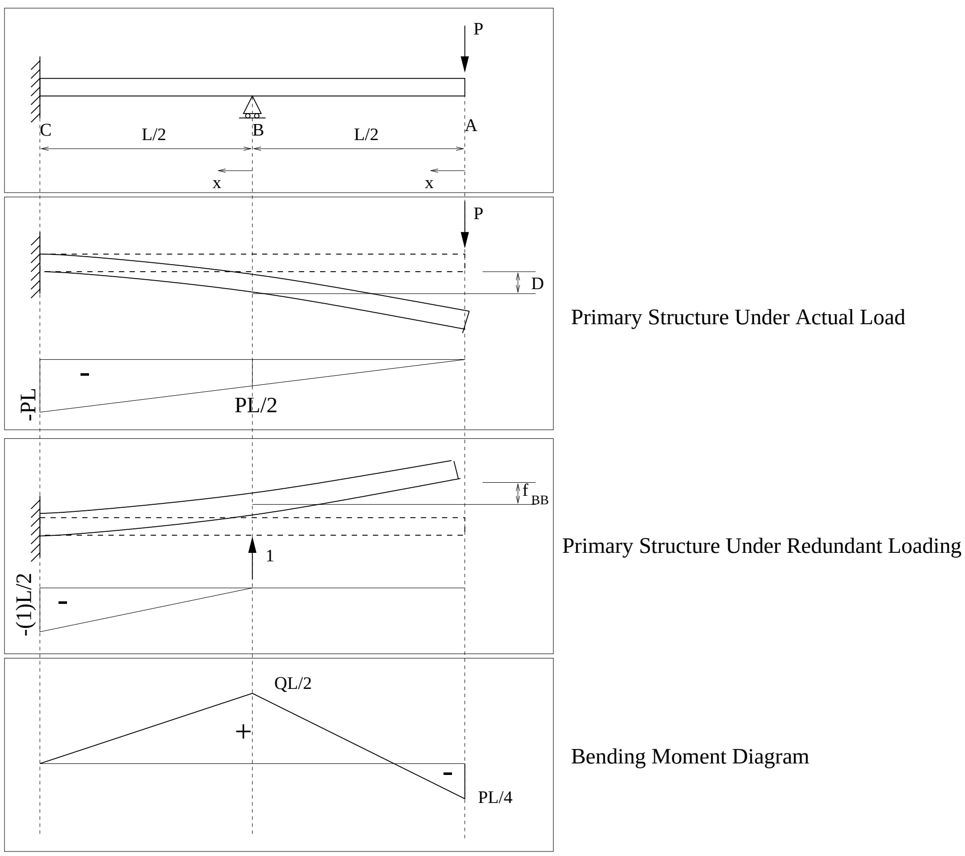


Figure 10.2: Propped Cantilever Beam

$$= \frac{1}{EI}\int_0^{L/2}\left(\frac{PL}{2}x + Px^2\right)dx \tag{10.6-c}$$

$$= \frac{1}{EI}\left[\frac{PLx^2}{4} + \frac{Px^3}{3}\right]\Bigg|_0^{L/2} \tag{10.6-d}$$

$$= \frac{5}{48}\frac{PL^3}{EI} \tag{10.6-e}$$

3. We then apply a unit load at point B and solve for the displacement at B using the virtual force method

$$1f_{BB} = \int \delta\overline{M}\frac{\overline{M}}{EI}dx \tag{10.7-a}$$

$$= \int_0^{L/2}(x)\frac{x}{EI}dx \tag{10.7-b}$$

$$= \frac{(1)L^3}{24EI} \tag{10.7-c}$$

4. Then we argue that the displacement at point B is zero, and hence the displacement f_{BB} should be multiplied by R_B such that

$$R_B f_{BB} = D \tag{10.8}$$

to ensure compatibility of displacements, hence

$$R_B = \frac{D}{f_{BB}} = \frac{\frac{5}{48}PL^3EI}{\frac{(1)L^3}{24EI}} = \boxed{\frac{5}{2}P} \tag{10.9}$$

10.2 The Force/Flexibility Method

8 Based on the previous two illustrative examples, we now seek to develop a general method for the *linear elastic* analysis of statically indeterminate structures.

1. Identify the degree of static indeterminancy (exterior and/or interior) n.
2. Select n redundant unknown forces and/or couples in the loaded structure along with n corresponding releases (angular or translation).
3. The n releases render the structure statically determinate, and it is called the *primary structure*.
4. Determine the n displacements in the primary structure (with the load applied) corresponding to the releases, D_i.
5. Apply a unit force at each of the releases j on the primary structure, without the external load, and determine the displacements in all releases i, we shall refer to these displacements as the *flexibility coefficients*, f_{ij}, i.e. displacement at release i due to a unit force at j
6. Write the compatibility of displacement relation

$$\boxed{\underbrace{\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}}_{[\mathbf{f}]} \underbrace{\begin{Bmatrix} R_1 \\ R_2 \\ \cdots \\ R_n \end{Bmatrix}}_{\mathbf{R}} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \\ \cdots \\ D_n \end{Bmatrix}}_{\mathbf{D}} = \underbrace{\begin{Bmatrix} D_1^0 \\ D_2^0 \\ \cdots \\ D_n^0 \end{Bmatrix}}_{\mathbf{D}_i^0}} \tag{10.10}$$

and

$$\boxed{[\mathbf{R}] = [\mathbf{f}]^{-1}\left\{\mathbf{D} - \mathbf{D}_i^0\right\}} \tag{10.11}$$

Note that $\mathbf{D}_i^0$ is the vector of initial displacements, which is usually zero unless we have an initial displacement of the support (such as support settlement).

7. The reactions are then obtained by simply inverting the flexibility matrix.

9 Note that from Maxwell-Betti's reciprocal theorem, the flexibility matrix $[\mathbf{f}]$ is always symmetric.

10.3 Short-Cut for Displacement Evaluation

10 Since deflections due to flexural effects must be determined numerous times in the flexibility method, Table 10.1 may simplify some of the evaluation of the internal strain energy. *You are strongly discouraged to use this table while still at school!.*

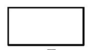
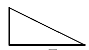
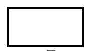
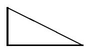
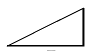
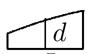
	$g_2(x)$		
$g_1(x)$	a  L	a  L	a  b L
c  L	Lac	$\frac{Lac}{2}$	$\frac{Lc(a+b)}{2}$
c  L	$\frac{Lac}{2}$	$\frac{Lac}{3}$	$\frac{Lc(2a+b)}{6}$
 c L	$\frac{Lac}{2}$	$\frac{Lac}{6}$	$\frac{Lc(a+2b)}{6}$
c  d L	$\frac{La(c+d)}{2}$	$\frac{La(2c+d)}{6}$	$\frac{La(2c+d)+Lb(c+2d)}{6}$
c  e L	$\frac{La(c+4d+e)}{6}$	$\frac{La(c+2d)}{6}$	$\frac{La(c+2d)+Lb(2d+e)}{6}$

Table 10.1: Table of $\int_0^L g_1(x)g_2(x)dx$

10.4 Examples

■ Example 10-1: Steel Building Frame Analysis, (White et al. 1976)

A small, mass-produced industrial building, Fig. 10.3, is to be framed in structural steel with a typical cross section as shown below. The engineer is considering three different designs for the frame: (a) for poor or unknown soil conditions, the foundations for the frame may not be able to develop any dependable horizontal forces at its bases. In this case the idealized base conditions are a hinge at one of the bases and a roller at the other; (b) for excellent soil conditions with properly designed foundations, the bases of the frame legs will have no tendency to move horizontally, and the idealized base condition is that of hinges at both points A and D; and (c) a design intermediate to the above cases, with a steel tie member capable of carrying only tension running between points A and D in the floor of the building. The foundations would not be expected to provide any horizontal restraint for this latter case, and the hinge-roller details at points A and D would apply.

Critical design loads for a frame of this type are usually the gravity loads (dead load + snow load) and the combination of dead load and wind load. We will restrict our attention to the first combination, and will use a snow load of 30 psf and an estimated total dead load of 20 psf. With frames spaced at 15 ft on centers along the length of the building, the design load is 15 (30 + 20) = 750 lb/ft.

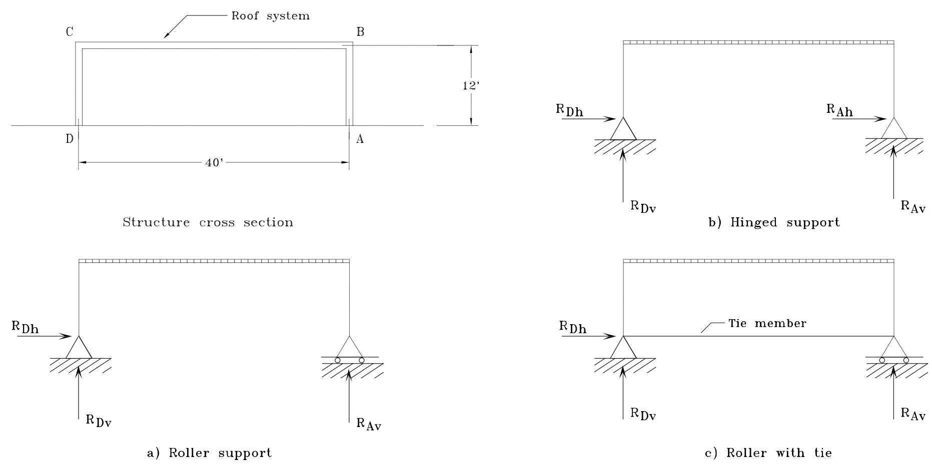


Figure 10.3:

If the frame is made of steel beam sections 21 in. deep and weighing 62 lb/ft of length (W 21 × 62), and the tie member for design (c) is tentatively chosen as a 2-in.^{2} bar, determine the bending moment diagrams for the three designs and discuss the alternate solutions.
Solution:

Structure a This frame is statically determinate since it has three possible unknown external forces acting on it, and the bending moment is shown in Fig. 10.6-a.

Structure b Hinging both legs of the frame results in another unknown force, making the structure statically indeterminate to the first degree (one redundant).
1. A lateral release at point A is chosen. with the redundant shearing force R_1. The displacement D_{1Q} in the primary structure, as a result of the real loading, is shown in Figure 10.4-a. D_{1Q} is computed by virtual work.
2. The virtual force system in Figure 10.4-b produces a virtual bending moment $\delta\overline{M}$, which is uniform

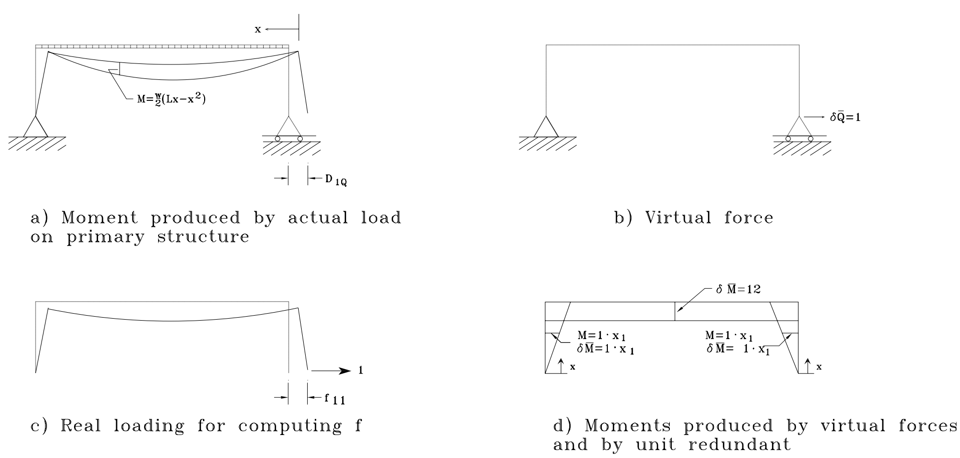


Figure 10.4:

across the top member of the frame. The virtual moment acting through the real angle changes gives the internal work term

$$\int \delta\overline{M} d\phi = \int_0^{40} \delta\overline{M} \frac{M}{EI} dx \tag{10.12}$$

3. Equating this to the external virtual work of $1 \cdot D_{1Q}$, we have

$$1 \cdot D_{1Q} = \int_0^{40} \frac{(12)(1/2)(.75)(Lx - x^2)}{EI} dx \tag{10.13}$$

or

$$D_{1Q} = \frac{48,000}{EI} \text{ k ft}^3 \rightarrow \tag{10.14}$$

4. The equation of consistent displacement is $D_{1Q} + f_{11}R_1 = 0$. The flexibility coefficient f_{11} is computed by applying a unit horizontal force at the release and determining the displacement at the same point.
5. It is seen that the real loading and the virtual loading are identical for this calculation, and

$$1 \cdot f_{11} = 2\left[\int_0^{12} \frac{x^2 dx}{EI} + \int_0^{20} \frac{12^2 dx}{EI}\right] \tag{10.15}$$

or

$$f_{11} = \frac{6,912}{EI} \text{ k ft}^3 \rightarrow \tag{10.16}$$

6. Solving for R_1

$$\frac{1}{EI}[48,000 + 6,912R_1] = 0 \tag{10.17}$$

or

$$R_1 = \boxed{6.93 \text{ k} \leftarrow} \tag{10.18}$$

7. The bending moment diagram is given below

Structure c The frame with the horizontal tie between points A and D has three unknown external forces. However, the structure is statically indeterminate to the first degree since the tie member provides one degree of internal redundancy.
8. The logical release to choose is a longitudinal release in the tie member, with its associated longitudinal displacement and axial force.
9. The primary structure Fig. 10.5 is the frame with the tie member released. The compatibility

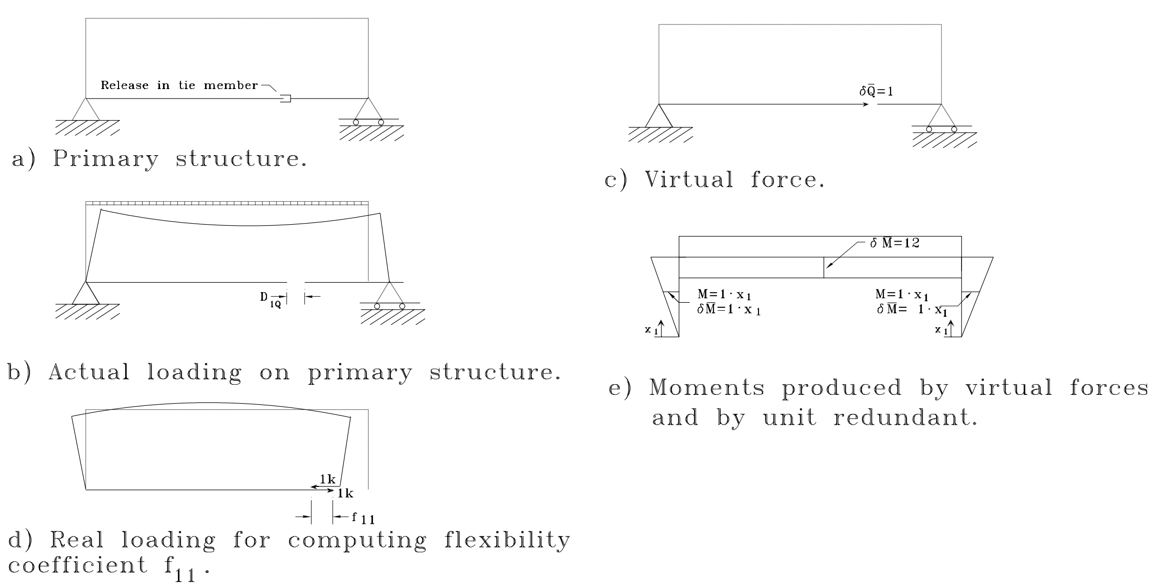


Figure 10.5:

equation is based on the fact that the displacement at the release must be zero; that is, the relative displacement of the two sections of the tie at the point of release must be zero, or

$$D_{1Q} + f_{11}R_1 = 0 \tag{10.19}$$

where

D_{1Q}	=	displacement at release 1 in the primary structure, produced by the loading,
f_{11}	=	relative displacement at release 1 for a unit axial force in the tie member,
R_1	=	force in the tie member in the original structure.

10. Virtual work is used to determine both displacement terms.

11. The value of D_{1Q} is identical to the displacement D_{1Q} computed for structure (b) because the tie member has no forces (and consequently no deformations) in the primary structure. Thus

$$D_{1Q} = \frac{(48,000)(1,728)}{(30 \cdot 10^3)(1,327)} = 2.08 \text{ in}\rightarrow \tag{10.20}$$

12. The flexibility coefficient f_{11} is composed of two separate effects: a flexural displacement due to the flexibility of the frame, and the axial displacement of the stressed tie member. The virtual and real loadings for this calculation are shown in Figure 10.5. The virtual work equation is

$$1 \cdot f_{11} = 2\left[\int_0^{12} \frac{x_1^2 dx_1}{EI} + \int_0^{20} \frac{(12)^2 dx}{EI}\right] + \delta\overline{P}\frac{pL}{EA} \tag{10.21-a}$$

$$= \frac{6,912}{EI} + \frac{1(1)(40)}{EA} \tag{10.21-b}$$

$$= \frac{(6,912)(1,728)}{(30 \cdot 10^3)(1,327)} + \frac{40(12)}{(30 \cdot 10^3)(2)} \tag{10.21-c}$$

$$= 0.300 + 0.008 = 0.308 \tag{10.21-d}$$

and

$$f_{11} = 0.308\text{in./k} \tag{10.22}$$

13. The equation of consistent deformation is

$$D_{1Q} + f_{11}R_1 = 0 \tag{10.23}$$

or

$$2.08 - .308R_1 = 0 \tag{10.24}$$

or

$$R_1 = \boxed{6.75 \text{ k (tension)}} \tag{10.25}$$

14. The two displacement terms in the equation must carry opposite signs to account for their difference in direction.

Comments The bending moment in the frame, Figure 10.6-c differs only slightly from that of structure (b). In other words, the tie member has such high axial stiffness that it provides nearly as much restraint as the foundation of structure (b). Frames with tie members are used widely in industrial buildings. A lesson to be learned here is that it is easy to provide high stiffness through an axially loaded member.

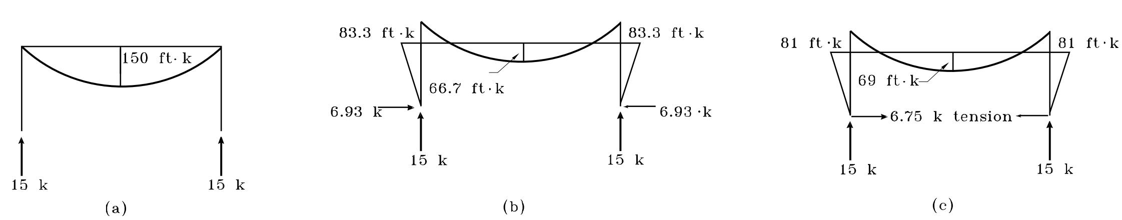


Figure 10.6:

The maximum moment in frames (b) and (c) is about 55% of the maximum moment in frame (a). This effect of continuity and redundancy is typical – the positive bending moments in the members are lowered while the joint moments increase and a more economical design can be realized.

Finally, we should notice that the vertical reactions at the bases of the columns do not change with the degree of horizontal restraint at the bases. A question to ponder is "Does this type of reaction behavior occur in all frames, or only in certain geometrical configurations?" ■

■ **Example 10-2: Analysis of Irregular Building Frame, (White et al. 1976)**

The structural steel frame for the Church of the Holy Spirit, Penfield, New York is shown below. In this example we will discuss the idealization of the structure and then determine the forces and bending moments acting on the frame.
Solution:

1. A sectional view of the building is given in Figure 10.8-a.

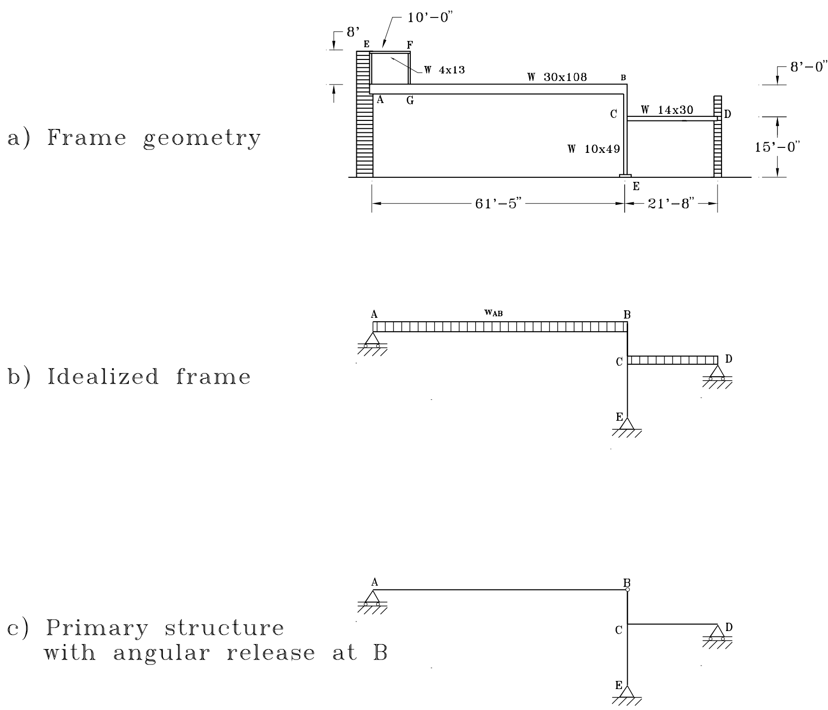
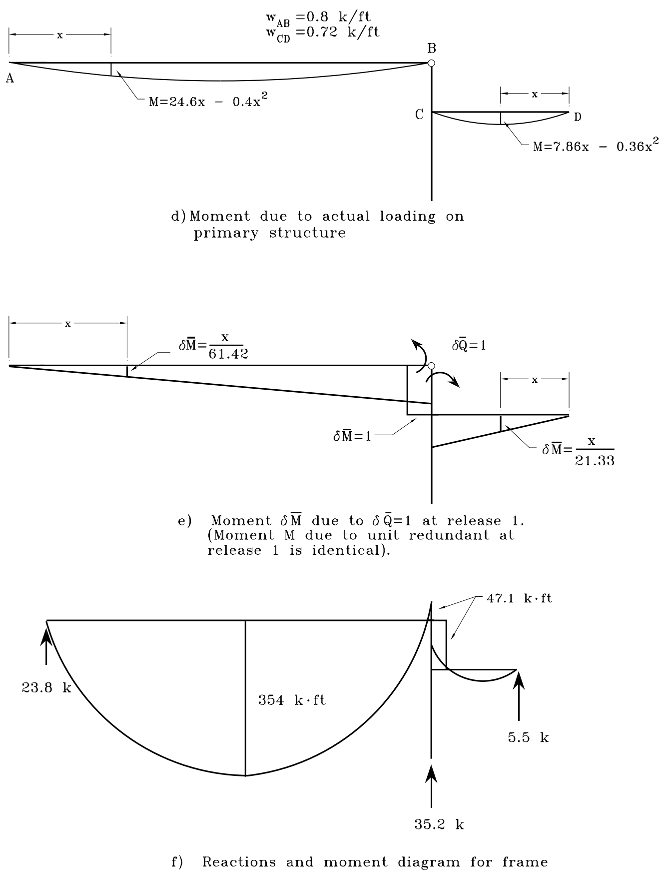


Figure 10.7:

The two main horizontal members of the frame are supported at points A and D by masonry walls.
2. The connection used at these points is not intended to transmit axial forces from the frame to the wall; accordingly, the axial forces in the horizontal members are assumed to be zero and the joints at A and D are idealized as rollers that transmit vertical forces only.
3. The base joint E is designed to resist both horizontal and vertical loads, but not moment, and is assumed to be a hinge.
4. Finally, joints B and C are designed to provide continuity and will be taken as rigid; that is, the angles of intersection of the members at the joint do not change with applied loading.
5. The frame is simplified for analysis by removing the small 4-in. wide flange members EF and FG and replacing their load effect by applying the roof load which acts on EF directly to the segment AG.
6. The idealized frame is shown in Figure 10.8-b.
7. The dead load on the higher portion of the frame is $w_{AB} = 25$ psf times the frame spacing of 13.33 ft, or $w_{AB} = 25(13.33) = 334$ lb/ft along the frame.
8. The dead load on CD is less because the weight of the frame member is substantially smaller, and the dead load is about 19 psf, or $w_{CD} = 19(13.33) = 254$ lb/ft of frame.
9. Snow load is 35 psf over both areas, or $w = 35(13.33) = 467$ lb/ft.
10. Total loads are then



w_{AB} =0.8 k/ft
w_{CD} =0.72 k/ft
x
B
A
M=24.6x − 0.4x^2
C
x
D
M=7.86x − 0.36x^2
d) Moment due to actual loading on primary structure
x
$\delta\overline{M}=\frac{x}{61.42}$
$\delta\overline{Q}$=1
x
$\delta\overline{M}$=1
$\delta\overline{M}=\frac{x}{21.33}$
e) Moment $\delta\overline{M}$ due to $\delta\overline{Q}$=1 at release 1. (Moment M due to unit redundant at release 1 is identical).
47.1 k·ft
23.8 k
354 k·ft
5.5 k
35.2 k
f) Reactions and moment diagram for frame

Figure 10.8:

Member AB: $w = 334 + 467 = 800$ lb/ft $= 0.8$ k/ft
Member CD: $w = 254 + 467 = 720$ lb/ft $= 0.72$ k/ft

11. The frame has four unknown reaction components and therefore has one redundant. Although several different releases are possible, we choose an angular (bending) release at point B.
12. The resulting primary structure is shown in Figure 10.8-c, where the redundant quantity R_1 is the bending moment at point B.
13. The equation of compatibility is

$$\theta_{1Q} + \theta_{11} R_1 = 0 \tag{10.26}$$

where θ_{1Q} is the relative angular rotation corresponding to release 1 as produced by the actual loading, and θ_{11} is the flexibility coefficient for a unit moment acting at the release.
14. From virtual work we have

$$1 \cdot \theta_{1Q} = \int \delta\overline{M} d\phi = \int \delta\overline{M} \frac{M}{EI} dx \tag{10.27}$$

and

$$1 \cdot \theta_{11} = \int \delta\overline{M} d\phi = \int \delta\overline{M} \frac{M}{EI} dx \tag{10.28}$$

where m is a real unit load and $\delta\overline{M}$ and M are defined in Figure 10.8-d and e.
15. Then

$$\theta_{1Q} = \frac{1}{(EI)_{AB}} \int_0^{61.42} \frac{x}{61.42}(24.6x - 0.4x^2)dx \tag{10.29-a}$$

$$+ \frac{1}{(EI)_{CD}} \int_0^{21.33} \frac{x}{21.33}(7.68x - 0.36x^2)dx \tag{10.29-b}$$

$$= \frac{7750}{(EI)_{AB}} + \frac{295}{(EI)_{CD}} \tag{10.29-c}$$

16. with $I_{AB} = 4,470$ in.^{4} and $I_{CD} = 290$ in.^{4}

$$\theta_{1Q} = \frac{1.733}{E} + \frac{1.0107}{E} = \frac{2.75}{E} \tag{10.30}$$

17. Similarly

$$\theta_{11} = \frac{1}{(EI)_{AB}} \int_0^{61.42} \left(\frac{x}{61.42}\right)^2 dx \tag{10.31-a}$$

$$+ \frac{1}{(EI)_{CD}} \int_0^{21.33} \left(\frac{x}{21.33}\right)^2 dx + \frac{1}{(EI)_{BC}} \int_0^{8} (1)^2 dx \tag{10.31-b}$$

18. with $I_{BC} = 273$ in^4

$$\theta_{11} = \frac{20.5}{(EI)_{AB}} + \frac{7.11}{(EI)_{CD}} + \frac{8.0}{(EI)_{BC}} = \frac{0.0585}{E} \tag{10.32}$$

Note that the numerators of θ_{1Q} and θ_{11} have the units k-ft^2/in^4.
19. Applying the compatibility equation,

$$\frac{2.75}{E} + \frac{0.0585}{E} R_1 = 0 \tag{10.33}$$

and the bending moment at point B is $\boxed{R_1 = -47.0 \text{ ft-k}}$. The reactions and moments in the structure are given in Figure 10.8-f. ■

■ Example 10-3: Redundant Truss Analysis, (White et al. 1976)

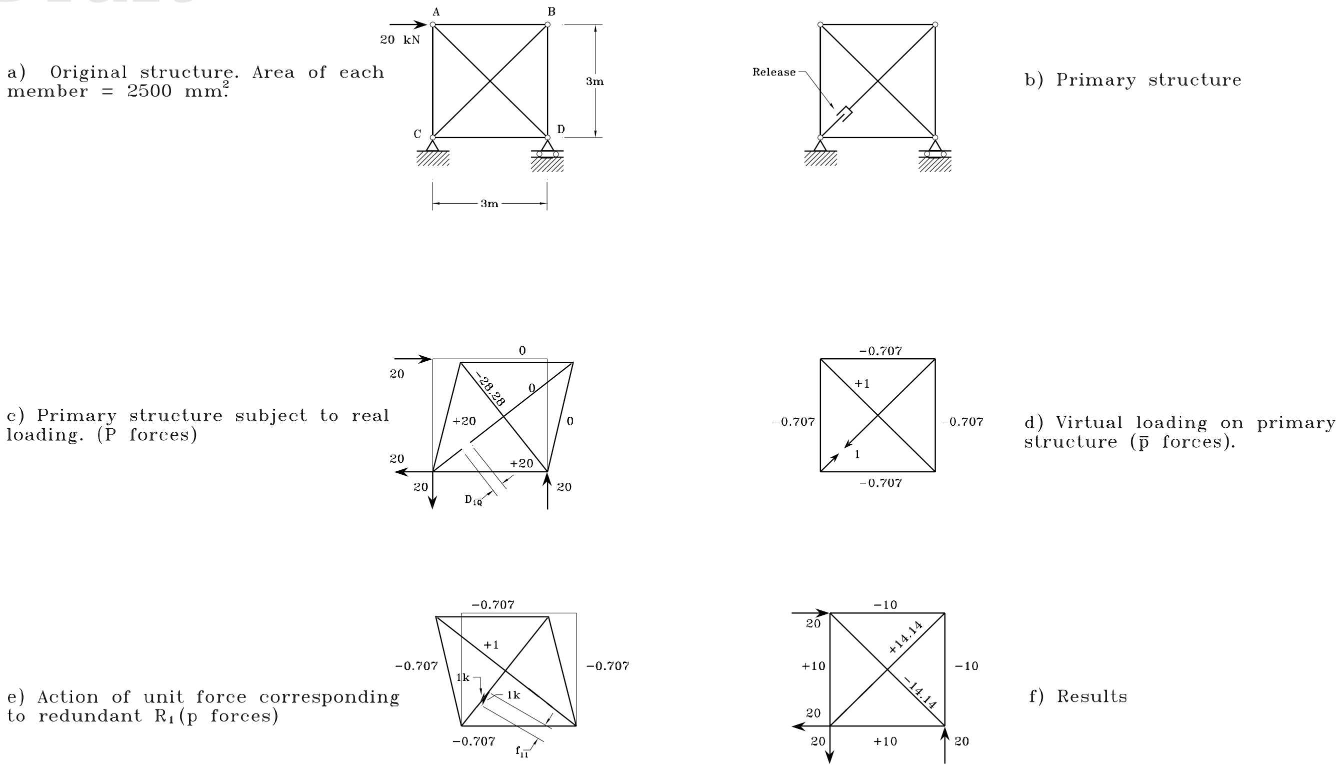


Figure 10.9:

Determine the bar forces in the steel truss shown below using the force method of analysis. The truss is part of a supporting tower for a tank, and the 20 kN horizontal load is produced by wind loading on the tank.

Solution:

1. Applying the criteria for indeterminacy, $2 \times 4 = 8$ equations, 6 members + 3 reactions $\Rightarrow$ one degree of indeterminacy. A longitudinal release in any of the six bars may be chosen.
2. Because the truss members carry only axial load, a longitudinal release is identical to actually cutting the member and removing its axial force capability from the truss.
3. In analyzing trusses with double diagonals it is both convenient and customary to select the release in one of the diagonal members because the resulting primary structure will be the conventional truss form to which we are accustomed.
4. Choosing the diagonal member BC for release, we cut it and remove its axial stiffness from the structure. The primary structure is shown in Figure 10.9-b.
5. The analysis problem reduces to applying an equation of compatibility to the changes in length of the release member. The relative displacement D_{1Q} of the two cut ends of member BC, as produced by the real loading, is shown in Figure 10.9-c.
6. The displacement is always measured along the length of the redundant member, and since the redundant is unstressed at this stage of the analysis, the displacement D_{1Q} is equal to the relative displacement of joint B with respect to joint C.
7. This displacement must be eliminated by the relative displacements of the cut ends of member BC when the redundant force is acting in the member. The latter displacement is written in terms of the axial flexibility coefficient f_{11}, and the desire equation of consistent deformation is

$$D_{1Q} + f_{11} R_1 = 0 \tag{10.34}$$

8. The quantity D_{1Q} is given by

$$1 \cdot D_{1Q} = \Sigma \delta \overline{P} (\Delta l) = \Sigma \delta \overline{P} (PL/AE) \tag{10.35}$$

where $\delta\overline{P}$ and P are given in Figure 10.9-d and c, respectively.
9. Similarly,

$$f_{11} = \Sigma\delta\overline{P}(\overline{P}L/AE) \quad (10.36)$$

10. Evaluating these summations in tabular form:

Member	P	$\delta\overline{P}$	L	$\delta\overline{P}PL$	$\delta\overline{P}\overline{P}L$
AB	0	-0.707	3	0	1.5
BC	0	-0.707	3	0	1.5
CD	$+20$	-0.707	3	-42.42	1.5
AC	$+20$	-0.707	3	-42.42	1.5
AD	-28.28	$+1$	4.242	-119.96	4.242
BC	0	$+1$	4.242	0	4.242
				-204.8	14.484

11. Since $A =$ constant for each member

$$D_{1Q} = \Sigma\delta\overline{P}\frac{PL}{AE} = -\frac{-204.8}{AE} \text{and} f_{11} = \frac{14.484}{AE} \quad (10.37)$$

then

$$\frac{1}{AE}[-204.8 + 14.484R_1] = 0 \quad (10.38)$$

12. The solution for the redundant force value is $\boxed{R_1 = 14.14 \text{ kN}}$.
13. The final values for forces in each of the truss members are given by superimposing the forces due to the redundant and the forces due to the real loading.
14. The real loading forces are shown in Figure 10.9-c while the redundant force effect is computed by multiplying the member forces in Figure 10.9-d by 2.83, the value of the redundant.
15. It is informative to compare the member forces from this solution to the approximate analysis obtained by assuming that the double diagonals each carry half the total shear in the panel. The comparison is given in Figure 10.10; it reveals that the approximate analysis is the same as the exact

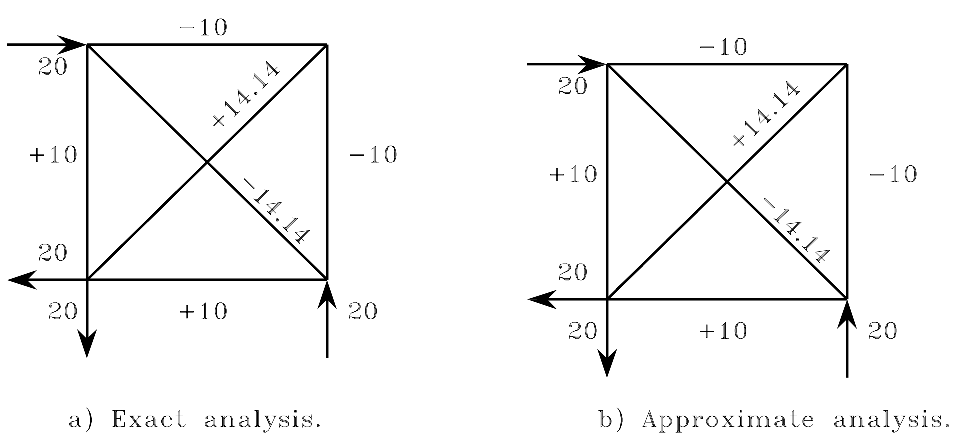


Figure 10.10:

analysis for this particular truss. The reason for this is that the stiffness provided by each of the diagonal members (against "shear" deformation of the rectangular panel) is the same, and therefore they each carry an equal portion of the total shear across the panel.
16. How would this structure behave if the diagonal members were very slender? ■

■ **Example 10-4: Truss with Two Redundants, (White et al. 1976)**

Another panel with a second redundant member is added to the truss of Example 10-3 and the new truss is supported at its outermost lower panel points, as shown in Figure 10.11. The truss, which is similar in form to trusses used on many railway bridges, is to be analyzed for bar forces under the given loading.

Solution:

1. The twice redundant truss is converted to a determinate primary structure by releasing two members of the truss; we choose two diagonals (DB and BF).
2. Releasing both diagonals in a single panel, such as members AE and DB, is inadmissible since it leads to an unstable truss form.
3. The member forces and required displacements for the real loading and for the two redundant forces

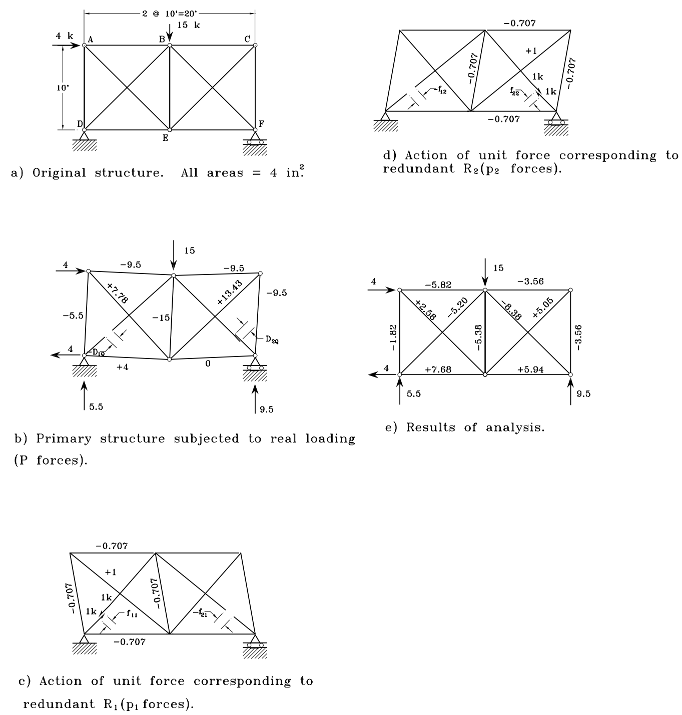


Figure 10.11:

in members DB and BF are given in Figure 10.11.
4. Although the real loading ordinarily stresses all members of the entire truss, we see that the unit forces corresponding to the redundants stres only those members in the panel that contains the redundant; all other bar forces are zero.
5. Recognizing this fact enables us to solve the double diagonal truss problem more rapidly than a frame with multiple redundants.
6. The virtual work equations for computing the six required displacements (two due to load and four flexibilities) are

$$1 \cdot D_{1Q} = \Sigma\delta\overline{P}_1\left(\frac{PL}{AE}\right) \tag{10.39-a}$$

$$1 \cdot D_{2Q} = \Sigma\delta\overline{P}_2\left(\frac{PL}{AE}\right) \tag{10.39-b}$$

$$1 \cdot f_{11} = \Sigma\delta\overline{P}_1\left(\frac{\overline{P}_1 L}{AE}\right) \tag{10.39-c}$$

$$1 \cdot f_{21} = \Sigma\delta\overline{P}_2\left(\frac{\overline{P}_1 L}{AE}\right) \tag{10.39-d}$$

$$f_{12} = f_{21} \text{ by the reciprocal theorem} \tag{10.39-e}$$

$$1 \cdot f_{22} = \Sigma\delta\overline{P}_2\frac{\overline{P}_2L}{AE} \tag{10.39-f}$$

7. If we assume tension in a truss member as postive, use tensile unit loads when computing the flexibility coefficients corresponding to the redundants, and let all displacement terms carry their own signs, then in the solution for the redundants a positive value of force indicates tension while a negative value means the member is in compression.
8. The calculation of f_{22} involves only the six members in the left panel of the truss; f_{21} involves only member BE.
9. The simple procedures used for performing the displacement analyses, as summarized in tabular form in Table 10.2, leads one quickly to the compatibility equations which state that the cut ends of both

Member	P	$\overline{P}_1$	$\overline{P}_2$	L	D_{1Q} δP_1PL	D_{2Q} δP_2PL	f_{11} δP_1P_1L	f_{21} δP_2P_1L	f_{22} δP_2P_2L
AB	-9.5	-0.707	0	120	+806	0	60	0	0
BC	-9.5	0	-0.707	120	0	+806	0	0	60
CF	-9.5	0	-0.707	120	0	+806	0	0	60
EF	0	0	-0.707	120	0	0	0	0	60
DE	+4	-0.707	0	120	-340	0	60	0	0
AD	-5.5	-0.707	0	120	+466	0	60	0	0
AE	+7.78	+1	0	170	+1,322	0	170	0	0
BE	.15	-0.707	-0.707	120	+1,272	+1272	60	60	60
CE	+13.43	0	+1	170	0	+2,280	0	0	170
BD	0	+1	0	170	0	0	170	0	0
BF	0	0	+1	170	0	0	0	0	170
					+3,528	+5,164	+580	+60	+580

Table 10.2:

redundant members must match (there can be no gaps or overlaps of members in the actual structure).
10. The equations are

$$D_{1Q} + f_{11}R_1 + f_{12}R_2 = 0 \tag{10.40-a}$$
$$D_{2Q} + f_{21}R_1 + f_{22}R_2 = 0 \tag{10.40-b}$$

or

$$\frac{1}{AE}\begin{bmatrix} 580 & 60 \\ 60 & 580 \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = -\frac{1}{AE}\begin{bmatrix} 3,528 \\ 5,164 \end{bmatrix} \tag{10.41}$$

and

$$R_1 = \boxed{5.20 \text{ k}} \tag{10.42-a}$$
$$R_2 = \boxed{-8.38 \text{ k}} \tag{10.42-b}$$

11. The final set of forces in the truss is obtained by adding up, for each member, the three separate effects. In terms of the forces shown in Figure 10.11 and Table 10.2, the force in any member is given by $F = P + R_1\overline{P}_1 + R_2\overline{P}_2$. The final solution is given in Figure 10.11-e. The truss is part of a supporting tower for a tank, and the 20 kN horizontal load is produced by wind loading on the tank.
12. NOTE: A mixture of internal redundant forces and external redundant reactions is no more difficult than the preceding example. Consider the two-panel truss of this example modified by the addition of another reaction component at joint E, Figure 10.12. The three releases for this truss can be chosen from a number of possible combinations: diagonals DB and BF and the reaction at E; the same two diagonals and the reaction at F; the same diagonals and the top chord member BC, etc. The only requirement to be met is that the primary structure be stable and statically determinate.
13. For any set of releases there are four new displacement components: the displacement at the third release resulting from the actual load on the primary structure, and the three flexibility coefficients f_{31}, f_{32}, and f_{33}. Judicious choice of releases often results in a number of the flexibility coefficients being zero ■

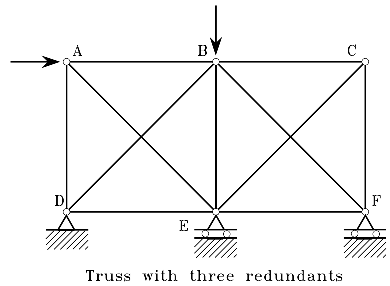


Figure 10.12:

■ **Example 10-5: Analysis of Nonprismatic Members, (White et al. 1976)**

The nonprismatic beam of Figure 10.13-a is loaded with an end moment M_A at its hinged end A. Determine the moment induced at the fixed end B by this loading.

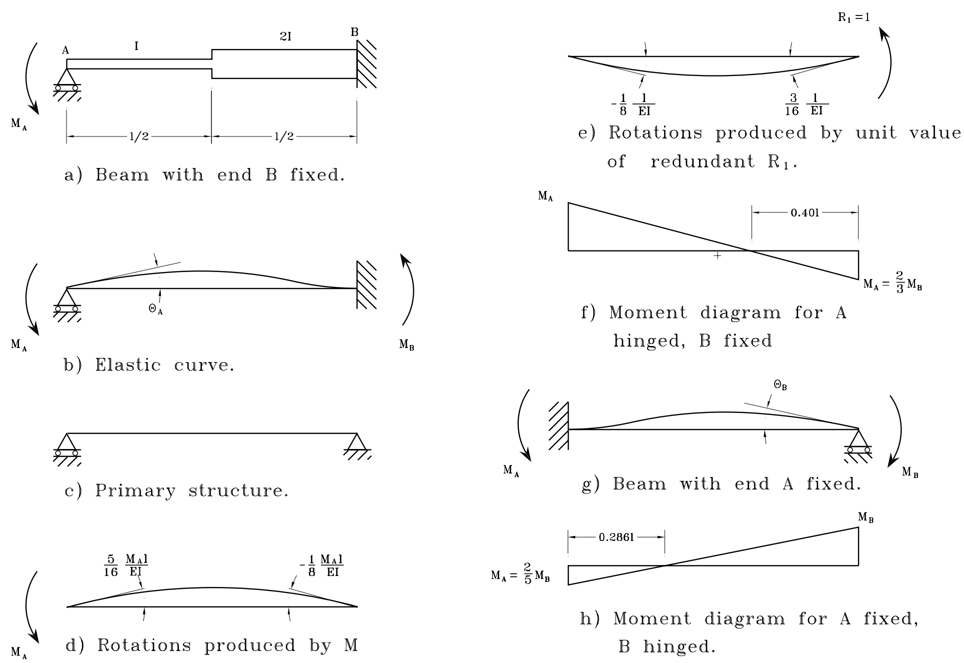


Figure 10.13:

Solution:

1. The beam has one redundant force; we select M_B as the redundant R_1, and obtain the primary structure shown in Figure 10.13-c. It can be shown that the flexibility co-efficients for unit moments applied at each end are those shown in Fig. 10.13-d and e, with a sign convention of counterclockwise as positive.
2. The equation of consistent displacements at B is

$$-\frac{M_A l}{8EI}+\frac{3}{16}\frac{l}{EI}R_1=0 \tag{10.43}$$

and the value of M_B is

$$M_B = R_1 = \frac{2}{3} M_A \tag{10.44}$$

3. The resulting moment diagram is given in Figure 10.13-f. We note that the inflection point is $0.40l$ from the fixed end. If the beam had a uniform value of I across its span, the inflection point would be $L/3$ from the fixed end. Thus the inflection point shifts toward the section of reduced stiffness.
4. The end rotation θ_A is given by

$$\theta_a = \frac{5}{16}\frac{M_A l}{EI} - \frac{1}{8}\left(\frac{2}{3}M_A\right)\frac{l}{EL} = \frac{11}{48}\frac{M_A l}{EI} \tag{10.45}$$

5. The ratio of applied end moment to rotation. M_A/θ_A, is called the *rotational stiffness* and is

$$\frac{M_A}{\theta_A} = \frac{48}{11}\frac{EI}{l} = 4.364\frac{EI}{l} \tag{10.46}$$

6. If we now reverse the boundary conditions, making A fixed and B hinged, and repeat the analysis for an applied moment M_B, the resulting moment diagram will be as given in Figure 10.13-h. The moment induced at end A is only 40% of the applied end moment M_B. The inflection point is $0.286l$ from the fixed end A. The corresponding end rotation θ_B in Figure 10.13-g is

$$\theta_B = \frac{11}{80}\frac{M_B l}{EI} \tag{10.47}$$

7. The rotational stiffness $\frac{M_B}{\theta_B}$ is

$$\frac{M_B}{\theta_B} = \frac{80}{11}\frac{EI}{l} = 7.272\frac{EI}{l} \tag{10.48}$$

8. A careful comparison of the rotational stiffnesses, and of the moment diagrams in Figures 10.13-f and h, illustrate the fact that flexural sections of increased stiffness attract more moment, and that inflection points always shift in the direction of decreased stiffness.
9. The approach illustrated here may be used to determine moments and end rotations in any type of nonprismatic member. The end rotations needed in the force analysis may be calculated by either virtual work or moment area (or by other methods). Complex variations in EI are handled by numerical integration of the virtual work equation or by approximating the resultant M/EI areas and their locations in the moment area method. ■

■ Example 10-6: Fixed End Moments for Nonprismatic Beams, (White et al. 1976)

The beam of example 10-5, with both ends fixed, is loaded with a uniform load w, Figure 10.14-a. Determine the fixed end moments M_A and M_B.
Solution:

1. The beam has two redundant forces and we select M_A and M_B. Releasing these redundants, R_1 and R_2, the primary structure is as shown in Figure 10.14-c.
2. The equations of consistent deformations are

$$D_{1Q} + f_{11}R_1 + f_{12}R_2 = 0 \tag{10.49-a}$$
$$D_{2Q} + f_{21}R_1 + f_{22}R_2 = 0 \tag{10.49-b}$$

where R_1 is M_A and R_2 is M_B.
3. The values of D_{1Q} and D_{2Q}, the end rotations produced by the real loading on the primary structure, can be computed by the virtual work method.
4. The flexibility coefficients are also separately derived (not yet in these notes) and are given in Figures 10.14-d and e of the previous example.

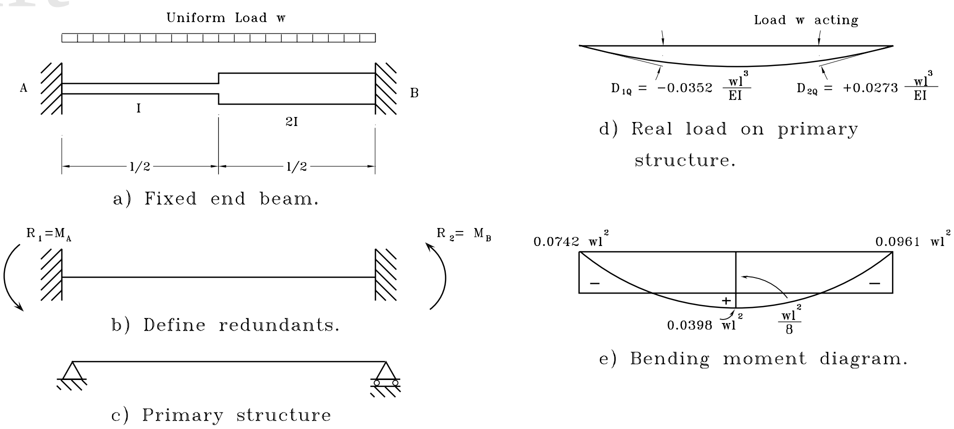


Figure 10.14:

5. We define counterclockwise end moments and rotations a positive and obtain

$$\frac{l}{EI}\begin{bmatrix} \frac{5}{16} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{16} \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \frac{wl^3}{EI}\begin{bmatrix} -0.352 \\ +0.0273 \end{bmatrix} \tag{10.50}$$

from which

$$R_1 = M_A = \boxed{0.0742wl^2} \tag{10.51-a}$$

$$R_2 = M_B = \boxed{-0.0961wl^2} \tag{10.51-b}$$

6. The stiffer end of the beam attracts 30% more than the flexible end.
7. For a prismatic beam with constant I, the fixed end moments are equal in magnitude ($M_A = -M_B = wl^2/12$) and intermediate in value between the two end moments determined above.
8. Fixed end moments are an essential part of indeterminate analysis based on the displacement (stiffness) method and will be used extensively in the Moment Distribution method.
■

■ Example 10-7: Rectangular Frame; External Load, (White et al. 1976)

Solution:

1. The structure is statically indeterminate to the third degree, and the displacements (flexibility terms) are shown in Fig. 10.15
2. In order to evaluate the 9 flexility terms, Fig. 10.16 we refer to Table 10.3
3. Substituting $h = 10$ ft, $L = 20$ ft, and $EI_b = EI_c = EI$, the flexibility matrix then becomes

$$[\mathbf{f}] = \frac{1}{EI}\begin{bmatrix} 2,667 & 3,000 & -300 \\ 3,000 & 6,667 & -400 \\ -300 & -400 & 40 \end{bmatrix} \tag{10.52}$$

and the vector of displacements for the primary structure is

$$\{\mathbf{D}\} = \frac{1}{EI}\left\{\begin{array}{c} -12,833 \\ -31,333 \\ 1,800 \end{array}\right\} \tag{10.53}$$

where the units are kips and feet.

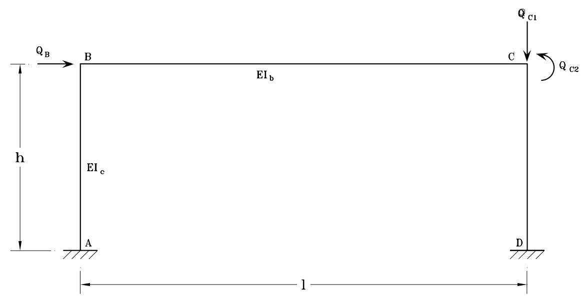


Figure 10.15:

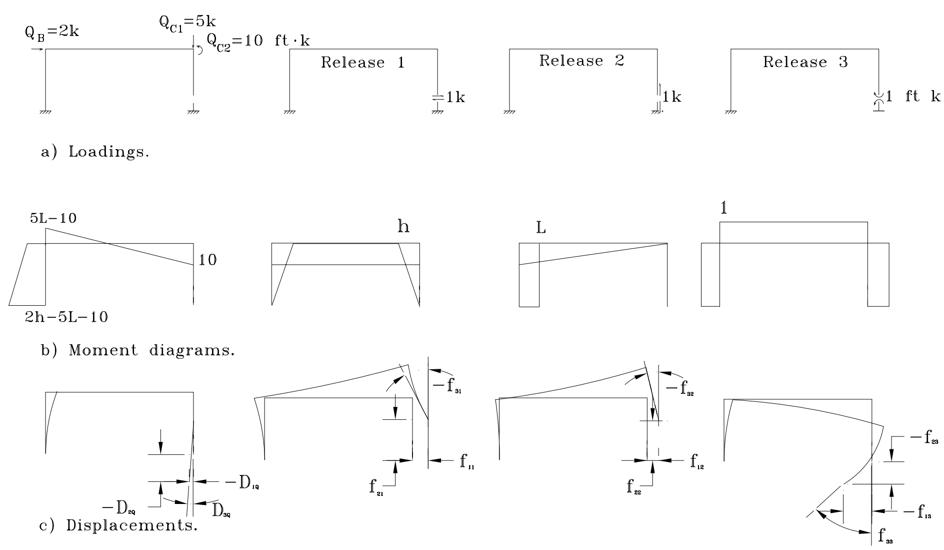


Figure 10.16: Definition of Flexibility Terms for a Rigid Frame

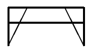
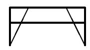
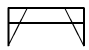
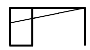
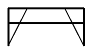
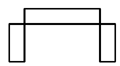
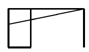
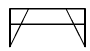
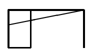
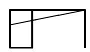
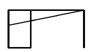
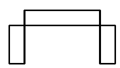
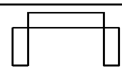
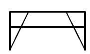
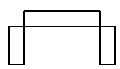
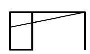
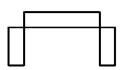
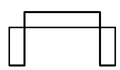
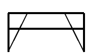
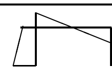
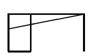
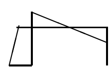
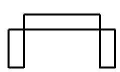
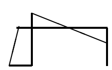
D	$\delta\overline{M}$	M	$\int \delta\overline{M} M dx$ AB	BC	CD	$\int \delta\overline{M} \frac{M}{EI} dx$ Total
f_{11}			$+\frac{h^3}{3}$	$+Lh^2$	$+\frac{h^3}{3}$	$+\frac{2h^3}{3EI_c} + \frac{Lh^2}{EI_b}$
f_{12}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{13}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{21}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{22}			$+L^2h$	$+\frac{L^3}{3}$	0	$+\frac{L^2h}{EI_c} + \frac{L^3}{3EI_b}$
f_{23}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{31}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{32}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{33}			$+h$	$+L$	$+h$	$+\frac{2h}{EI_c} + \frac{L}{EI_b}$
D_{1Q}			$-\frac{h^2(2h+15L-30)}{6}$	$+\frac{Lh(20-5L)}{2}$	0	$-\frac{h^2(2h+15L-30)}{6EI_c} + \frac{Lh(20-5L)}{2EI_b}$
D_{2Q}			$-\frac{Lh(2h+10L-20)}{2}$	$+\frac{L^2(30-10L)}{6}$	0	$-\frac{Lh(2h+10L-20)}{2EI_c} + \frac{L^2(30-10L)}{6EI_b}$
D_{3Q}			$+\frac{h(2h+10L-20)}{2}$	$-\frac{L(20-5L)}{2}$	0	$-\frac{h(2h+10L-20)}{2EI_c} - \frac{L(20-5L)}{2EI_b}$

Table 10.3: Displacement Computations for a Rectangular Frame

4. The inverse of the flexibility matrix is

$$[\mathbf{f}]^{-1} = 10^{-3}EI \begin{bmatrix} 2.40 & 0.00 & 18.00 \\ 0.00 & 0.375 & 3.750 \\ 18.000 & 3.750 & 197.5 \end{bmatrix} \tag{10.54}$$

5. Hence the reactions are determined from

$$\{\mathbf{R}\} = \left\{ \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \right\} = 10^{-3}EI \begin{bmatrix} 2.40 & 0.00 & 18.00 \\ 0.00 & 0.375 & 3.750 \\ 18.000 & 3.750 & 197.5 \end{bmatrix} \frac{1}{EI} \left\{ \begin{array}{c} -12,833 \\ -31,333 \\ 1,800 \end{array} \right\} = \boxed{\left\{ \begin{array}{c} -1.60 \\ +5.00 \\ -7.00 \end{array} \right\}} \tag{10.55}$$

■

■ Example 10-8: Frame with Temperature Effectsand Support Displacements, (White et al. 1976)

The single bay frame, of example 10-7, has a height $h = 10$ ft and span $L = 20$ ft and its two suports rigidly connected and is constructed of reinforced concrete. It supports a roof and wall partitions in such a manner that a linear temperature variation occurs across the depth of the frame members when inside and outside temperatures differ. Assume the member depth is constant at 1 ft, and that the structure was built with fixed bases A and D at a temperature of 85°F. The temperature is now 70°F inside and 20°F outside. We wish to determine the reactions at D under these conditions. Assume that the coefficient of linear expansion of reinforced concrete is $\alpha = 0.0000055$/°F.
Solution:

1. Our analysis proceeds as before, using Equation 10.11 with the $[D]$ vector interpreted appropriately. The three releases shown in Fig. 10.16 will be used.
2. The first stage in the analysis is the computation of the relative displacements $D_{1\Delta}, D_{2\Delta}, D_{3\Delta}$ of the primary structure caused by temperature effects. These displacements are caused by two effects: axial shortening of the members because of the drop in average temperature (at middepth of the members), and curvature of the members because of the temperature gradient.
3. In the following discussion the contributions to displacements due to axial strain are denoted with a single prime ($'$) and those due to curvature by a double prime ($''$).
4. Consider the axial strain first. A unit length of frame member shortens as a result of the temperature decrease from 85°F to 45°F at the middepth of the member. The strain is therefore

$$\alpha\Delta T = (0.0000055)(40) = 0.00022 \tag{10.56}$$

5. The effect of axial strain on the relative displacements needs little analysis. The horizontal member shortens by an amount $(0.00022)(20) = 0.0044$ ft. The shortening of the vertical members results in no relative displacement in the vertical direction 2. No rotation occurs.
6. We therefore have $D'_{1\Delta} = -0.0044$ ft, $D'_{2\Delta} = 0$, and $D'_{3\Delta} = 0$.
7. The effect of curvature must also be considered. A frame element of length dx undergoes an angular strain as a result of the temperature gradient as indicated in Figure 10.17. The change in length at an extreme fiber is

$$\epsilon = \alpha\Delta T dx = 0.0000055(25)dx = 0.000138dx \tag{10.57}$$

8. with the resulting real rotation of the cross section

$$d\phi = \epsilon/0.5 = 0.000138dx/0.5 = 0.000276dx \text{ radians} \tag{10.58}$$

9. The relative displacements of the primary structure at D are found by the virtual force method.
10. A virtual force $\delta\overline{Q}$ is applied in the direction of the desired displacement and the resulting moment diagram $\delta\overline{M}$ determined.

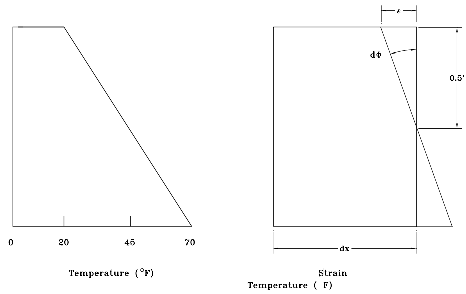


Figure 10.17:

11. The virtual work equation

$$\delta\overline{Q}D = \int \delta\overline{M}d\phi \tag{10.59}$$

is used to obtain each of the desired displacements D.
12. The results, which you should verify, are

$$D''_{1\Delta} = \boxed{0.0828 \text{ ft}} \tag{10.60-a}$$
$$D''_{2\Delta} = \boxed{0.1104 \text{ ft}} \tag{10.60-b}$$
$$D''_{3\Delta} = \boxed{-0.01104 \text{ radians}} \tag{10.60-c}$$

13. Combining the effects of axial and rotational strain, we have

$$\begin{array}{lclcl} D_{1\Delta} & = & D'_{1\Delta} + D''_{1\Delta} & = & 0.0784 \text{ ft} \\ D_{2\Delta} & = & D'_{2\Delta} + D''_{2\Delta} & = & 0.1104 \text{ ft} \\ D_{3\Delta} & = & D'_{3\Delta} + D''_{3\Delta} & = & -0.01104 \text{ radians} \end{array} \tag{10.61}$$

14. We now compute the redundants caused by temperature effects:

$$[R] = [f]^{-1}(-[D]) \tag{10.62}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^{-3}EI \begin{bmatrix} 18.0 & 3.75 & 197.5 \end{bmatrix} \begin{bmatrix} -0.0784 \\ -0.1104 \\ +0.01104 \end{bmatrix} = \begin{bmatrix} +0.0106 \\ +0.355 \end{bmatrix} \cdot 10^{-3}EI \tag{10.63}$$

where the units are feet and kips.
15. You should construct the moment diagram for this structure using the values of the redundants found in the analysis.
16. Notice that the stiffness term EI does not cancel out in this case. Internal forces and reactions in a statically indeterminate structure subject to effects other than loads (such as temperature) are dependent on the actual stiffnesses of the structure.
17. The effects of axial strain caused by forces in the members have been neglected in this analysis. This is usual for low frames where bending strain dominates behavior. To illustrate the significance of this assumption, consider member BC. We have found $R_1 = 10.6 \cdot 10^{-6}EI$ k. The tension in BC has this same value, resulting in a strain for the member of $10.6 \cdot 10^{-6}EI/EA$. For a rectangular member, $I/A = (bd^3/12)(bd) = d^2/12$. In our case $d = 1$ ft, therefore the axial strain is $10.6 \cdot 10^{-6}(0.0833) = 8.83 \cdot 10^{-7}$, which is several orders of magnitude smaller than the temperature strain computed for the same member. We may therefore rest assured that neglecting axial strain caused by forces does not affect the values of the redundants in a significant manner for this structure.
18. Now consider the effects of foundation movement on the same structure. The indeterminate frame behavior depends on a structure that we did not design: the earth. The earth is an essential part of nearly all structures, and we must understand the effects of foundation behavior on structural behavior.

For the purposes of this example, assume that a foundation study has revealed the possibility of a clockwise rotation of the support at D of 0.001 radians and a downward movement of the support at D of 0.12 ft. We wish to evaluate the redundants R_1, R_2, and R_3 caused by this foundation movement.
19. No analysis is needed to determine the values of $D_{1\Delta}, D_{2\Delta}$, and $D_{3\Delta}$ for the solution of the redundants. These displacements are found directly from the support movements, with proper consideration of the originally chosen sign convention which defined the positive direction of the relative displacements. From the given support displacements, we find $D_{1\Delta} = 0, D_{2\Delta} = +0.12$ ft, and $D_{3\Delta} = -0.001$ radians. Can you evaluate these quantities for a case in which the support movements occurred at A instead of D?
20. The values of the redundants is given by

$$[R] = [f]^{-1}(-[D]) \tag{10.64}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^{-3}EI \begin{bmatrix} 18.0 & 3.75 & 197.5 \end{bmatrix} \begin{bmatrix} -0.12 \\ +0.001 \end{bmatrix} = 10^{-6}EI \begin{bmatrix} 18.0 \\ -252.5 \end{bmatrix} \tag{10.65}$$

with units in kips and feet.
21. A moment diagram may now be constructed, and other internal force quantities computed from the now known values of the redundants. The redundants have been valuated separately for effects of temperature and foundation settlement. These effects may be combined with those due to loading using the principle of superposition. ■

■ Example 10-9: Braced Bent with Loads and Temperature Change, (White et al. 1976)

The truss shown in Figure 10.18 reperesents an internal braced bent in an enclosed shed, with lateral loads of 20 kN at the panel points. A temperature drop of 30°C may occur on the outer members (members 1-2, 2-3, 3-4, 4-5, and 5-6). We wish to analyze the truss for the loading and for the temperature effect.

Solution:

1. The first step in the analysis is the definition of the two redundants. The choice of forces in diagonals 2-4 and 1-5 as redundants facilitates the computations because some of the load effects are easy to analyze. Figure **??**-b shows the definition of R_1 and R_2.
2. The computations are organized in tabular form in Table 10.4. The first column gives the bar forces P in the primary structure caused by the actual loads. Forces are in kN. Column 2 gives the force in each bar caused by a unit load (1 kN) corresponding to release 1. These are denoted p_1 and also represent the bar force $\bar{q}_1/\delta\overline{Q}_1$ caused by a virtual force $\delta\overline{Q}_1$ applied at the same location. Column 3 lists the same quantity for a unit load and for a virtual force $\delta\overline{Q}_2$ applied at release 2. These three columns constitute a record of the truss analysis needed for this problem.
3. Column 4 gives the value of L/EA for each bar in terms of L_c/EA_c of the vertical members. This is useful because the term L/EA cancels out in some of the calculations.
4. The method of virtual work is applied directly to compute the displacements D_{1Q} and D_{2Q} corresponding to the releases and caused by the actual loads. Apply a virtual force $\delta\overline{Q}_1$ at release 1. The internal virtual forces $\bar{q}_1$ are found in column 2. The internal virtual work $\bar{q}_1\Delta l$ is found in column 5 as the product of columns 1, 2, and 4. The summation of column 5 is $D_{1q} = -122.42\ L_c/EA_c$. Similarly, column 6 is the product of columns 1, 3, and 4, giving $D_{2Q} = -273.12\ L_c/EA_c$.
5. The same method is used to compute the flexibilities f_{ij}. In this case the real loading is a unit load corresponding first to release 1 leading to f_{11}, and f_{21}, and then to release 2 leading to f_{12} and f_{22}. Column 7 shows the computation for f_{11}. It is the product of column 2 representing force due to the real unit load with column 2 representing force due to a virtual load $\delta\overline{Q}_1$ at the same location (release 1) multiplied by column 4 to include the L_c/EA_c term. Column 8 derives from columns 2, 3, and 4 and leads to f_{21}. Columns 9 and 10 are the computations for the remaining flexibilities.

	P	p_1	p_2	L/EA	D_{1Q} $\bar{q}_1PL/EA$	D_{2Q} $\bar{q}_2PL/EA$	f_{11} $\bar{q}_1p_1L/EA$	f_{21} $\bar{q}_2p_2L/EA$	f_{12} $\bar{q}p_2L/EA$	f_{22} $\bar{q}_2p_2L/EA$	$\Delta/_{temp}$	$D_{1\Delta}$ $\bar{q}_1\Delta/$	$D_{2\Delta}$ $\bar{q}_2\Delta/$
multiply by				L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c	$10^{-4}L_c$	$10^{-4}L_c$
1-2	60.0	0	-0.707	1	0	-42.42	0	0	0	0.50	-0.0003	0	2.12
2-3	20.00	-0.707	0	1	-14.14	0	0.50	0	0	0	-0.0003	2.12	0
3-4	0	-0.707	0	2	0	0	1.00	0	0	0	-0.0003	2.12	0
4-5	0	-0.707	0	1	0	0	0.50	0	0	0	-0.0003	2.12	0
5-6	-20.00	0	-0.707	1	0	14.14	0	0	0	0.50	-0.0003	0	2.12
6-1	40.00	0	-0.707	2	0	-56.56	0	0	0	1.00	0	0	0
2-5	20.00	-0.707	-0.707	2	-28.28	-28.28	1.00	1.00	1.00	1.00	0	0	0
1-5	0	0	1.00	2.828	0	0	0	0	0	2.83	0	0	0
2-6	-56.56	0	1.00	2.828	0	-160.00	0	0	0	2.83	0	0	0
2-4	0	1.00	0	2.828	0	0	2.83	0	0	0	0	0	0
3-5	-28.28	1.00	0	2.838	-80.00	0	2.83	0	0	0	0	0	0
					-122.42	-273.12	8.66	1.00	1.00	8.66		6.36	4.24

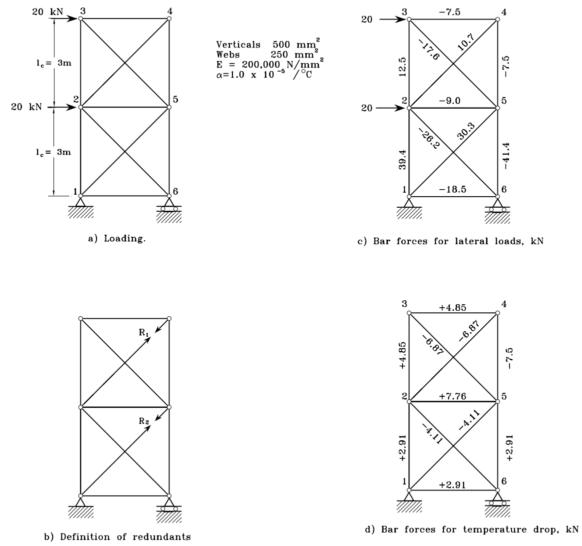


Figure 10.18:

6. We have assumed that a temperature drop of 30°C occurs in the outer members. The corresponding length changes are found in column 11. Again using the virtual work method, column 12 tabulates the internal virtual work of virtual forces $\bar{q}_1$ through displacements Δl where for each bar, $\Delta l = \alpha l \Delta T$. Column 12 is therefore the product of columns 2 and 11. The summation of the elements of column 12 is the displacement D_1 corresponding to release 1. Column 13 repeats this process for D_2 corresponding to release 2.
7. The tabulated information provides the necessary terms for a matrix solution of the problem. We have

$$f = \begin{bmatrix} 8.66 & 1.00 \\ 1.00 & 8.66 \end{bmatrix} L_c/EA_c \tag{10.66-a}$$

$$D_q = \begin{bmatrix} -122.42 \\ -273.12 \end{bmatrix} L_c/EA_c \tag{10.66-b}$$

$$D_\Delta = \begin{bmatrix} 6.36 \\ 4.24 \end{bmatrix} (10)^{-4} L_c \tag{10.66-c}$$

therefore

$$f^{-1} = \begin{bmatrix} 0.117 & -0.0134 \\ -0.0134 & 0.117 \end{bmatrix} EA_c/L_c \tag{10.67}$$

8. The redundant forces due to the applied loading are

$$R = f^{-1}(-D_Q) \tag{10.68-a}$$

$$= \begin{bmatrix} -0.0134 & 0.117 \end{bmatrix} EA_c/L_c \begin{bmatrix} 122.42 \\ 273.12 \end{bmatrix} L_c/EA_c = \begin{bmatrix} 10.66 \\ 30.32 \end{bmatrix} \tag{10.68-b}$$

9. thus $R_1 = 10.66$ kN, $R_2 = 30.32$ kN.
10. The redundant forces due to the temperature drop are $R = f^{-1}(-D_\Delta)$

$$= \begin{bmatrix} -0.0134 & 0.117 \end{bmatrix} EA_c/L_c \begin{bmatrix} -6.36 \\ -4.24 \end{bmatrix} 10^{-4} L_c = \begin{bmatrix} -6.87 \\ -4.11 \end{bmatrix} 10^{-5} EA_c$$

11. Thus with $E = 200$ kN/mm^2, $A_c = 500$ mm^2, we have

$$R_1 = 6.87(10^{-5})(200)(500) = -6.87\text{kN} \tag{10.69-a}$$

$$R_2 = 4.11(10^{-5})(200)(500) = -4.11\text{kN} \tag{10.69-b}$$

12. Using the redundant forces from each of these analyses, the remainder of the bar forces are computed by simple equilibrium. The information in Table 10.4 contains the basis for such computations. The bar force in any bar is the force of column 1 added to that in column 2 multiplied by R_1 plus that in column 3 multiplied by R_2. This follows from the fact that columns 2 and 3 are bar forces caused by a force of unity corresponding to each of the redundants. The results of the calculations are shown in Figure **??**-c for the applied loading and **??**-d for the temperature drop. The forces caused by the temperature drop are similar in magnitude to those caused by wind load in this example. Temperature differences, shrinkage, support settlement, or tolerance errors can cause important effects in statically indeterminate structures. These stresses are self-limiting, however, in the sense that if they cause yielding or some ductile deformation failure does not necessarily follow, rather relief from th ■

Chapter 11

KINEMATIC INDETERMINANCY; STIFFNESS METHOD

11.1 Introduction

11.1.1 Stiffness vs Flexibility

1 There are two classes of structural analysis methods, Table 11.1:

Flexibility: where the primary unknown is a force, where equations of equilibrium are the starting point, static indeterminancy occurs if there are more unknowns than equations, and displacements of the entire structure (usually from virtual work) are used to write an equation of compatibility of displacements in order to solve for the redundant forces.

Stiffness: method is the counterpart of the flexibility one. Primary unknowns are displacements, and we start from expressions for the forces written in terms of the displacements (at the element level) and then apply the equations of equilibrium. The structure is considered to be *kinematically indeterminate* to the nth degree where n is the total number of independent displacements. From the displacements, we then compute the internal forces.

	Flexibility	**Stiffness**
Primary Variable (d.o.f.)	Forces	Displacements
Indeterminancy	Static	Kinematic
Force-Displacement	Displacement(Force)/Structure	Force(Displacement)/Element
Governing Relations	Compatibility of displacement	Equilibrium
Methods of analysis	"Consistent Deformation"	Slope Deflection; Moment Distribution

Table 11.1: Stiffness vs Flexibility Methods

2 In the flexibility method, we started by releasing as many redundant forces as possible in order to render the structure *statically determinate*, and this made it quite flexible. We then applied an appropriate set of forces such that *kinematic constraints* were satisifed.

3 In the stiffness method, we follow a different approach, we stiffen the structure by constraining all the displacements, hence making it *kinematically determinate*, and then we will release all the constraints in such a way to satisfy *equilibrium*.

4 In the slope deflection method, all constraints are released *simultaneously*, thus resulting in a linear system of n equations with n unknowns. In the Moment Distribution method, we release the constraints *one at a time* and essentially solve for the system of n equations *iteratively*.

11.1.2 Sign Convention

5 The sign convention in the stiffness method is different than the one previously adopted in structural analysis/design, Fig. 12.3.

6 In the stiffness method the sign convention adopted is consistent with the prevailing coordinate system. Hence, we define a positive moment as one which is counter-clockwise at the end of the element, Fig. 12.3.

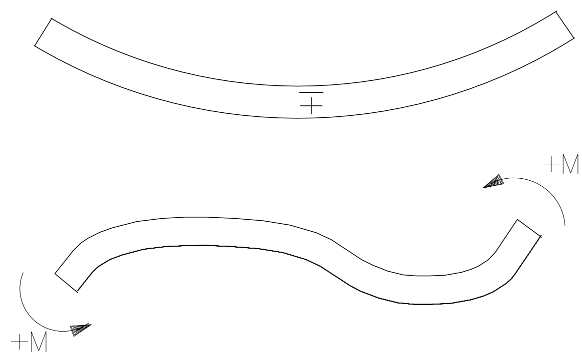


Figure 11.1: Sign Convention, Design and Analysis

11.2 Degrees of Freedom

7 A degree of freedom (d.o.f.) is an independent generalized nodal displacement of a node.

8 The displacements must be linearly independent and thus not related to each other. For example, a roller support on an inclined plane would have three displacements (rotation θ, and two translations u and v), however since the two displacements are kinematically constrained, we only have two independent displacements, Fig. 12.5.

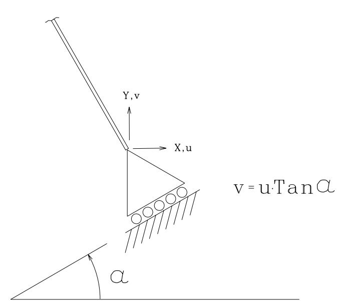


Figure 11.2: Independent Displacements

9 We note that we have been referring to *generalized* displacements, because we want this term to include translations as well as rotations. Depending on the type of structure, there may be none, one or more than one such displacement.

10 The types of degrees of freedom for various types of structures are shown in Table 12.4

Type		Node 1	Node 2
1 Dimensional			
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4
2 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6
3 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1,	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}

Table 11.2: Degrees of Freedom of Different Structure Types Systems

11 Fig. 12.4 also shows the geometric (upper left) and elastic material (upper right) properties associated with each type of element.

11.2.1 Methods of Analysis

12 There are three methods for the stiffness based analysis of a structure

Slope Deflection: (Mohr, 1892) Which results in a system of n linear equations with n unknowns, where n is the degree of kinematic indeterminancy (i.e. total number of independent displacements/rotation).

Moment Distribution: (Cross, 1930) which is an iterative method to solve for the n displacements and corresponding internal forces in flexural structures.

Direct Stiffness method: ($\tilde{1}960$) which is a formal statement of the stiffness method and cast in matrix form is by far the most powerful method of structural analysis.

The first two methods lend themselves to hand calculation, and the third to a computer based analysis.

11.3 Kinematic Relations

11.3.1 Force-Displacement Relations

13 Whereas in the flexibility method we sought to obtain a displacement in terms of the forces (through virtual work) for an entire structure, our starting point in the stiffness method is to develop a set of relationship for the *force in terms of the displacements for a single element.*

$$\left\{\begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\right\} = \left[\begin{array}{cccc} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array}\right] \left\{\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}\right\} \tag{11.1}$$

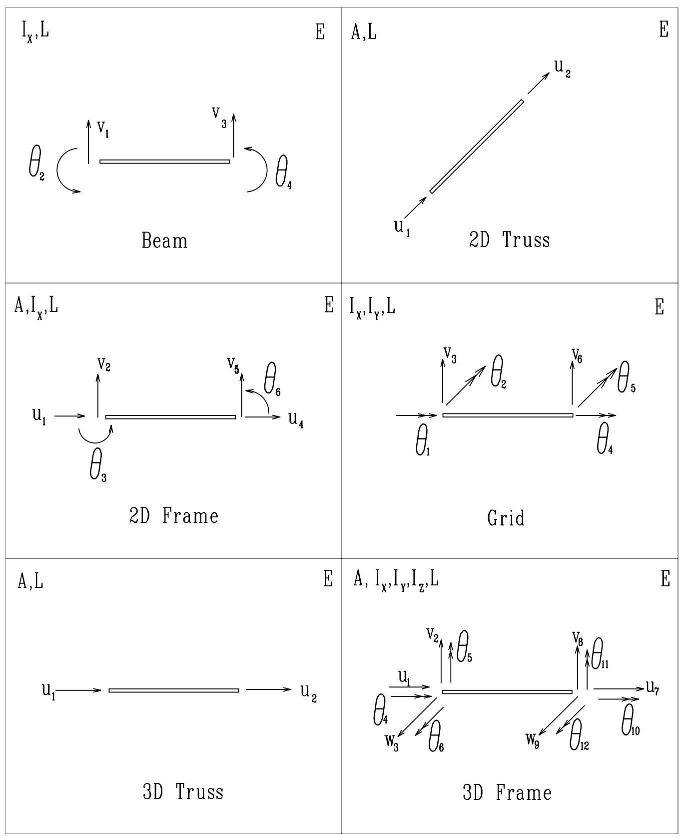


Figure 11.3: Total Degrees of Freedom for various Type of Elements

14 We start from the differential equation of a beam, Fig. 11.4 in which we have all positive known displacements, we have from strength of materials

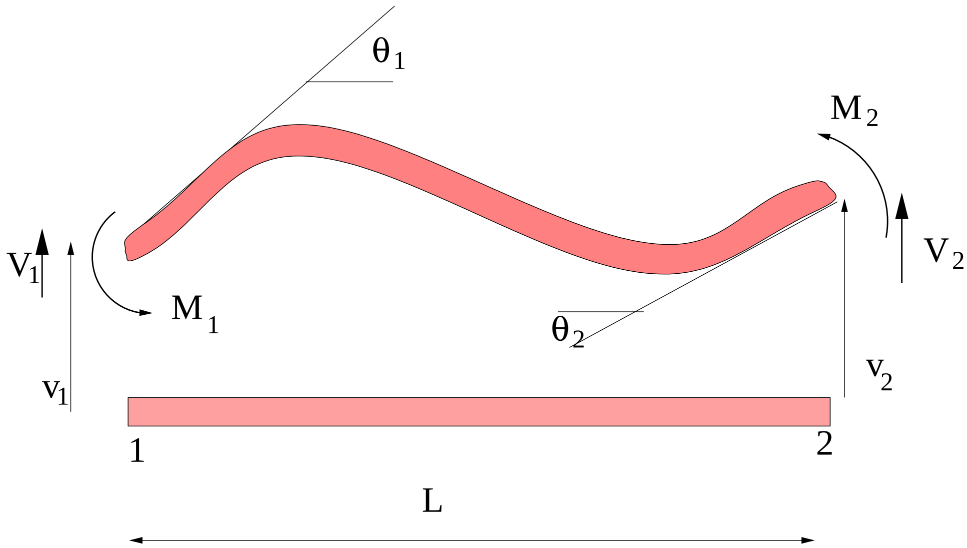


Figure 11.4: Flexural Problem Formulation

$$M = -EI\frac{d^2v}{dx^2} = M_1 - V_1x + m(x) \tag{11.2}$$

where $m(x)$ is the moment applied due to the applied load only. It is positive when counterclockwise.

15 Integrating twice

$$\begin{aligned} -EIv' &= M_1x - \frac{1}{2}V_1x^2 + f(x) + C_1 && (11.3) \\ -EIv &= \frac{1}{2}M_1x^2 - \frac{1}{6}V_1x^3 + g(x) + C_1x + C_2 && (11.4) \end{aligned}$$

where $f(x) = \int m(x)dx$, and $g(x) = \int f(x)dx$.

16 Applying the boundary conditions at $x = 0$

$$\left.\begin{array}{rcl} v' & = & \theta_1 \\ v & = & v_1 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} C_1 & = & -EI\theta_1 \\ C_2 & = & -EIv_1 \end{array}\right. \tag{11.5}$$

17 Applying the boundary conditions at $x = L$ and combining with the expressions for C_1 and C_2

$$\left.\begin{array}{rcl} v' & = & \theta_2 \\ v & = & v_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -EI\theta_2 & = & M_1L - \frac{1}{2}V_1L^2 + f(L) - EI\theta_1 \\ -EIv_2 & = & \frac{1}{2}M_1L^2 - \frac{1}{6}V_1L^3 + g(L) - EI\theta_1L - EIv_1 \end{array}\right. \tag{11.6}$$

18 Since equilibrium of forces and moments must be satisfied, we have:

$$V_1 + q + V_2 = 0 \qquad M_1 - V_1L + m(L) + M_2 = 0 \tag{11.7}$$

where $q = \int_0^L p(x)dx$, thus

$$V_1 = \frac{(M_1 + M_2)}{L} + \frac{1}{L}m(L) \qquad V_2 = -(V_1 + q) \tag{11.8}$$

19 Substituting V_1 into the expressions for θ_2 and v_2 in Eq. 11.6 and rearranging

$$\left\{\begin{array}{rl} M_1 - M_2 & = \frac{2EI_z}{L}\theta_1 + \frac{2EI_z}{L}\theta_2 + m(L) - \frac{2}{L}f(L) \\ 2M_1 - M_2 & = \frac{6EI_z}{L}\theta_1 - \frac{6EI_z}{L^2}v_1 - \frac{6EI_z}{L^2}v_2 + m(L) - \frac{6}{L^2}g(L) \end{array}\right. \tag{11.9}$$

20 Solving those two equations, we obtain:

$$\boxed{\begin{aligned} M_1 &= \underbrace{\frac{2EI_z}{L}(2\theta_1+\theta_2)-\frac{6EI_z}{L^2}(v_2-v_1)}_{I}+\underbrace{M_1^F}_{II} \quad (11.10)\\ M_2 &= \underbrace{\frac{2EI_z}{L}(\theta_1+2\theta_2)-\frac{6EI_z}{L^2}(v_2-v_1)}_{I}+\underbrace{M_2^F}_{II} \quad (11.11)\end{aligned}}$$

where

$$M_1^F = \frac{2}{L^2}\left[Lf(L)-3g(L)\right] \quad (11.12\text{-a})$$

$$M_2^F = -\frac{1}{L^2}\left[L^2m(L)-4Lf(L)+6g(L)\right] \quad (11.12\text{-b})$$

M_1^F and M_2^F are the *fixed end moments* for $\theta_1=\theta_2=0$ and $v_1=v_2=0$, that is *fixed end moments*. They can be obtained either from the analysis of a fixed end beam, or more readily from the preceding two equations.

21 In Eq. 11.10 and 11.11 we observe that the moments developed at the end of a member are caused by: I) end rotation and displacements; and II) fixed end members.

22 Finally, we can substitute those expressions in Eq. 11.8

$$\boxed{\begin{aligned} V_1 &= \underbrace{\frac{6EI_z}{L^2}(\theta_1+\theta_2)-\frac{12EI_z}{L^3}(v_2-v_1)}_{I}+\underbrace{V_1^F}_{II} \quad (11.13)\\ V_2 &= \underbrace{-\frac{6EI_z}{L^2}(\theta_1+\theta_2)+\frac{12EI_z}{L^3}(v_2-v_1)}_{I}+\underbrace{V_2^F}_{II} \quad (11.14)\end{aligned}}$$

where

$$V_1^F = \frac{6}{L^3}\left[Lf(L)-2g(L)\right] \quad (11.15\text{-a})$$

$$V_2^F = -\left[\frac{6}{L^3}\left[Lf(L)-2g(L)\right]+q\right] \quad (11.15\text{-b})$$

23 The relationships just derived enable us now to determine the *stiffness matrix* of a beam element.

$$\boxed{\left\{\begin{array}{c} V_1\\ M_1\\ V_2\\ M_2\end{array}\right\} = \underbrace{\begin{array}{c c} & \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array}\\ \begin{array}{c} V_1\\ M_1\\ V_2\\ M_2\end{array} & \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L}\\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2}\\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}\end{array}\right]\end{array}}_{\mathbf{k}^e}\left\{\begin{array}{c} v_1\\ \theta_1\\ v_2\\ \theta_2\end{array}\right\}} \quad (11.16)$$

11.3.2 Fixed End Actions

24 As mentioned above, the end actions developed in a member involve the end displacements, rotations, and the in-span loads. In-spans loads exhibit themselves in the form of fixed-end forces.

25 The fixed-end actions can be determined from the equations derived above, or by analyzing a fixed-end beam under the applied loads.

26 Note that in both cases, the load has to be assumed *positive*, i.e. pointing up for a beam.

27 The equations derived for calculating the fixed-end actions can be summarized as follows. We recall that, with the x axis directed to the right, positive loads and shear forces act upward and positive

moments are counterclockwise. To calculate the fixed end actions the only thing we need is an expression for the moment of the applied loads (without the end reactions) in the analysis sign convention. Thus with

$$m(x) = \text{moment due to the applied loads at section } x \tag{11.17-a}$$
$$f(x) = \int m(x)dx \tag{11.17-b}$$
$$g(x) = \int f(x)dx \tag{11.17-c}$$
$$q = \int p(x)dx = \text{ total load on the span} \tag{11.17-d}$$

and

$$\boxed{\begin{aligned} M_1^F &= \frac{2}{L^2}\left[Lf(L) - 3g(L)\right] && (11.18)\\ M_2^F &= -\frac{1}{L^2}\left[L^2 m(L) - 4Lf(L) + 6g(L)\right] && (11.19)\\ V_1^F &= \frac{6}{L^3}\left[Lf(L) - 2g(L)\right] && (11.20)\\ V_2^F &= -\frac{6}{L^3}\left[Lf(L) - 2g(L)\right] - q && (11.21)\end{aligned}}$$

11.3.2.1 Uniformly Distributed Loads

28 For a uniformly distributed load w over the entire span,

$$m(x) = -\frac{1}{2}wx^2; \quad f(x) = -\frac{1}{6}wx^3; \quad g(x) = -\frac{1}{24}wx^4; \quad q = wL \tag{11.22}$$

29 Substituting

$$M_1^F = \frac{2}{L^2}\left[L\left(-\frac{1}{6}wL^3\right) - 3\left(-\frac{1}{24}wL^4\right)\right] = \boxed{-\frac{wL^2}{12}} \tag{11.23-a}$$
$$M_2^F = -\frac{1}{L^2}\left[L^2\left(-\frac{1}{2}wL^2\right) - 4L\left(-\frac{1}{6}wL^3\right) + 6\left(-\frac{1}{24}wL^4\right)\right] = \boxed{\frac{wL^2}{12}} \tag{11.23-b}$$
$$V_1^F = \frac{6}{L^3}\left[L\left(-\frac{1}{6}wL^3\right) - 2\left(-\frac{1}{24}wL^4\right)\right] = \boxed{-\frac{wL}{2}} \tag{11.23-c}$$
$$V_2^F = -\frac{6}{L^3}\left[L\left(-\frac{1}{6}wL^3\right) - 2\left(-\frac{1}{24}wL^4\right)\right] - wL = \boxed{-\frac{wL}{2}} \tag{11.23-d}$$

11.3.2.2 Concentrated Loads

30 For a concentrated load we can use the unit step function to find $m(x)$. For a concentrated load P acting at a from the left-hand end with $b = L - a$,

$$\begin{array}{rclcrcl} m(x) &=& -P(x-a)H_a & \text{gives} & m(L) &=& -Pb \\ f(x) &=& -\frac{1}{2}P(x-a)^2H_a & & f(L) &=& -\frac{1}{2}Pb^2 \\ g(x) &=& -\frac{1}{6}P(x-a)^3H_a & & g(L) &=& -\frac{1}{6}Pb^3 \end{array} \tag{11.24}$$

where we define $H_a = 0$ if $x < a$, and $H_a = 1$ if $x \geq a$.

31 and

$$q = P \tag{11.25-a}$$
$$M_1^F = \frac{2}{L^2}\left[L\left(-\frac{1}{2}Pb^2\right) - 3\left(-\frac{1}{6}Pb^3\right)\right] == \boxed{-\frac{Pb^2a}{L^2}} \tag{11.25-b}$$

$$M_2^F = -\frac{1}{L^2}\left[L^2(-Pb) - 4L\left(-\frac{1}{2}Pb^2\right) + 6\left(-\frac{1}{6}Pb^3\right)\right] = \frac{Pb}{1^2}\left(L^2 - 2Lb + b^2\right) \quad (11.25\text{-c})$$
$$= \boxed{\frac{Pba^2}{L^2}} \quad (11.25\text{-d})$$
$$V_1^F = \frac{6}{L^3}\left[L\left(-\frac{1}{2}Pb^2\right) - 2\left(-\frac{1}{6}Pb^3\right)\right] = -\frac{Pb^2}{L^3}(3L-2b) = \boxed{-\frac{Pb^2}{L^3}(3a+b)} \quad (11.25\text{-e})$$
$$V_2^F = -\left(\frac{6}{L^3}\left[L\left(-\frac{1}{2}Pb^2\right) - 2\left(-\frac{1}{6}Pb^3\right)\right] + P\right) = \boxed{-\frac{Pa^2}{L^3}(a+3b)} \quad (11.25\text{-f})$$

32 If the load is applied at midspan ($a = B = L/2$), then the previous equation reduces to

$$\boxed{\begin{aligned} M_1^F &= -\frac{PL}{8} && (11.26)\\ M_2^F &= \frac{PL}{8} && (11.27)\\ V_1^F &= -\frac{P}{2} && (11.28)\\ V_2^F &= -\frac{P}{2} && (11.29)\end{aligned}}$$

11.4 Slope Deflection; Direct Solution

11.4.1 Slope Deflection Equations

33 In Eq. 11.10 and 11.11 if we let $\Delta = v_2 - v_1$ (relative displacement), $\psi = \Delta/L$ (rotation of the chord of the member), and $K = I/L$ (stiffness factor) then the end equations are:

$$\boxed{\begin{aligned} M_1 &= 2EK(2\theta_1 + \theta_2 - 3\psi) + M_1^F && (11.30)\\ M_2 &= 2EK(\theta_1 + 2\theta_2 - 3\psi) + M_2^F && (11.31)\end{aligned}}$$

34 Note that ψ will be positive if counterclockwise, negative otherwise.

35 From Eq. 11.30 and 11.31, we note that if a node has a displacement Δ, then *both* moments in the adjacent element will have the same sign. However, the moments in elements on each side of the node will have different sign.

11.4.2 Procedure

36 To illustrate the general procedure, we consider the two span beam in Fig. 11.5 under the applied load, we will have three rotations θ_1, θ_2, and θ_3 (i.e. *three degrees of freedom*) at the supports. Separating the spans from the supports, we can write the following equilibrium equations for each support

$$M_{12} = 0 \quad (11.32\text{-a})$$
$$M_{21} + M_{23} = 0 \quad (11.32\text{-b})$$
$$M_{32} = 0 \quad (11.32\text{-c})$$

37 The three equilibrium equations in turn can be expressed in terms of the three unknown rotations, thus we can analyze this structure (note that in the slope deflection this will always be the case).

38 Using equations 11.30 and 11.31 we obtain

$$M_{12} = 2EK_{12}(2\theta_1 + \theta_2) + M_{12}^F \quad (11.33\text{-a})$$
$$M_{21} = 2EK_{12}(\theta_1 + 2\theta_2) + M_{21}^F \quad (11.33\text{-b})$$
$$M_{23} = 2EK_{23}(2\theta_2 + \theta_3) \quad (11.33\text{-c})$$
$$M_{32} = 2EK_{23}(\theta_2 + 2\theta_3) \quad (11.33\text{-d})$$

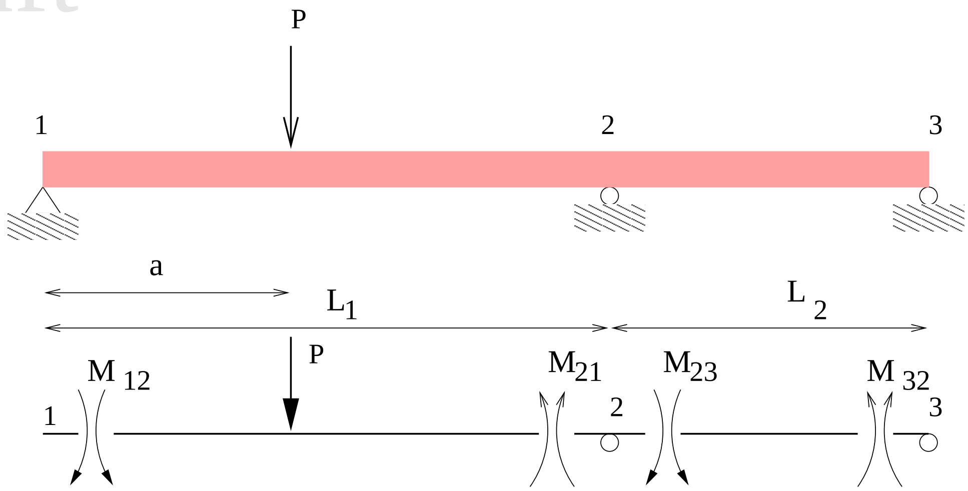


Figure 11.5: Illustrative Example for the Slope Deflection Method

39 Substituting into the equations of equilibrium, we obtain

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ K_{12} & 2(K_{12}+K_{23}) & K_{23} \\ 0 & 1 & 2 \end{bmatrix}}_{\text{Stiffness Matrix}} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{M_{12}^F}{2EK_{12}} \\ -\frac{M_{21}^F}{2E} \\ 0 \end{Bmatrix} \quad (11.34)$$

Where the fixed end moment can be separately determined.

40 Once the rotations are determined, we can then determine the moments from the slope deflection equation Eq. 11.30.

41 The computational requirements of this method are far less than the one involved in the flexibility method (or method of consistent deformation).

11.4.3 Algorithm

42 Application of the slope deflection method requires the following steps:

1. Sketch the deflected shape.
2. Identify all the unknown support degrees of freedom (rotations and deflections).
3. Write the equilibrium equations at all the supports in terms of the end moments.
4. Express the end moments in terms of the support rotations, deflections and fixed end moments.
5. Substitute the expressions obtained in the previous step in the equilibrium equations.
6. Solve the equilibrium equations to determine the unknown support rotation and/or deflections.
7. Use the slope deflection equations to determine the end moments.
8. Draw the moment diagram, careful about the difference in sign convention between the slope deflection moments and the moment diagram.

11.4.4 Examples

■ Example 11-1: Propped Cantilever Beam, (Arbabi 1991)

Find the end moments for the beam of Fig. 11.6

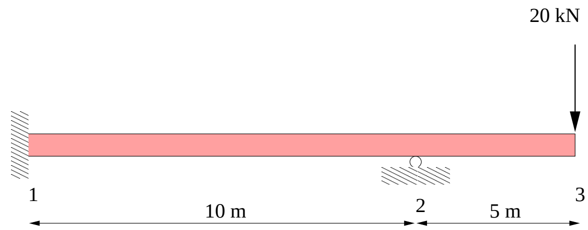


Figure 11.6: Slope Deflection; Propped Cantilever Beam

Solution:

1. The beam is *kinematically indeterminate* to the third degree ($\theta_2,\ \Delta_3,\ \theta_3$), however by replacing the the overhang by a fixed end moment equal to 100 kN.m at support 2, we reduce the degree of kinematic indeterminancy to one (θ_2).
2. The equilbrium relation is

$$M_{21} - 100 = 0 \tag{11.35}$$

3. The members end moments in terms of the rotations are (Eq. 11.30 and 11.31)

$$M_{12} = 2EK_{12}\left(2\theta_1 + \theta_2\right) = \frac{2}{10}EI\theta_2 \tag{11.36-a}$$

$$M_{21} = 2EK_{12}\left(\theta_1 + 2\theta_2\right) = \frac{4}{10}EI\theta_2 \tag{11.36-b}$$

4. Substituting into the equilibrium equations

$$\theta_2 = \frac{10}{4EI}M_{21} = \frac{250}{EI} \tag{11.37}$$

or

$$M_{12} = \frac{2}{10}EI\theta_2 = \frac{2}{10}EI\frac{10}{4EI}M_{21} = \frac{(2)EI(50)}{(10)EI} = \boxed{50 \text{ kN.m}} \tag{11.38}$$

■

■ Example 11-2: Two-Span Beam, Slope Deflection, (Arbabi 1991)

Draw the moment diagram for the two span beam shown in Fig. 11.8 **Solution:**

1. The unknowns are θ_1, θ_2, and θ_3
2. The equilibrium relations are

$$M_{21} + M_{23} = 0 \tag{11.39-a}$$

$$M_{32} = 0 \tag{11.39-b}$$

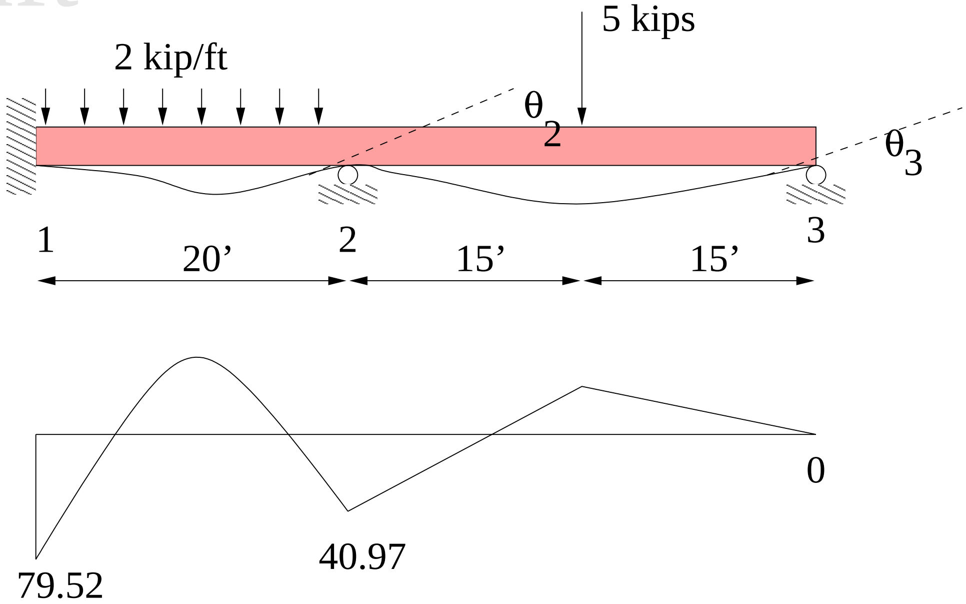


Figure 11.7: Two-Span Beam, Slope Deflection

3. The fixed end moments are given by Eq. 11.23-b and 11.26

$$M_{12}^F = -M_{21}^F = -\frac{wL^2}{12} = -\frac{(-2)(20)^2}{12} = 66.67 \text{ k.ft} \tag{11.40-a}$$

$$M_{23}^F = -M_{32}^F = -\frac{PL}{8} = -\frac{(-5)(30)}{8} = 18.75 \text{ k.ft} \tag{11.40-b}$$

4. The members end moments in terms of the rotations are (Eq. 11.30 and 11.31)

$$M_{12} = 2EK_{12}(\theta_2) + M_{12}^F = \frac{2EI}{L_1}\theta_2 + M_{12}^F = \frac{EI}{10}\theta_2 + 66.67 \tag{11.41-a}$$

$$M_{21} = 2EK_{12}(2\theta_2) + M_{21}^F = \frac{4EI}{L_1}\theta_2 + M_{21}^F = \frac{EI}{5}\theta_2 - 66.67 \tag{11.41-b}$$

$$\begin{aligned} M_{23} &= 2EK_{23}(2\theta_2 + \theta_3) + M_{23}^F = \frac{2EI}{L_2}(2\theta_2 + \theta_3) + M_{23}^F \\ &= \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + 18.75 \end{aligned} \tag{11.41-c}$$

$$\begin{aligned} M_{32} &= 2EK_{23}(\theta_2 + 2\theta_3) + M_{32}^F = \frac{2EI}{L_2}(\theta_2 + 2\theta_3) + M_{32}^F \\ &= \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 - 18.75 \end{aligned} \tag{11.41-d}$$

(11.41-e)

5. Substituting into the equilibrium equations

$$\frac{EI}{5}\theta_2 - 66.67 + \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + 18.75 = 0 \tag{11.42-a}$$

$$\frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 - 18.75 = 0 \tag{11.42-b}$$

or

$$\underbrace{EI\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{Stiffness Matrix}} \left\{ \begin{array}{c} \theta_2 \\ \theta_3 \end{array} \right\} = \left\{ \begin{array}{c} 718.8 \\ 281.25 \end{array} \right\} \tag{11.43}$$

which will give $EI\theta_2 = 128.48$ and $EI\theta_3 = 76.38$
6. Substituting for the moments

$$\begin{aligned} M_{12} &= 12.85 + 66.67 = \boxed{79.52 \text{ k.ft}} && (11.44\text{-a}) \\ M_{21} &= \frac{128.48}{5} - 66.67 = \boxed{-40.97 \text{ k.ft}} && (11.44\text{-b}) \\ M_{23} &= \frac{128.48}{7.5} + \frac{76.38}{15} + 18.75 = \boxed{40.97 \text{ k.ft}}\surd && (11.44\text{-c}) \\ M_{32} &= \frac{128.48}{15} + \frac{76.38}{7.5} - 18.75 = \boxed{0 \text{ k.ft}}\surd && (11.44\text{-d}) \end{aligned}$$

43 The final moment diagram is also shown in Fig. 11.8. We note that the midspan moment has to be separately computed from the equations of equilibrium in order to complete the diagram. ■

■ Example 11-3: Two-Span Beam, Slope Deflection, Initial Deflection, (Arbabi 1991)

Determine the end moments for the previous problem if the middle support settles by 6 inches, Fig. 11.8.

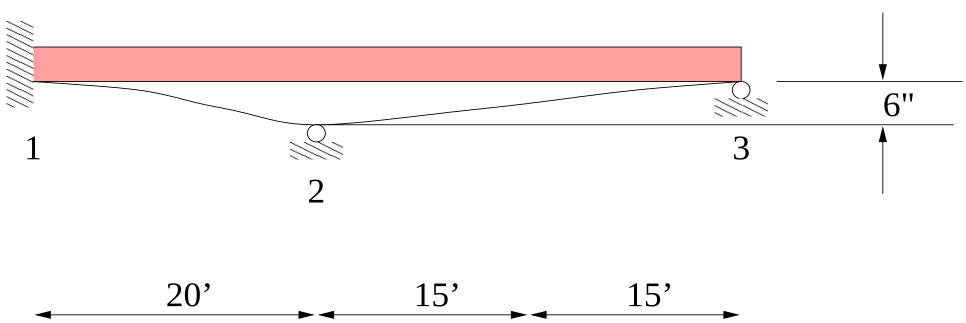


Figure 11.8: Two Span Beam, Slope Deflection, Moment Diagram

Solution:

1. Since we are performing a linear elastic analysis, we can separately analyze the beam for support settlement, and then add then add the moments to those due to the applied loads.
2. The unknowns are θ_1, θ_2, and θ_3
3. The equilibrium relations are

$$\begin{aligned} M_{21} + M_{23} &= 0 && (11.45\text{-a}) \\ M_{32} &= 0 && (11.45\text{-b}) \end{aligned}$$

4. The members end moments in terms of the rotations are (Eq. 11.30 and 11.31)

$$\begin{aligned} M_{12} &= 2EK_{12}\left(\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{EI}{10}\theta_2 + \frac{3EI}{400} && (11.46\text{-a}) \\ M_{21} &= 2EK_{12}\left(2\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{EI}{5}\theta_2 + \frac{3EI}{400} && (11.46\text{-b}) \\ M_{23} &= 2EK_{23}\left(2\theta_2 + \theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + \frac{EI}{300} && (11.46\text{-c}) \\ M_{32} &= 2EK_{23}\left(\theta_2 + 2\theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 + \frac{EI}{300} && (11.46\text{-d}) \end{aligned}$$

5. Substituting into the equilibrium equations

$$\frac{EI}{5}EI\theta_2 + \frac{3EI}{400} + \frac{EI}{15}\theta_3 + \frac{EI}{300} = 0 \quad (11.47\text{-a})$$

$$\frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 + \frac{5EI}{300} = 0 \quad (11.47\text{-b})$$

or

$$\underbrace{EI\begin{bmatrix} 100 & 20 \\ 20 & 40 \end{bmatrix}}_{\text{Stiffness Matrix}}\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = EI\begin{Bmatrix} -\frac{13}{4} \\ -1 \end{Bmatrix} \quad (11.48)$$

which will give $\theta_2 = -\frac{5.5}{180} = -0.031$ radians and $\theta_3 = \frac{-1+\frac{5.5}{9}}{40} = -0.0097$ radians

6. Thus the *additional* moments due to the settlement are

$$M_{12} = \frac{EI}{10}(-0.031) + \frac{3EI}{400} = \boxed{0.0044\text{EI}} \quad (11.49\text{-a})$$

$$M_{21} = \frac{EI}{5}(-0.031) + \frac{3EI}{400} = \boxed{0.0013\text{EI}} \quad (11.49\text{-b})$$

$$M_{23} = \frac{EI}{7.5}(-0.031) + \frac{EI}{15}(0.0097) + \frac{EI}{300} = \boxed{0.0015\text{EI}} \quad (11.49\text{-c})$$

$$M_{32} = \frac{EI}{15}\theta_2 + \frac{EI}{7.5}(0.0097) + \frac{EI}{300} = \boxed{0.}\surd \quad (11.49\text{-d})$$

■

■ **Example 11-4:** *dagger* **Frames, Slope Deflection, (Arbabi 1991)**

Determine the end moments for the frame shown in Fig. 11.9.

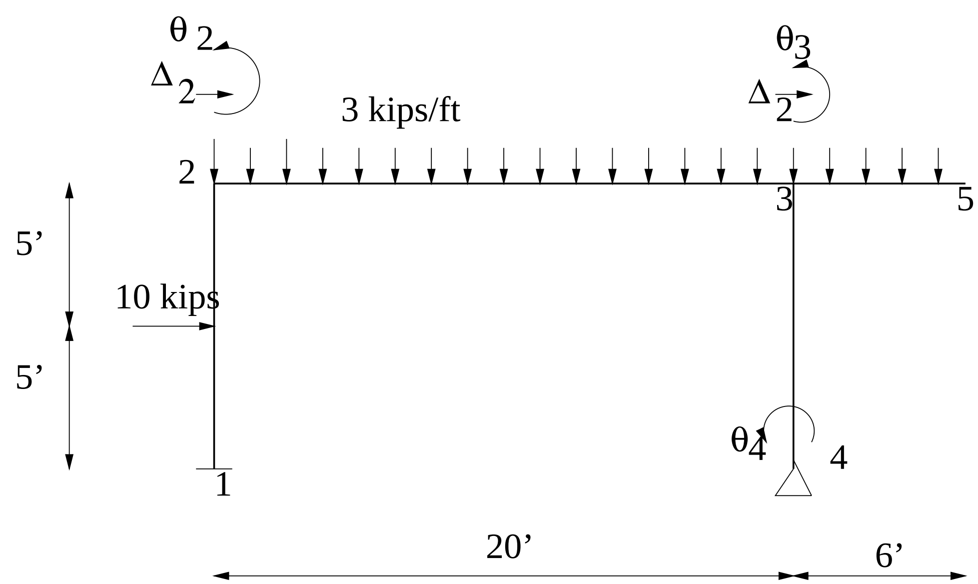


Figure 11.9: Frame Analysis by the Slope Deflection Method

Solution:

1. The effect of the 35 cantilever can be included by replacing it with its end moment.

$$M_3 = -wL\frac{L}{2} = -(3)(6)(3) = -54 \text{ k.ft} \quad (11.50)$$

2. The unknowns displacements and rotations are

- Δ_2 and θ_2 at joint 2.
- θ_3 and θ_4 at joints 3 and 4.

We observe that due to the lack of symmetry, there will be a lateral displacement in the frame, and neglecting axial deformations, $\Delta_2 = \Delta_3$.

3. The equilibrium relations are

$$\begin{aligned} M_{21} + M_{23} &= 0 & (11.51\text{-a}) \\ M_{32} + M_{34} &= -54 & (11.51\text{-b}) \\ M_{43} &= 0 & (11.51\text{-c}) \\ V_{12} + V_{43} - 10 &= 0 & (11.51\text{-d}) \end{aligned}$$

Thus we have four unknown displacements and four equations. However, the last two equations are in terms of the shear forces, and we need to have them in term of the end moments, this can be achieved through the following equilibrium relations

$$\begin{aligned} V_{12} &= \frac{M_{12} + M_{21} + 50}{L_{12}} & (11.52\text{-a}) \\ V_{43} &= \frac{M_{34} + M_{43}}{L_{34}} & (11.52\text{-b}) \end{aligned}$$

Hence, all four equations are now in terms of the moments.

4. The fixed end moments fro member 23 are

$$\begin{aligned} M_{21}^F &= -\frac{PL}{8} = -\frac{-(10)(10)}{8} = 12.5 \text{ k.ft} & (11.53\text{-a}) \\ M_{23}^F &= -\frac{wL^2}{12} = -\frac{-(3)(20)^2}{12} = 100 \text{ k.ft} & (11.53\text{-b}) \end{aligned}$$

5. The members end moments in terms of the rotations are (Eq. 11.30 and 11.31)

$$\begin{aligned} M_{12} &= 2EK_{12}\left(\theta_2 - 3\frac{\Delta_2}{L_{12}}\right) + M_{12}^F = 0.2EI(\theta_2 - 0.3\Delta_2) + 12.5 & (11.54\text{-a}) \\ M_{21} &= 2EK_{12}\left(2\theta_2 - 3\frac{\Delta_2}{L_{21}}\right) + M_{21}^F = 0.2EI(2\theta_2 - 0.3\Delta_2) - 12.5 & (11.54\text{-b}) \\ M_{23} &= 2EK_{23}\left(2\theta_2 + \theta_3\right) + M_{23}^F = 0.1EI(2\theta_2 + \theta_3) + 100. & (11.54\text{-c}) \\ M_{32} &= 2EK_{32}\left(\theta_2 + 2\theta_3\right) + M_{32}^F = 0.1EI(\theta_2 + 2\theta_3) - 100. & (11.54\text{-d}) \\ M_{34} &= 2EK_{34}\left(2\theta_3 + \theta_4 - \frac{3\Delta_2}{L_{34}}\right) = 0.2EI(2\theta_3 + \theta_4 - 0.3\Delta_2) & (11.54\text{-e}) \\ M_{43} &= 2EK_{43}\left(\theta_3 + 2\theta_4 - \frac{3\Delta_2}{L_{34}}\right) = 0.2EI(\theta_3 + 2\theta_4 - 0.3\Delta_2) & (11.54\text{-f}) \end{aligned}$$

6. Substituting into the equilibrium equations and dividing by EI

$$\begin{aligned} 6\theta_2 + \theta_3 - 0.6\Delta_2 &= -\frac{875}{EI} & (11.55\text{-a}) \\ \theta_2 + 6\theta_3 + 2\theta_4 - 0.6\Delta_2 &= \frac{460}{EI} & (11.55\text{-b}) \\ \theta_3 + 2\theta_4 - 0.3\Delta_2 &= 0 & (11.55\text{-c}) \end{aligned}$$

and the last equilibrium equation is obtained by substituting V_{12} and V_{43} and multiplying by $10/EI$:

$$\theta_2 + \theta_3 + \theta_4 - 0.4\Delta_2 = -\frac{83.3}{EI} \tag{11.56}$$

or

$$\underbrace{EI\begin{bmatrix} 6 & 1 & 0 & -0.6 \\ 1 & 6 & 2 & -0.6 \\ 0 & 1 & 2 & -0.3 \\ 1 & 1 & 1 & -0.4 \end{bmatrix}}_{\text{Stiffness Matrix}} \left\{ \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_4 \\ \Delta_2 \end{array} \right\} = \frac{1}{EI} \left\{ \begin{array}{c} -875 \\ 460 \\ 0 \\ -83.3 \end{array} \right\} \tag{11.57}$$

which will give

$$\left\{ \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_4 \\ \Delta_2 \end{array} \right\} = \frac{1}{EI} \left\{ \begin{array}{c} -294.8 \\ 68.4 \\ -240.6 \\ -1,375.7 \end{array} \right\} \tag{11.58}$$

7. Substitution into the slope deflection equations gives the end-moments

$$\left\{ \begin{array}{c} M_{12} \\ M_{21} \\ M_{23} \\ M_{32} \\ M_{34} \\ M_{43} \end{array} \right\} = \boxed{\left\{ \begin{array}{c} 36.0 \\ -47.88 \\ 47.88 \\ -115.80 \\ 61.78 \\ 0 \end{array} \right\}} \tag{11.59}$$

■

11.5 Moment Distribution; Indirect Solution

11.5.1 Background

43 The moment distribution is essentially a variation of the slope deflection method, however rather than solving a system of n linear equations directly, the solution is achieved iteratively through a successive series of operations.

44 The method starts by locking all the joints, and then unlock each joint in succession, the internal moments are then "distributed" and balanced until all the joints have rotated to their final (or nearly final) position.

45 In order to better understand the method, some key terms must first be defined.

11.5.1.1 Sign Convention

46 The sign convention is the same as the one adopted for the slope deflection method, counter-clockwise moment at*element's end* is positive.

11.5.1.2 Fixed-End Moments

47 Again fixed end moments are the same set of forces defined in the slope deflection method for a beam which is rigidly connected at both ends.

48 Consistent with the sign convention, the fixed end moments are the moments caused by the applied load at the end of the beam (assuming it is rigidly connected).

11.5.1.3 Stiffness Factor

49 We define the *stiffness factor* as *the moment required to rotate the end of a beam by a unit angle of one radian, while the other end is fixed.* From Eq. 11.10, we set $\theta_2 = v_1 = v_2 = 0$, and $\theta_1 = 1$, this will

yield

$$\boxed{K = \frac{4EI}{L} \text{ Far End Fixed}} \tag{11.60}$$

50 We note that this is slightly different than the definition given in the slope deflection method (I/L).

51 If the far end of the beam is hinged rather than fixed, then we will have a *reduced stiffness factor*. From Eq. 11.10 and 11.11, with $M_2 = v_1 = v_2 = 0$, we obtain

$$M_2 = \frac{2EI_z}{L}(\theta_1 + 2\theta_2) = 0 \tag{11.61-a}$$

$$\Rightarrow \theta_2 = -\frac{\theta_1}{2} \tag{11.61-b}$$

Substituting into M_1

$$\left.\begin{array}{rcl} M_1 & = & \frac{2EI_z}{L}(2\theta_1 + \theta_2) \\ \theta_2 & = & -\frac{\theta_1}{2} \end{array}\right\} \boxed{M_1 = \frac{3EI}{L}\theta_1 \text{ Far End Pinned}} \tag{11.62}$$

52 Comparing this reduced stiffness $\frac{3EI}{L}$ with the stiffness of a beam, we define the *reduced stiffness factor*

$$\boxed{\text{Reduced Stiffness Factor} = \frac{K_{red}}{K} = \frac{3}{4} \text{ Far End Pinned}} \tag{11.63}$$

11.5.1.4 Distribution Factor (DF)

53 If a member is applied to a fixed-connection joint where there is a total of n members, then from equilibrium:

$$M = M_1 + M_2 + \cdots + M_n \tag{11.64}$$

However, from Eq. 11.30, and *assuming the other end of the member to be fixed*, then

$$M = K_1\theta + K_2\theta + \cdots + K_n\theta \tag{11.65}$$

or

$$\boxed{DF_i = \frac{M_i}{M} = \frac{K_i}{\Sigma K_i}} \tag{11.66}$$

54 Hence if a moment M is applied at a joint, then the portion of M carried by a member connected to this joint is proportional to the distribution factor. The stiffer the member, the greater the moment carried.

55 Similarly, $DF = 0$ for a fixed end, and $DF = 1$ for a pin support.

11.5.1.5 Carry-Over Factor

56 Again from Eq. 11.30, and 11.31

$$M_1 = 2EK(2\theta_1 + \theta_2 - 3\psi) + M_1^F \tag{11.67-a}$$

$$M_2 = 2EK(\theta_1 + 2\theta_2 - 3\psi) + M_2^F \tag{11.67-b}$$

we observe that if one end of the beam is restrained ($\theta_2 = \psi = 0$), and there is no member load, then the previous equations reduce to

$$\left.\begin{array}{rcl} M_1 & = & 2EK(2\theta_1) \\ M_2 & = & 2EK(\theta_1) \end{array}\right\} \Rightarrow M_2 = \frac{1}{2}M_1 \tag{11.68}$$

57 Hence in this case the carry-over factor represents the fraction of M that is "carried over" from the rotating end to the fixed one.

$$\boxed{CO = \frac{1}{2} \text{ Far End Fixed}} \tag{11.69}$$

58 If the far end is pinned there is no carry over.

11.5.2 Procedure

59 The general procedure of the Moment Distribution method can be described as follows:

1. Constrain all the rotations and translations.
2. Apply the load, and determine the fixed end moments (which may be caused by element loading, or support translation).
3. At any given joint i equilibrium is not satisfied $M^F_{left} \neq M^F_{right}$, and the net moment is M_i
4. We enforce equilibrium by applying at the node $-M_i$, in other words we *balance* the forces at the node.
5. How much of M_i goes to each of the elements connected to node i depends on the *distribution factor*.
6. But by applying a portion of $-M_i$ to the end of a beam, while the other is still constrained, from Eq. 11.30, half of that moment must also be *carried over* to the other end.
7. We then lock node i, and move on to node j where these operations are repeated
   (a) Sum moments
   (b) Balance moments
   (c) Distribute moments (K, DF)
   (d) Carry over moments (CO)
   (e) lock node
8. repeat the above operations until all nodes are balanced, then *sum* all moments.
9. The preceding operations can be easily carried out through a proper tabulation.

60 If an end node is hinged, then we can use the *reduced stiffness factor* and we will not carry over moments to it.

61 Analysis of frame with unsymmetric loading, will result in lateral displacements, and a two step analysis must be performed (see below).

11.5.3 Algorithm

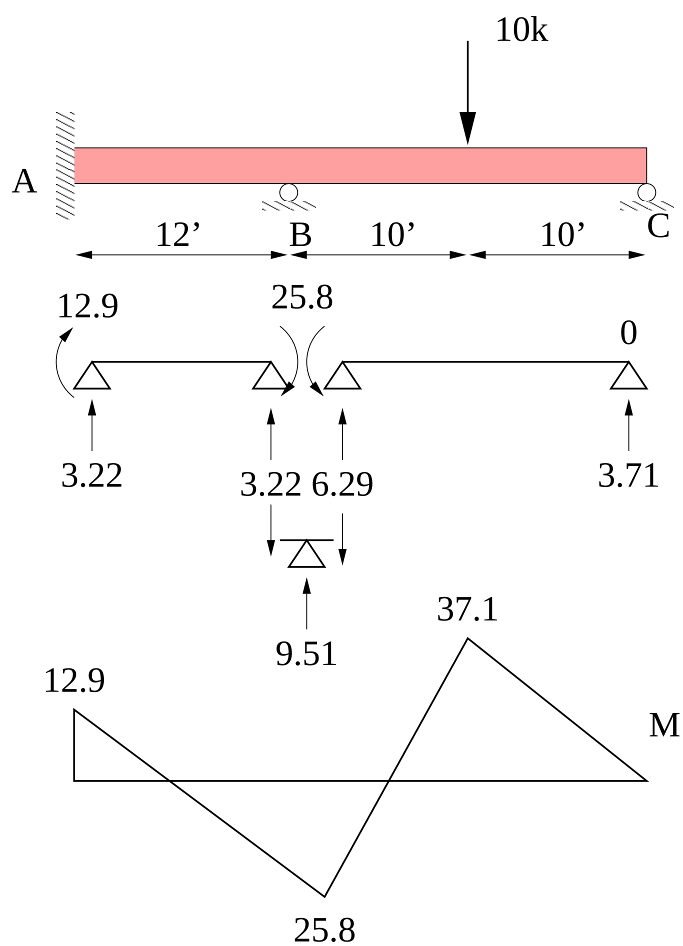
1. Calculate the stiffness ($K = 4EI/L$, however this can often be simplified to I/L) factor for all the members and the distribution factors at all the joints.
2. If a member AB is pinned at B, then $K^{AB} = 3EI/L$, and $K^{BA} = 4EI/L$. Thus, we must apply the reduced stiffness factor to K^{AB} *only* and not to K^{BA}.
3. The carry-over factor is $\frac{1}{2}$ for members with constant cross-section.
4. Find the fixed-end moments for all the members. Note that even if the end of a member is pinned, determine the fixed end moments as if it was fixed.
5. Start out by fixing all the joints, and release them one at a time.

6. If a node is pinned, start by balancing this particular node. If no node is pinned, start from either end of the structure.
7. Distribute the unbalanced moment at the released joint
8. Carry over the moments to the far ends of the members (unless it is pinned).
9. Fix the joint, and release the next one.
10. Continue releasing joints until the distributed moments are insignificant. If the last moments carried over are small and cannot be distributed, it is better to discard them so that the joints remain in equilibrium.
11. Sum up the moments at each end of the members to obtain the final moments.

11.5.4 Examples

■ **Example 11-5: Continuous Beam, (Kinney 1957)**

Solve for moments at A and B by moment distribution, using (a) the ordinary method, and (b) the simplified method.



Solution:

1. For this example the fixed-end moments are computed as follows:

$$M_{BC}^F = \frac{PL}{8} = \frac{(10)(20)}{8} = +25.0 \text{ k.ft} \quad (11.70\text{-a})$$

$$M_{CB}^F = -25.0 \text{ k.ft} \quad (11.70\text{-b})$$

2. Since the relative stiffness is given in each span, the distribution factors are

$$DF_{AB} = \frac{K_{AB}}{\Sigma K} = \frac{5}{\infty + 5} = 0, \quad (11.71\text{-a})$$

$$DF_{BA} = \frac{K_{BA}}{\Sigma K} = \frac{5}{8} = 0.625, \quad (11.71\text{-b})$$

$$DF_{BC} = \frac{K_{BC}}{\Sigma K} = \frac{3}{8} = 0.375, \quad (11.71\text{-c})$$

$$DF_{CB} = \frac{K_{CB}}{\Sigma K} = \frac{3}{3} = 1. \quad (11.71\text{-d})$$

3. The balancing computations are shown below.

Joint	A	B		C	Balance	CO
Member	AB	BA	BC	CB		
K	5	5	3	3		
DF	0	0.625	0.375	1		
FEM			+25.0	-25.0		
			+12.5	+25.0	C	BC
	-11.7	-23.4	-14.1	-7.0	B	AB; CB
			+3.5	+7.0	C	BC
	-1.1	-2.2	-1.3	-0.6	B	AB; CB
			+0.3	+0.6	C	BC
	-0.1	-0.2	-0.1		B	AB
Total	**-12.9**	**-25.8**	**+25.8**	**0**		

4. The above solution is that referred to as the *ordinary method*, so named to designate the manner of handling the balancing at the simple support at C. It is known, of course, that the final moment must be zero at this support because it is simple.
5. Consequently, the first step is to balance the fixed-end moment at C to zero. The carry-over is then made immediately to B. When B is balanced, however, a carry-over must be made back to C simply because the relative stiffness of BC is based on end C of this span being fixed. It is apparent, however, that the moment carried back to C (in this case, -7.0) cannot exist at this joint. Accordingly, it is immediately balanced out, and a carry-over is again made to B, this carry-over being considerably smaller than the first. Now B is again balanced, and the process continues until the numbers involved become too small to have any practical value.
6. Alternatively, we can use the *simplified method*. It was previously shown that if the support at C is simple and a moment is applied at B, then the resistance of the span BC to this moment is reduced to three-fourths of the value it would have had with C fixed. Consequently, if the relative stiffness of span BC is reduced to three-fourths of the value given, it will not be necessary to carry over to C.

Joint	A	B		C	Balance	CO
Member	AB	BA	BC	CB		
K	5	5	$\frac{3}{4} \times 3 = 2.25$	3.00		
DF	0	0.69	0.31	1		
FEM			+25.0	-25.0		
			+12.5	+25.0	C	BC
	-12.9	-25.8	-11.7		B	AB
Total	**-12.9**	**-25.8**	**+25.8**	**0**		

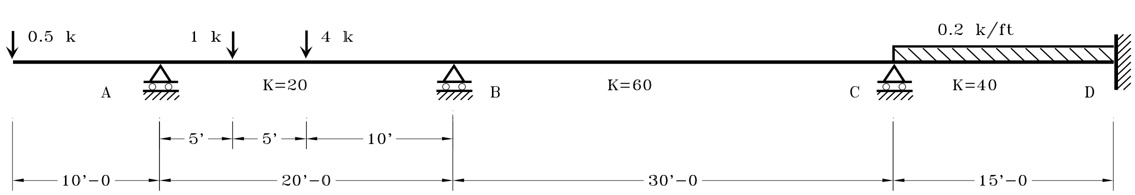
7. From the standpoint of work involved, the advantage of the simplified method is obvious. It should always be used when the external (terminal) end of a member rests on a simple support, but it does not apply when a structure is continuous at a simple support. Attention is called to the fact that when the opposite end of the member is simply supported, the reduction factor for stiffness is always $\frac{3}{4}$ for a prismatic member but a variable quantity for nonprismatic members.
8. One valuable feature of the tabular arrangement is that of dropping down one line for each balancing operation and making the carry-over on the same line. This practice clearly indicates the order of balancing the joints, which in turn makes it possible to check back in the event of error. Moreover, the placing of the carry-over on the same line with the balancing moments definitely decreases the chance of omitting a carry-over.
9. The correctness of the answers may in a sense be checked by verifying that $\Sigma M = 0$ at each joint. However, *even though the final answers satisfy this equation at every joint, this in no way a check on the initial fixed-end moments.* These fixed-end moments, therefore, should be checked with great care before beginning the balancing operation. Moreover, it occasionally happens that compensating errors are made in the balancing, and these errors will not be apparent when checking $\Sigma M = 0$ at each joint.
10. To draw the final shear and moment diagram, we start by drawing the free body diagram of each beam segment with the computed moments, and then solve from statics for the reactions:

$$12,9 + 25.8 - 12V_A = 0 \Rightarrow V_A = R_A = 3.22 \text{ k} \downarrow \quad (11.72\text{-a})$$
$$V_A + V_B^L = 0 \Rightarrow V_B^L = 3.22 \text{ k} \uparrow \quad (11.72\text{-b})$$
$$25.8 + (10)(10) - 20V_B^R = 0 \Rightarrow V_B^R = 6.29 \text{ k} \uparrow \quad (11.72\text{-c})$$
$$6.29 + V_C - 10 = 0 \Rightarrow V_C = R_C 3.71 \text{ k} \uparrow \quad (11.72\text{-d})$$
$$-V_B^L - V_B^R + R_B = 0 \Rightarrow R_B = 9.51 \text{ k} \uparrow \quad (11.72\text{-e})$$
$$\text{Check: } R_A + R_B + R_C - 10 = -3.22 + 9.51 + 3.71 - 10 = 0\surd \quad (11.72\text{-f})$$
$$M_{BC}^+ = (3.71)(10) = 37.1 \text{ k.ft} \quad (11.72\text{-g})$$

■

■ Example 11-6: Continuous Beam, Simplified Method, (Kinney 1957)

Using the simplified method of moment distribution, find the moments in the following continuous beam. The values of I as indicated by the various values of K, are different for the various spans. Determine the values of reactions, draw the shear and bending moment diagrams, and sketch the deflected structure.



Solution:

1. Fixed-end moments:

$$M_{AO}^F = -(0.5)(10) = -5.0 \text{ k.ft} \quad (11.73)$$

For the 1 k load:

$$M_{AB}^F = \frac{Pab^2}{L^2} = \frac{(1)(5)(15^2)}{20^2} = +2.8 \text{ k.ft} \quad (11.74\text{-a})$$
$$M_{BA}^F = \frac{Pa^2b}{L^2} = \frac{(1)(5^2)(15)}{20^2} = -0.9 \text{ k.ft} \quad (11.74\text{-b})$$

For the 4 k load:

$$M_{AB}^F = \frac{PL}{8} = \frac{(4)(20)}{8} = +10.0 \text{ k.ft} \tag{11.75-a}$$

$$M_{BA}^F = -10.0 \text{ k.ft} \tag{11.75-b}$$

For the uniform load:

$$M_{CD}^F = \frac{wL^2}{12} = \frac{(0.2)(15^2)}{12} = +3.8 \text{ k.ft} \tag{11.76-a}$$

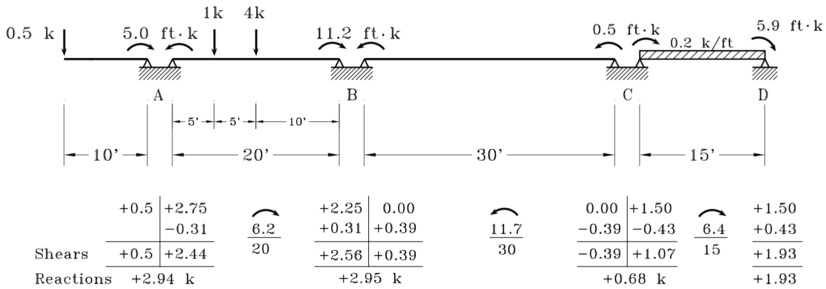
$$M_{DC}^F = -3.8 \text{ k.ft} \tag{11.76-b}$$

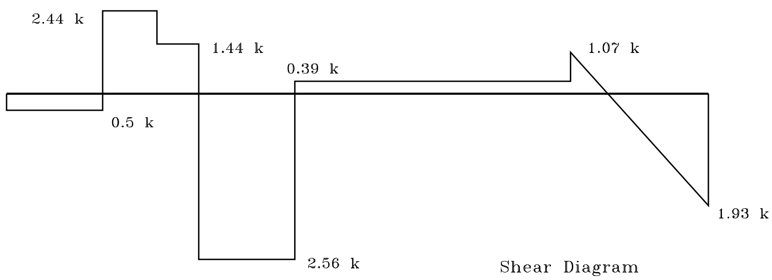
2. The balancing operation is shown below

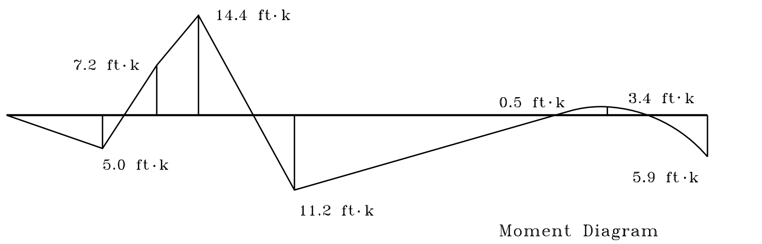
Joint	A		B		C		D	Balance	CO
Member	AO	AB	BA	BC	CB	CD	DC		
K	0	20	$\frac{3}{4}(20) = 15$	60	60	40	40		
DF	0	1	0.2	0.8	0.6	0.4	0		
FEM	-5.0	+12.8	-10.9			+3.8	-3.8		
		-7.8	-3.9					A	BA
			+2.9	+11.9	+5.9			B	CB
				-2.9	-5.8	-3.9	-1.9	C	DC; BC
			+0.6	+2.3	+1.1			B	CB
				-0.3	-0.7	-0.4	-0.2	C	DC; BC
			+0.1	+0.2				B	
Total	**-5.0**	**+5.0**	**-11.2**	**+11.2**	**+0.5**	**-0.5**	**-5.9**		

3. The only new point in this example is the method of handling the overhanging end. It is obvious that the final internal moment at A must be 5.0 k.ft and, accordingly, the first step is to balance out 7.8 k.ft of the fixed-end moment at AB, leaving the required 5.0 k.ft for the internal moment at AB. Since the relative stiffness of BA has been reduced to three-fourths of its original value, to permit considering the support at A as simple in the balancing, no carry-over from B to A is required.

4. The easiest way to determine the reactions is to consider each span as a free body. End shears are first determined as caused by the loads alone on each span and, following this, the end shears caused by the end moments are computed. These two shears are added algebraically to obtain the net end shear for each span. An algebraic summation of the end shears at any support will give the total reaction.



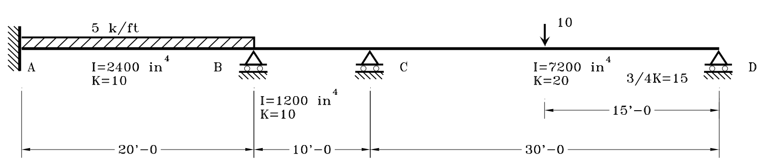




■

■ **Example 11-7: Continuous Beam, Initial Settlement, (Kinney 1957)**

For the following beam find the moments at A, B, and C by moment distribution. The support at C settles by 0.1 in. Use $E = 30,000$ k/in^2.



Solution:

1. Fixed-end moments: Uniform load:

$$M^F_{AB} = \frac{wL^2}{12} = \frac{(5)(20^2)}{12} = +167 \text{ k.ft} \tag{11.77-a}$$

$$M^F_{BA} = -167 \text{ k.ft} \tag{11.77-b}$$

Concentrated load:

$$M^F_{CD} = \frac{PL}{8} = \frac{(10)(30)}{8} = +37.5 \text{ k.ft} \tag{11.78-a}$$

$$M^F_{DC} = -37.5 \text{ k.ft} \tag{11.78-b}$$

Moments caused by deflection:

$$M^F_{BC} = \frac{6EI\Delta}{L^2} = \frac{6(30,000)(1,200)(0.1)}{(120)^2} = +1,500 \text{ k.in} = +125 \text{ k.ft} \quad (11.79\text{-a})$$

$$M^F_{CB} = +125 \text{ k.ft} \quad (11.79\text{-b})$$

$$M^F_{CD} = \frac{6EI\Delta}{L^2} = \frac{(6)(30,000)(7,200)(0.1)}{(360)^2} = 1,000 \text{ k.in} = -83 \text{ k.ft} \quad (11.79\text{-c})$$

$$M^F_{DC} = -83 \text{ k.ft} \quad (11.79\text{-d})$$

2. Moment distribution

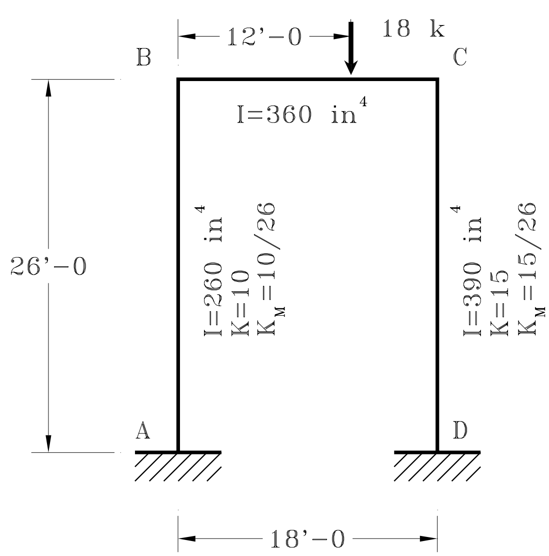
Joint	A	B		C		D	Balance	CO
Member	AB	BA	BC	CB	CD	DC		
K	10	10	10	10	$\frac{3}{4}(20) = 15$	20		
DF	0	0.5	0.5	0.4	0.6	1		
FEM Load	+167	-167			+38	-38		
FEM Δ			+125	+125	-83	-83		
					+60	+121	D	CD
			-28	-56	-84		C	BC
	+17	+35	+35	+17			B	AB; CB
			-3	-7	-10		C	BC
	+1	+2	+1				B	
Total	**+185**	**-130**	**+130**	**+79**	**-79**	**0**		

The fixed-end moments caused by a settlement of supports have the same sign at both ends of each span adjacent to the settling support. The above computations have been carried to the nearest k.ft, which for moments of the magnitudes involved, would be sufficiently close for purposes of design.

■

■ Example 11-8: Frame, (Kinney 1957)

Find all moments by moment distribution for the following frame Draw the bending moment diagram and the deflected structure.



Solution:

1. The first step is to perform the usual moment distribution. The reader should fully understand that this balancing operation adjusts the internal moments at the ends of the members by a series of corrections as the joints are considered to rotate, until $\Sigma M = 0$ at each joint. The reader should also realize that *during this balancing operation no translation of any joint is permitted*.
2. The fixed-end moments are

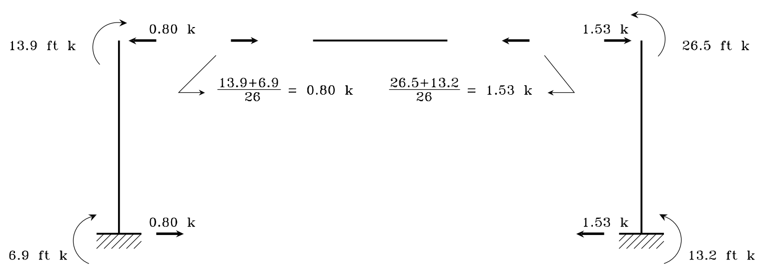
$$M^F_{BC} = \frac{(18)(12)(6^2)}{18^2} = +24 \text{ k.ft} \quad (11.80\text{-a})$$

$$M^F_{CB} = \frac{(18)(6)(12^2)}{18^2} = -48 \text{ k.ft} \quad (11.80\text{-b})$$

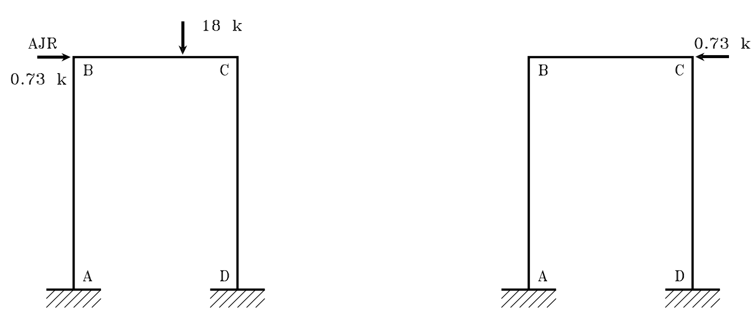
3. Moment distribution

Joint	A	B		C		D	Balance	CO
Member	AB	BA	BC	CB	CD	DC		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM			+24.0	-48.0			FEM	
			+13.7	+27.4	+20.6	+10.3	C	DC; BC
	-6.3	-12.6	-25.1	-12.5			B	AB; CB
			+3.6	+7.1	+5.4	+2.7	C	BC; DC
	-0.6	-1.2	-2.4	-1.2			B	AB; CB
			+0.3	+0.7	+0.5	+0.02	C	BC; DC
		-0.1	-0.2				B	
Total	**-6.9**	**-13.9**	**+13.9**	**-26.5**	**+26.5**	**+13.2**		

4. The *final moments listed in the table are correct only if there is no translation of any joint.* It is therefore necessary to determine whether or not, with the above moments existing, there is any tendency for side lurch of the top of the frame.
5. If the frame is divided into three free bodies, the result will be as shown below.



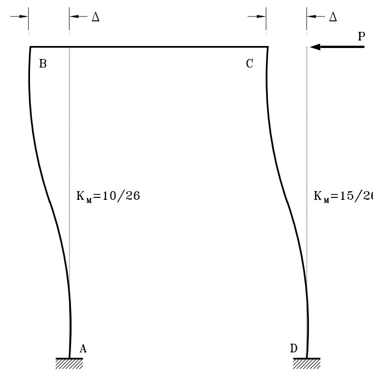
Inspection of this sketch indicates that if the moments of the first balance exist in the frame, there is a net force of $1.53 - 0.80 = 0.73$ k tending to sway the frame to the left. In order to prevent side-sway, and thus allow these moments to exist (temporarily, for the purpose of the analysis), it is necessary that an imaginary horizontal force be considered to act to the right at B or C. This force is designated as the *artificial joint restraint* (abbreviated as AJR) and is shown below.



6. This illustration now shows the complete load system which would have to act on the structure if the final moments of the first balance are to be correct. The AJR, however, cannot be permitted to remain,

and thus its effect must be cancelled. This may be accomplished by finding the moments in the frame resulting from a force equal but opposite to the AJR and applied at the top.

7. Although it is not possible to make a direct solution for the moments resulting from this force, they may be determined indirectly. Assume that some unknown force P acts on the frame, as shown below



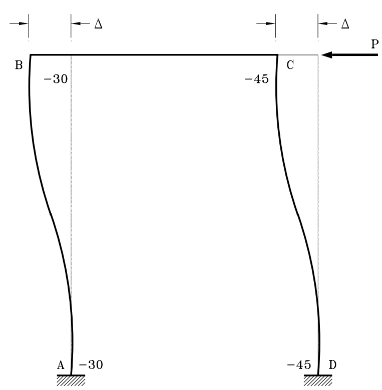
and causes it to deflect laterally to the left, without joint rotation, through some distance Δ. Now, regardless of the value of P and the value of the resulting Δ, the fixed-end moments induced in the ends of the columns must be proportional to the respective values of K_M.

62 Recalling that the fixed end moment is $M^F = 6EI\frac{\Delta}{L^2} = 6EK_m\Delta$, where $K_m = \frac{I}{L^2} = \frac{K}{L}$ we can write

$$\Delta = \frac{M^F_{AB}}{6EK_m} = \frac{M^F_{DC}}{6EK_m} \quad (11.81\text{-a})$$

$$\Rightarrow \frac{M^F_{AB}}{M^F_{DC}} = \frac{K_m^{AB}}{K_m^{DC}} = \frac{10}{15} \quad (11.81\text{-b})$$

63 These fixed-end moments could, for example, have the values of -10 and -15 k.ft or -20 and -30, or -30 and -45, or any other combination so long as the above ratio is maintained. The proper procedure is to choose values for these fixed-end moments of approximately the same order of magnitude as the original fixed-end moments due to the real loads. This will result in the same accuracy for the results of the balance for the side-sway correction that was realized in the first balance for the real loads. Accordingly, it will be assumed that P, and the resulting Δ, are of such magnitudes as to result in the fixed-end moments shown below



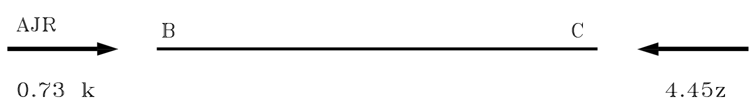
8. Obviously, $\Sigma M = 0$ is not satisfied for joints B and C in this deflected frame. Therefore these joints must rotate until equilibrium is reached. The effect of this rotation is determined in the distribution below

Joint	A	B		C		D	Balance	CO
Member	AB	BA	BC	CB	CD	DC		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM	-30.0	-30.0			-45.0	-45.0		
			+12.9	+25.8	+19.2	+9.6	C	BC; DC
	+2.8	+5.7	+11.4	+5.7			B	AB; CB
			-1.6	-3.3	-2.4	-1.2	C	BC; DC
	+0.2	+0.5	+1.1	+0.5			B	AB; CB
				-0.3	-0.2	-0.1	C	
Total	**-27.0**	**-23.8**	**+23.8**	**+28.4**	**-28.4**	**-36.7**		

9. During the rotation of joints B and C, as represented by the above distribution, *the value of* Δ *has remained constant, with* P *varying in magnitude as required to maintain* Δ.
10. It is now possible to determine the final value of P simply by adding the shears in the columns. The shear in any member, without external loads applied along its length, is obtained by adding the end moments algebraically and dividing by the length of the member. The final value of P is the force necessary to maintain the deflection of the frame after the joints have rotated. In other words, it is the force which will be consistent with the displacement and internal moments of the structure as determined by the second balancing operation. Hence this final value of P will be called the *consistent joint force* (abbreviated as CJF).
11. The consistent joint force is given by

$$CJF = \frac{+27.0+23.8}{26} + \frac{28.4+36.7}{26} = 1.95 + 2.50 = 4.45 \text{ k} \tag{11.82}$$

and inspection clearly indicates that the CJF must act to the left.
12. Obviously, then, the results of the last balance above are moments which will exist in the frame when a force of 4.45 k acts to the left at the top level. It is necessary, however, to determine the moments resulting from a force of 0.73 k acting to the left at the top level, and some as yet unknown factor "z" times 4.45 will be used to represent this force acting to the left.



13. The free body for the member BC is shown above. $\Sigma H = 0$ must be satisfied for this figure, and if forces to the left are considered as positive, the result is $4.45z - 0.73 = 0$, and

$$z = +0.164. \tag{11.83}$$

If this factor $z = +0.164$ is applied to the moments obtained from the second balance, the result will be the moments caused by a force of 0.73 k acting to the left at the top level. If these moments are then added to the moments obtained from the first balance, the result will be the final moments for the frame, the effect of the AJR having been cancelled. This combination of moments is shown below.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
M from 1^{st} balance	-6.9	-13.9	+13.9	-26.5	+26.5	+13.2
$z \times$ M from 2^{nd} balance	-4.4	-3.9	+3.9	+4.7	-4.7	-6.0
Final moments	**-11.3**	**-17.8**	**+17.8**	**-21.8**	**+21.8**	**+7.2**

14. If the final moments are correct, the shears in the two columns of the frame should be equal and opposite to satisfy $\Sigma H = 0$ for the entire frame. This check is expressed as

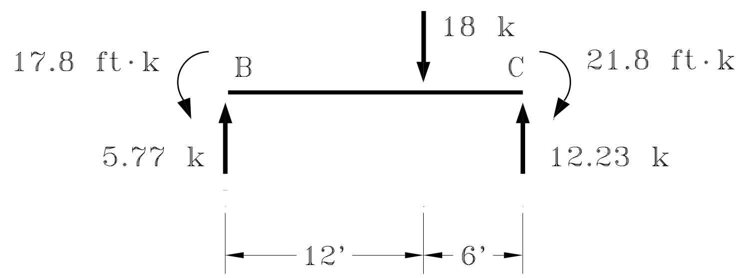
$$\frac{+11.3+17.8}{26} + \frac{-21.8-7.2}{26} = 0, \tag{11.84}$$

and

$$+1.12 - 1.11 = 0(\text{nearly}) \tag{11.85}$$

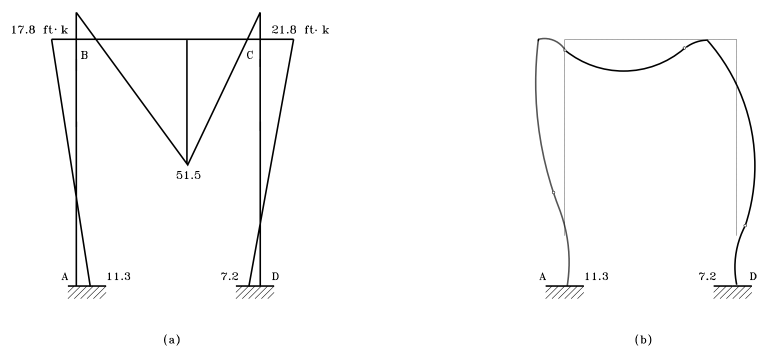
The signs of all moments taken from the previous table have been reversed to give the correct signs for the end moments external to the columns. It will be remembered that the moments considered in moment distribution are always internal for each member. However, the above check actually considers each column as a free body and so external moments must be used.

15. The moment under the 18 k load is obtained by treating BC as a free body:



$$M_{18} = (5.77)(12) - 17.8 = +51.5 \text{ k.ft} \tag{11.86}$$

16. The direction of side-lurch may be determined from the obvious fact that the frame will always lurch in a direction opposite to the AJR. If required, the magnitude of this side lurch may be found. The procedure which follows will apply.

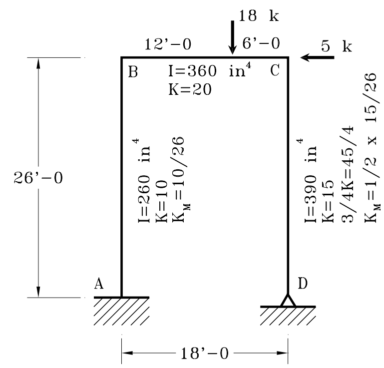


A force P of sufficient magnitude to result in the indicated column moments and the lurch Δ was applied to the frame. During the second balance this value of Δ was held constant as the joints B and C rotated, and the value of P was considered to vary as necessary. The final value of P was found to be 4.45 k. Since Δ was held constant, however, its magnitude may be determined from the equation $M = 6EI\Delta/L^2$, where M is the fixed-end moment for either column, I is the moment of inertia of that column, and L is the length. This Δ will be the lurch for 4.45 k acting at the top level. For any other force acting horizontally, Δ would vary proportionally and thus the final lurch of the frame would be the factor z multiplied by the Δ determined above.

■

■ **Example 11-9: Frame with Side Load, (Kinney 1957)**

Find by moment distribution the moments in the following frame



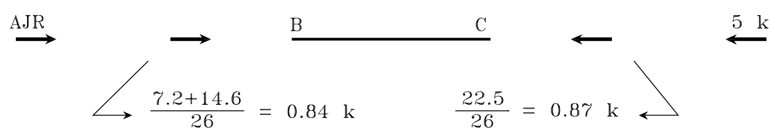
Solution:

The first balance will give the results shown

AB	BA	BC	CB	CD	DC
-7.2	-14.6	+14.6	-22.5	-22.5	0

A check of the member BC as a free body for $\Sigma H = 0$ will indicate that an AJR is necessary as follows:

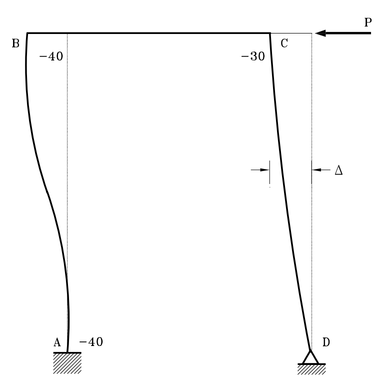
$$AJR + 0.84 - 0.87 - 5.0 = 0 \tag{11.87}$$



from which

$$AJR = +5.03 \text{ in the direction assumed} \tag{11.88}$$

The values of K_M for the two columns are shown, with K_M for column CD being $K/2L$ because of the pin at the bottom. The horizontal displacement Δ of the top of the frame is



assumed to cause the fixed-end moments shown there. These moments are proportional to the values of K_M and of approximately the same order of magnitude as the original fixed-end moments due to the real loads. The results of balancing out these moments are

AB	BA	BC	CB	CD	DC
-34.4	-28.4	+28.4	+23.6	-23.6	0

$$CJF = \frac{+34.4 + 28.4 + 23.6}{26} = 3.32 \text{ k} \tag{11.89}$$

and

$$5.03 - z(3.32) = 0, \tag{11.90}$$

from which

$$z = +1.52. \tag{11.91}$$

The final results are

	AB	BA	BC	CB	CD	DC
M from 1^{st} balance	-7.2	-14.6	+14.6	-22.5	+22.5	0
$z\times$ M 2^{nd} balance	-52.1	-43.0	+43.0	+35.8	-35.8	0
Final moments	**-59.3**	**-57.6**	**+57.6**	**+13.3**	**-13.3**	**0**

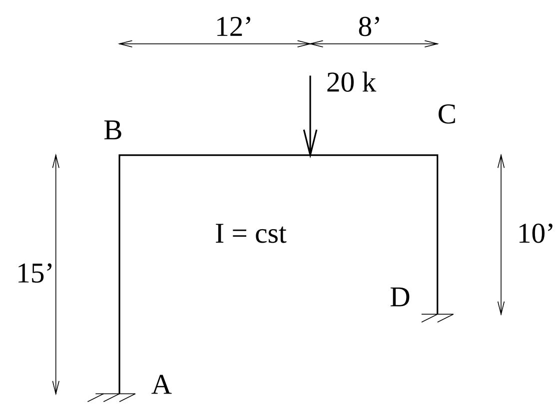
If these final moments are correct, the sum of the column shears will be 5.0 k: Sum of column shears:

$$\Sigma V = \frac{59.3 + 57.6 + 13.3}{26} = 5.01 \text{ k} \tag{11.92}$$

The 5 k horizontal load acting at C enters into the problem only in connection with the determination of the AJR. If this load had been applied to the column CD between the ends, it would have resulted in initial fixed-end moments in CD and these would be computed in the usual way. In addition, such a load would have entered into the determination of the AJR, since the horizontal reaction of CD against the right end of BC would have been computed by treating CD as a free body. ■

■ **Example 11-10: Moment Distribution on a Spread-Sheet**

Analyse the following frame



Solution:

Sheet1

Artificially Restrained Structure

	A	B		C		D		
	AB	**BA**	**BC**	**CB**	**CD**	**DC**	Balance	CO
Lenght	15	15	20	20	10	10		
EI	30	30	30	30	30	30	"arbitrary"	
K	2.0	2.0	1.5	1.5	3.0	3.0		
DF	1.0	0.6	0.4	0.3	0.7	1.0		
FEM			38.4	-57.6				
	-11.0	-21.9	-16.5	-8.2			B	AB, CB
			11.0	21.9	43.9	21.9	C	BC, DC
	-3.1	-6.3	-4.7	-2.4			B	AB, CB
			0.4	0.8	1.6	0.8	C	BC, DC
	-0.1	-0.2	-0.2	-0.1			B	AB, CB
			0.0	0.0	0.1	0.0	C	BC, DC
Total	**-14.2**	**-28.4**	**28.5**	**-45.5**	**45.5**	**22.8**	.	

	V		6EI/L^2	M^F
H_A	-2.84	AB	0.80	**-8**
H_D	6.826355685	DC	1.80	**-18**
AJR	**3.98**	Delta	-10	

Delta to the left, will cause -ve M^F

Frame sways to the right

Horizontal displacement Delta

	A	B		C		D		
	AB	**BA**	**BC**	**CB**	**CD**	**DC**	Balance	CO
Lenght	15	15	20	20	10	10		
EI	30	30	30	30	30	30	"arbitrary"	
K	2.0	2.0	1.5	1.5	3.0	3.0		
DF	1.0	0.6	0.4	0.3	0.7	1.0		
FEM	-8.0	-8.0			-18.0	-18.0		
	2.3	4.6	3.4	1.7			B	AB, CB
			2.7	5.4	10.9	5.4	C	BC, DC
	-0.8	-1.6	-1.2	-0.6			B	AB, CB
			0.1	0.2	0.4	0.2	C	BC, DC
		-0.1	0.0				B	AB, CB
Total	**-6.5**	**-5.0**	**5.0**	**6.8**	**-6.8**	**-12.4**	.	

	V
H_A	-0.77
H_D	-1.9
CJF	**-2.68**
z	**1.49**

Final Moments

	AB	BA	BC	CB	CD	DC
1st	-14.2	-28.4	28.5	-45.5	45.5	22.8
z 2nd	-9.6	-7.5	7.5	10.0	-10.0	-18.4
Final	**-23.9**	**-35.9**	**35.9**	**-35.5**	**35.5**	**4.4**

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Chapter 12

DIRECT STIFFNESS METHOD

12.1 Introduction

12.1.1 Structural Idealization

1 Prior to analysis, a structure must be idealized for a suitable mathematical representation. Since it is practically impossible (and most often unnecessary) to model every single detail, assumptions must be made. Hence, structural idealization is as much an art as a science. *Some* of the questions confronting the analyst include:

1. Two dimensional versus three dimensional; Should we model a single bay of a building, or the entire structure?
2. Frame or truss, can we neglect flexural stiffness?
3. Rigid or semi-rigid connections (most important in steel structures)
4. Rigid supports or elastic foundations (are the foundations over solid rock, or over clay which may consolidate over time)
5. Include or not secondary members (such as diagonal braces in a three dimensional analysis).
6. Include or not axial deformation (can we neglect the axial stiffness of a beam in a building?)
7. Cross sectional properties (what is the moment of inertia of a reinforced concrete beam?)
8. Neglect or not haunches (those are usually present in zones of high negative moments)
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations (In the analysis of a high rise building subjected to wind load, the moments should be amplified by the product of the axial load times the lateral deformation, $P-\Delta$ effects).
11. Time dependent effects (such as creep, which is extremely important in prestressed concrete, or cable stayed concrete bridges).
12. Partial collapse or local yielding (would the failure of a single element trigger the failure of the entire structure?).
13. Load static or dynamic (when should a dynamic analysis be performed?).
14. Wind load (the lateral drift of a high rise building subjected to wind load, is often the major limitation to higher structures).
15. Thermal load (can induce large displacements, specially when a thermal gradient is present.).
16. Secondary stresses (caused by welding. Present in most statically indeterminate structures).

12.1.2 Structural Discretization

2 Once a structure has been idealized, it must be discretized to lend itself for a mathematical representation which will be analyzed by a computer program. This discretization should uniquely define each node, and member.

3 The node is characterized by its nodal id (node number), coordinates, boundary conditions, and load (this one is often defined separately), Table 12.1. Note that in this case we have two nodal coordinates,

Node No.	Coor.		B. C.		
	X	Y	X	Y	Z
1	0.	0.	1	1	0
2	5.	5.	0	0	0
3	20.	5.	0	0	0
4	25.	2.5	1	1	1

Table 12.1: Example of Nodal Definition

and three degrees of freedom (to be defined later) per node. Furthermore, a 0 and a 1 indicate unknown or known displacement. Known displacements can be zero (restrained) or non-zero (as caused by foundation settlement).

4 The element is characterized by the nodes which it connects, and its group number, Table 12.2.

Element No.	From Node	To Node	Group Number
1	1	2	1
2	3	2	2
3	3	4	2

Table 12.2: Example of Element Definition

5 Group number will then define both element type, and elastic/geometric properties. The last one is a pointer to a separate array, Table 12.3. In this example element 1 has element code 1 (such as beam element), while element 2 has a code 2 (such as a truss element). Material group 1 would have different elastic/geometric properties than material group 2.

Group No.	Element Type	Material Group
1	1	1
2	2	1
3	1	2

Table 12.3: Example of Group Number

6 From the analysis, we first obtain the nodal displacements, and then the element internal forces. Those internal forces vary according to the element type. For a two dimensional frame, those are the axial and shear forces, and moment at each node.

7 Hence, the need to define two coordinate systems (one for the entire structure, and one for each element), and a sign convention become apparent.

12.1.3 Coordinate Systems

8 We should differentiate between 2 coordinate systems:

Global: to describe the structure nodal coordinates. This system can be arbitrarily selected provided it is a Right Hand Side (RHS) one, and we will associate with it upper case axis labels, X, Y, Z, Fig. 12.1 or 1,2,3 (running indeces within a computer program).

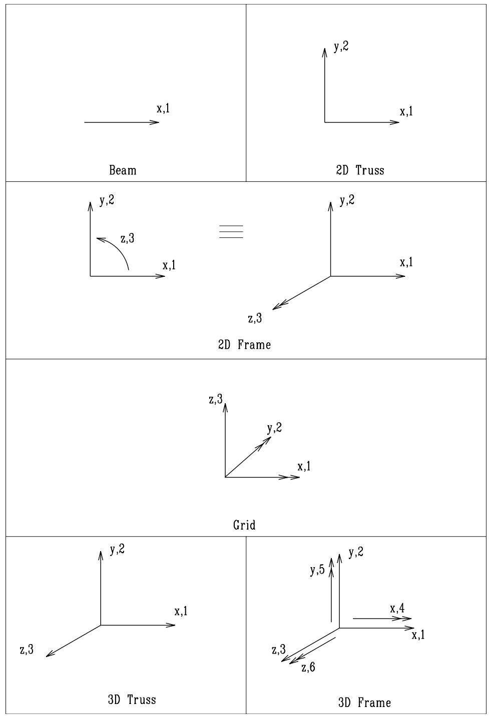

Figure 12.1: Global Coordinate System

Local: system is associated with each element and is used to describe the element internal forces. We will associate with it lower case axis labels, x, y, z (or 1,2,3), Fig. 12.2.

9 The x-axis is assumed to be along the member, and the direction is chosen such that it points from the 1st node to the 2nd node, Fig. 12.2.

10 Two dimensional structures will be defined in the X-Y plane.

12.1.4 Sign Convention

11 The sign convention in structural analysis is completely different than the one previously adopted in structural analysis/design, Fig. 12.3 (where we focused mostly on flexure and defined a positive moment as one causing "tension below". This would be awkward to program!).

12 In matrix structural analysis the sign convention adopted is consistent with the prevailing coordinate system. Hence, we define a positive moment as one which is counter-clockwise, Fig. 12.3

13 Fig. 12.4 illustrates the sign convention associated with each type of element.

14 Fig. 12.4 also shows the geometric (upper left) and elastic material (upper right) properties associated with each type of element.

12.1.5 Degrees of Freedom

15 A degree of freedom (d.o.f.) is an independent generalized nodal displacement of a node.

16 The displacements must be linearly independent and thus not related to each other. For example, a roller support on an inclined plane would have three displacements (rotation θ, and two translations u

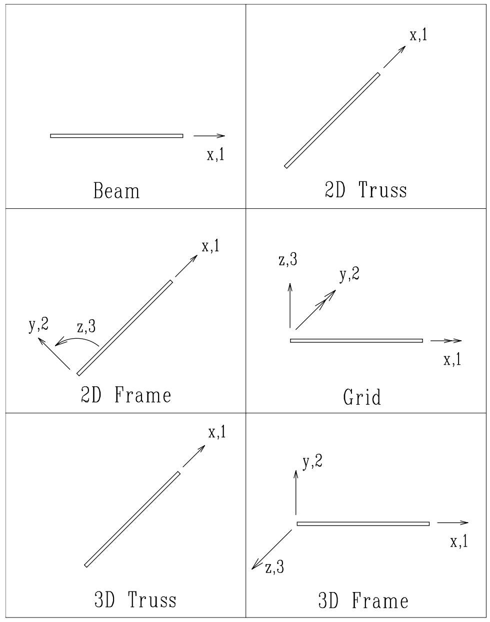

Figure 12.2: Local Coordinate Systems

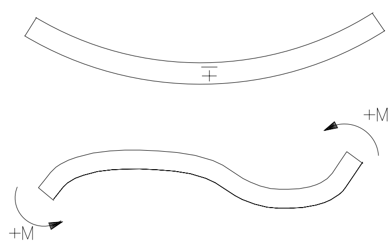

Figure 12.3: Sign Convention, Design and Analysis

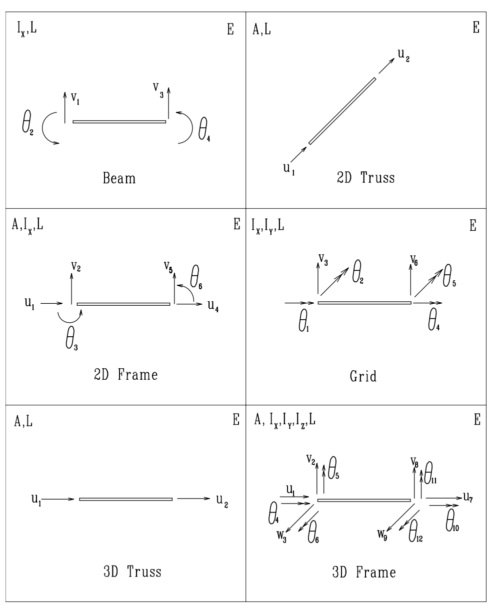

Figure 12.4: Total Degrees of Freedom for various Type of Elements

and v), however since the two displacements are kinematically constrained, we only have two independent displacements, Fig. 12.5.

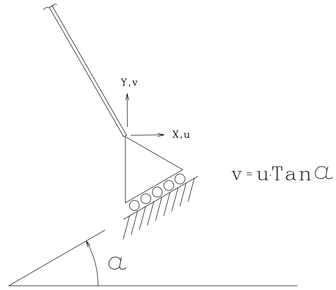

Figure 12.5: Dependent Displacements

17 We note that we have been referring to *generalized* displacements, because we want this term to include translations as well as rotations. Depending on the type of structure, there may be none, one or more than one such displacement. It is unfortunate that in most introductory courses in structural analysis, too much emphasis has been placed on two dimensional structures, and not enough on either three dimensional ones, or two dimensional ones with torsion.

18 In most cases, there is the same number of d.o.f in local coordinates as in the global coordinate system. One notable exception is the truss element. In local coordinate we can only have one axial deformation, whereas in global coordinates there are two or three translations in 2D and 3D respectively for each node.

19 Hence, it is essential that we understand the degrees of freedom which can be associated with the various types of structures made up of one dimensional rod elements, Table 12.4.

20 This table shows the degree of freedoms and the corresponding generalized forces.

21 We should distinguish between local and global d.o.f.'s. The numbering scheme follows the following simple rules:

Local: d.o.f. for a given element: Start with the first node, number the local d.o.f. in the same order as the subscripts of the relevant local coordinate system, and repeat for the second node.

Global: d.o.f. for the entire structure: Starting with the 1st node, number all the unrestrained global d.o.f.'s, and then move to the next one until all global d.o.f have been numbered, Fig. 12.6.

12.2 Stiffness Matrices

12.2.1 Truss Element

22 From strength of materials, the force/displacement relation in axial members is

$$\begin{aligned} \sigma &= E\epsilon \\ \underbrace{A\sigma}_{P} &= \frac{AE}{L}\underbrace{\Delta}_{1} \end{aligned} \tag{12.1}$$

Hence, for a unit displacement, the applied force should be equal to $\frac{AE}{L}$. From statics, the force at the other end must be equal and opposite.

23 The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node.

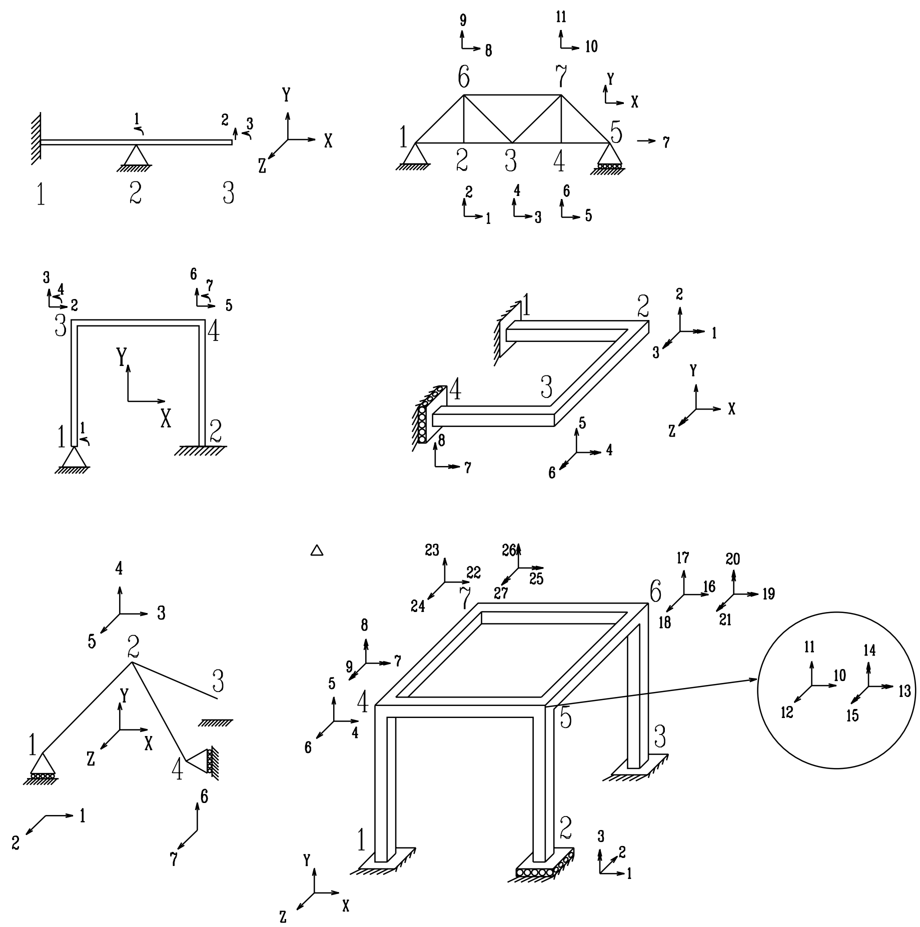

Figure 12.6: Examples of Active Global Degrees of Freedom

Type		Node 1	Node 2	$[\mathbf{k}]$ (Local)	$[\mathbf{K}]$ (Global)
1 Dimensional					
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\boldsymbol{\delta}\}$	u_1	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathbf{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\boldsymbol{\delta}\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}	2×2	6×6
	$\{\boldsymbol{\delta}\}$	u_1,	u_2		
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}	12×12	12×12
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}		

Table 12.4: Degrees of Freedom of Different Structure Types Systems

Hence, from Eq. 12.1, we have

$$\boxed{[\mathbf{k}^t] = \frac{AE}{L}\begin{array}{c} \\ p_1 \\ p_2 \end{array}\begin{array}{cc} u_1 & u_2 \\ \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right. & \left.\right] \end{array}} \tag{12.2}$$

12.2.2 Beam Element

24 Using Equations 11.10, 11.11, 11.13 and 11.14 we can determine the forces associated with each unit displacement.

$$[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \text{Eq. } 11.13(v_1=1) & \text{Eq. } 11.13(\theta_1=1) & \text{Eq. } 11.13(v_2=1) & \text{Eq. } 11.13(\theta_2=1) \\ \text{Eq. } 11.10(v_1=1) & \text{Eq. } 11.10(\theta_1=1) & \text{Eq. } 11.10(v_2=1) & \text{Eq. } 11.10(\theta_2=1) \\ \text{Eq. } 11.14(v_1=1) & \text{Eq. } 11.14(\theta_1=1) & \text{Eq. } 11.14(v_2=1) & \text{Eq. } 11.14(\theta_2=1) \\ \text{Eq. } 11.11(v_1=1) & \text{Eq. } 11.11(\theta_1=1) & \text{Eq. } 11.11(v_2=1) & \text{Eq. } 11.11(\theta_2=1) \end{bmatrix} \end{array} \tag{12.3}$$

25 The stiffness matrix of the beam element (neglecting shear and axial deformation) will thus be

$$\boxed{[\mathbf{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \end{array}} \tag{12.4}$$

26 We note that this is identical to Eq.11.16

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \underbrace{\begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \end{array}}_{\mathbf{k}^{(e)}} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{12.5}$$

12.2.3 2D Frame Element

27 The stiffness matrix of the two dimensional frame element is composed of terms from the truss and beam elements where $\mathbf{k}^b$ and $\mathbf{k}^t$ refer to the beam and truss element stiffness matrices respectively.

$$[\mathbf{k}^{2dfr}] = \begin{array}{c} \\ P_1 \\ V_1 \\ M_1 \\ P_2 \\ V_2 \\ M_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} k^t_{11} & 0 & 0 & k^t_{12} & 0 & 0 \\ 0 & k^b_{11} & k^b_{12} & 0 & k^b_{13} & k^b_{14} \\ 0 & k^b_{21} & k^b_{22} & 0 & k^b_{23} & k^b_{24} \\ k^t_{21} & 0 & 0 & k^t_{22} & 0 & 0 \\ 0 & k^b_{31} & k^b_{32} & 0 & k^b_{33} & k^b_{34} \\ 0 & k^b_{41} & k^b_{42} & 0 & k^b_{43} & k^b_{44} \end{bmatrix} \end{array} \tag{12.6}$$

Thus, we have:

$$\boxed{[\mathbf{k}^{2dfr}] = \begin{array}{c} \\ P_1 \\ V_1 \\ M_1 \\ P_2 \\ V_2 \\ M_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \end{array}} \tag{12.7}$$

12.2.4 Remarks on Element Stiffness Matrices

Singularity: All the derived stiffness matrices are singular, that is there is at least one row and one column which is a linear combination of others. For example in the beam element, row 4 = −row 1; and L times row 2 is equal to the sum of row 3 and 6. This singularity (not present in the flexibility matrix) is caused by the linear relations introduced by the equilibrium equations which are embedded in the formulation.

Symmetry: All matrices are symmetric due to Maxwell-Betti's reciprocal theorem, and the stiffness flexibility relation.

28 More about the stiffness matrix properties later.

12.3 Direct Stiffness Method

12.3.1 Orthogonal Structures

29 As a "vehicle" for the introduction to the stiffness method let us consider the problem in Fig 12.7-a, and recognize that there are only two unknown displacements, or more precisely, two global d.o.f: θ_1 and θ_2.

30 If we were to analyse this problem by the force (or flexibility) method, then

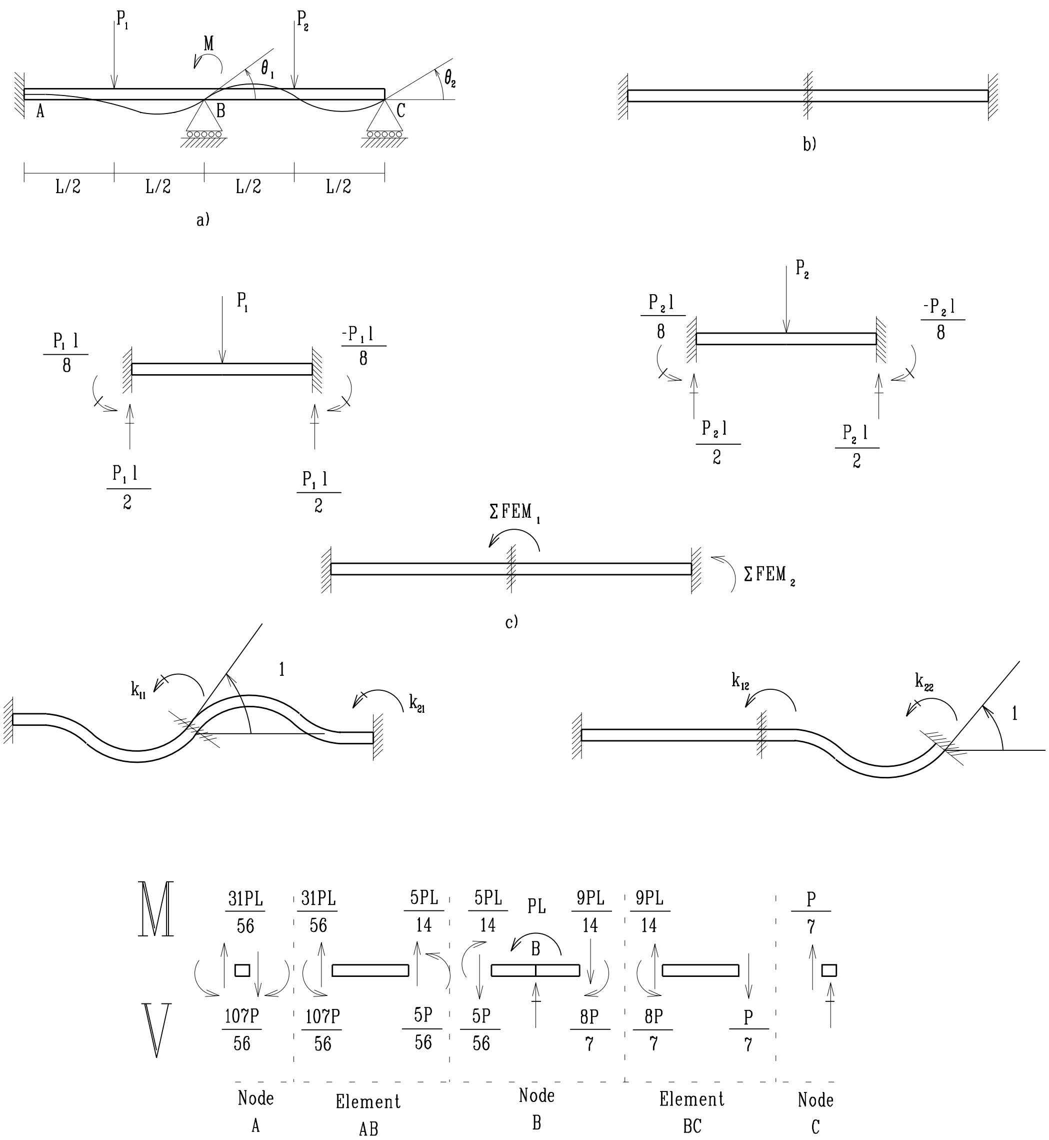

Figure 12.7: Problem with 2 Global d.o.f. θ_1 and θ_2

1. We make the structure statically determinate by removing *arbitrarily* two reactions (as long as the structure remains stable), and the beam is now statically determinate.
2. Assuming that we remove the two roller supports, then we determine the corresponding deflections due to the actual laod (Δ_B and Δ_C).
3. Apply a unit load at point B, and then C, and compute the deflections f_{ij} at note i due to a unit force at node j.
4. Write the compatibility of displacement equation

$$\begin{bmatrix} f_{BB} & f_{BC} \\ f_{CB} & f_{CC} \end{bmatrix} \left\{ \begin{array}{c} R_B \\ R_C \end{array} \right\} - \left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{12.8}$$

5. Invert the matrix, and solve for the reactions

31 We will analyze this simple problem by the stiffness method.

1. The first step consists in making it kinematically determinate (as opposed to statically determinate in the flexibility method). Kinematically determinate in this case simply means restraining all the d.o.f. and thus prevent joint rotation, Fig 12.7-b.
2. We then determine the fixed end actions caused by the element load, and sum them for each d.o.f., Fig 12.7-c: ΣFEM_1 and ΣFEM_2.
3. In the third step, we will apply a unit displacement (rotation in this case) at each degree of freedom at a time, and in each case we shall determine the reaction forces, K_{11}, K_{21}, and K_{12}, K_{22} respectively. Note that we use $[\mathbf{K}]$, rather than $\mathbf{k}$ since those are forces in the global coordinate system, Fig 12.7-d. Again note that we are focusing only on the reaction forces corresponding to a global degree of freedom. Hence, we are not attempting to determine the reaction at node A.
4. Finally, we write the equation of equilibrium at each node:

$$\boxed{\left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} = \left\{ \begin{array}{c} \Sigma\text{FEM}_1 \\ \Sigma\text{FEM}_2 \end{array} \right\} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\}} \tag{12.9}$$

32 Note that the FEM being on the right hand side, they are detemined as the reactions to the applied load. Strictly speaking, it is a load which should appear on the left hand side of the equation, and are the nodal equivalent loads to the element load (more about this later).

33 As with the element stiffness matrix, each entry in the global stiffness matrix K_{ij}, corresponds to the internal force along d.o.f. i due to a unit displacement (generalized) along d.o.f. j (both in global coordinate systems).

■ Example 12-1: Beam

Considering the previous problem, Fig. 12.7-a, let $P_1 = 2P$, $M = PL$, $P_2 = P$, and $P_3 = P$, Solve for the displacements.

Solution:

1. Using the previously defined sign convention:

$$\Sigma\text{FEM}_1 = \underbrace{-\frac{P_1L}{8}}_{BA} + \underbrace{\frac{P_2L}{8}}_{BC} = -\frac{2PL}{8} + \frac{PL}{8} = -\frac{PL}{8} \tag{12.10}$$

$$\Sigma\text{FEM}_2 = \underbrace{-\frac{PL}{8}}_{CB} \tag{12.11}$$

2. If it takes $\frac{4EI}{L}$ (k_{44}^{AB}) to rotate AB (Eq. 12.4) and $\frac{4EI}{L}$ (k_{22}^{BC}) to rotate BC, it will take a total force of $\frac{8EI}{L}$ to simultaneously rotate AB and BC, (Note that a rigid joint is assumed).
3. Hence, K_{11} which is the sum of the rotational stiffnesses at global d.o.f. 1. will be equal to $K_{11} = \frac{8EI}{L}$; similarly, $K_{21} = \frac{2EI}{L}$ (k_{42}^{BC}) .
4. If we now rotate dof 2 by a unit angle, then we will have $K_{22} = \frac{4EI}{L}$ (k_{22}^{BC}) and $K_{12} = \frac{2EI}{L}$ (k_{42}^{BC}) .
5. The equilibrium relation can thus be written as:

$$\underbrace{\left\{ \begin{array}{c} PL \\ 0 \end{array} \right\}}_{\mathbf{M}} = \underbrace{\left\{ \begin{array}{c} -\frac{PL}{8} \\ -\frac{PL}{8} \end{array} \right\}}_{FEM} + \underbrace{\left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]}_{\mathbf{K}} \underbrace{\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\}}_{\boldsymbol{\Delta}} \tag{12.12}$$

or

$$\left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} \tag{12.13}$$

We note that this matrix corresponds to the structure's stiffness matrix, and not the augmented one.
6. The two by two matrix is next inverted

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]^{-1} \left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left\{ \begin{array}{c} \frac{17}{112}\frac{PL^2}{EI} \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array} \right\} \tag{12.14}$$

7. Next we need to determine both the reactions and the internal forces.
8. Recall that for each element $\{\mathbf{p}\} = [\mathbf{k}]\{\boldsymbol{\delta}\}$, and in this case $\{\mathbf{p}\} = \{\mathbf{P}\}$ and $\{\boldsymbol{\delta}\} = \{\boldsymbol{\Delta}\}$ for element AB. The element stiffness matrix has been previously derived, Eq. 12.4, and in this case the global and local d.o.f. are the same.
9. Hence, the equilibrium equation for element AB, at the element level, can be written as:

$$\underbrace{\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right\}}_{\{\mathbf{p}\}} = \underbrace{\left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array} \right]}_{[\mathbf{k}]} \underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{17}{112}\frac{PL^2}{EI} \end{array} \right\}}_{\{\boldsymbol{\delta}\}} + \underbrace{\left\{ \begin{array}{c} \frac{2P}{2} \\ \frac{2PL}{8} \\ \frac{2P}{2} \\ -\frac{2PL}{8} \end{array} \right\}}_{\text{FEM}} \tag{12.15}$$

solving

$$\lfloor\ p_1 \quad p_2 \quad p_3 \quad p_4\ \rfloor = \lfloor\ \frac{107}{56}P \quad \frac{31}{56}PL \quad \frac{5}{56}P \quad \frac{5}{14}PL\ \rfloor \tag{12.16}$$

10. Similarly, for element BC:

$$\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \right\} = \left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} 0 \\ \frac{17}{112}\frac{PL^2}{EI} \\ 0 \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array} \right\} + \left\{ \begin{array}{c} \frac{P}{2} \\ \frac{PL}{8} \\ \frac{P}{2} \\ -\frac{PL}{8} \end{array} \right\} \tag{12.17}$$

or

$$\lfloor\ p_1 \quad p_2 \quad p_3 \quad p_4\ \rfloor = \lfloor\ \frac{7}{8}P \quad \frac{9}{14}PL \quad -\frac{P}{7} \quad 0\ \rfloor \tag{12.18}$$

11. This simple example calls for the following observations:

1. Node A has contributions from element AB only, while node B has contributions from both AB and BC.

   12. We observe that $p_3^{AB} \neq p_1^{BC}$ eventhough they both correspond to a shear force at node B, the difference betweeen them is equal to the reaction at B. Similarly, $p_4^{AB} \neq p_2^{BC}$ due to the externally applied moment at node B.

2. From this analysis, we can draw the complete free body diagram, Fig. 12.7-e and then the shear and moment diagrams which is what the Engineer is most interested in for design purposes.

■

12.3.2 Local and Global Element Stiffness Matrices ($[\mathbf{k}^{(e)}]$ $[\mathbf{K}^{(e)}]$)

34 In the previous section, in which we focused on orthogonal structures, the assembly of the structure's stiffness matrix $[\mathbf{K}^{(e)}]$ in terms of the element stiffness matrices was relatively straight-forward.

35 The determination of the element stiffness matrix in global coordinates, from the element stiffness matrix in local coordinates requires the introduction of a transformation.

36 This section will examine the 2D transformation required to obtain an element stiffness matrix in global coordinate system prior to assembly (as discussed in the next section).

37 Recalling that

$$\{\mathbf{p}\} = [\mathbf{k}^{(e)}]\{\boldsymbol{\delta}\} \tag{12.19}$$

$$\{\mathbf{P}\} = [\mathbf{K}^{(e)}]\{\boldsymbol{\Delta}\} \tag{12.20}$$

38 Let us define a transformation matrix $[\boldsymbol{\Gamma}^{(e)}]$ such that:

$$\{\boldsymbol{\delta}\} = [\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \tag{12.21}$$

$$\{\mathbf{p}\} = [\boldsymbol{\Gamma}^{(e)}]\{\mathbf{P}\} \tag{12.22}$$

Note that we use the same matrix $\boldsymbol{\Gamma}^{(e)}$ since both $\{\boldsymbol{\delta}\}$ and $\{\mathbf{p}\}$ are vector quantities (or tensors of order one).

39 Substituting Eqn. 12.21 and Eqn. 12.22 into Eqn. 12.19 we obtain

$$[\boldsymbol{\Gamma}^{(e)}]\{\mathbf{P}\} = [\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \tag{12.23}$$

premultiplying by $[\boldsymbol{\Gamma}^{(e)}]^{-1}$

$$\{\mathbf{P}\} = [\boldsymbol{\Gamma}^{(e)}]^{-1}[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]\{\boldsymbol{\Delta}\} \tag{12.24}$$

40 But since the rotation matrix is orthogonal, we have $[\boldsymbol{\Gamma}^{(e)}]^{-1} = [\boldsymbol{\Gamma}^{(e)}]^T$ and

$$\{\mathbf{P}\} = \underbrace{[\boldsymbol{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]}_{[\mathbf{K}^{(e)}]}\{\boldsymbol{\Delta}\} \tag{12.25}$$

$$\boxed{[\mathbf{K}^{(e)}] = [\boldsymbol{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\boldsymbol{\Gamma}^{(e)}]} \tag{12.26}$$

which is the general relationship between element stiffness matrix in local and global coordinates.

12.3.2.1 2D Frame

41 The vector rotation matrix is defined in terms of 9 direction cosines of 9 different angles. However for the 2D case, Fig. 12.8, we will note that four angles are interrelated ($l_{xX}, l_{xY}, l_{yX}, l_{yY}$) and can all be expressed in terms of a single one α, where α is the direction of the local x axis (along the member from the first to the second node) with respect to the global X axis. The remaining 5 terms are related to another angle, β, which is between the Z axis and the x-y plane. This angle is zero because we select an orthogonal right handed coordinate system. Thus, the rotation matrix can be written as:

$$[\gamma] = \begin{bmatrix} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos(\frac{\pi}{2}-\alpha) & 0 \\ \cos(\frac{\pi}{2}+\alpha) & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12.27}$$

and we observe that the angles are defined from the second subscript to the first, and that counterclockwise angles are positive.

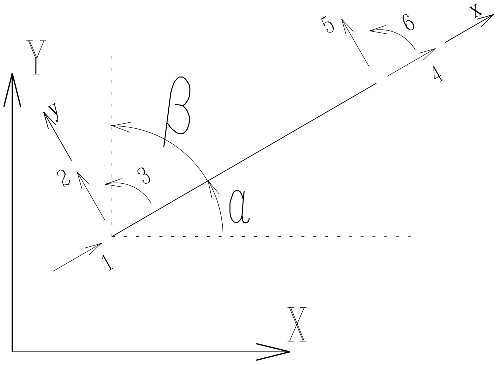

Figure 12.8: 2D Frame Element Rotation

42 The element rotation matrix $[\mathbf{\Gamma}^{(e)}]$ will then be given by

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \underbrace{\left[\begin{array}{rrr|rrr} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathbf{\Gamma}^{(e)}]} \left\{\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \\ P_6 \end{array}\right\} \tag{12.28}$$

12.3.3 Global Stiffness Matrix

43 The physical interpretation of the global stiffness matrix $\mathbf{K}$ is analogous to the one of the element, i.e. If all degrees of freedom are restrained, then K_{ij} corresponds to the force along global degree of freedom i due to a unit positive displacement (or rotation) along global degree of freedom j.

44 For instance, with reference to Fig. 12.9, we have three global degrees of freedom, Δ_1, Δ_2, and θ_3. and the global (restrained or structure's) stiffness matrix is

$$\mathbf{K} = \left[\begin{array}{ccc} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{array}\right] \tag{12.29}$$

and the first column corresponds to all the internal forces in the unrestrained d.o.f. when a unit displacement along global d.o.f. 1 is applied.

12.3.3.1 Structural Stiffness Matrix

45 The structural stiffness matrix is assembled only for those active dgrees of freedom which are active (i.e unrestrained). It is the one which will be inverted (or rather decomposed) to determine the nodal displacements.

12.3.3.2 Augmented Stiffness Matrix

46 The augmented stiffness matrix is expressed in terms of all the dof. However, it is partitioned into two groups with respective subscript 'u' where the displacements are known (zero otherwise), and t where the loads are known.

$$\boxed{\left\{\begin{array}{c} \mathbf{P}_t\surd \\ \hline \mathbf{R}_u? \end{array}\right\} = \left[\begin{array}{c|c} \mathbf{K}_{tt} & \mathbf{K}_{tu} \\ \hline \mathbf{K}_{ut} & \mathbf{K}_{uu} \end{array}\right] \left\{\begin{array}{c} \mathbf{\Delta}_t? \\ \hline \mathbf{\Delta}_u\surd \end{array}\right\}} \tag{12.30}$$

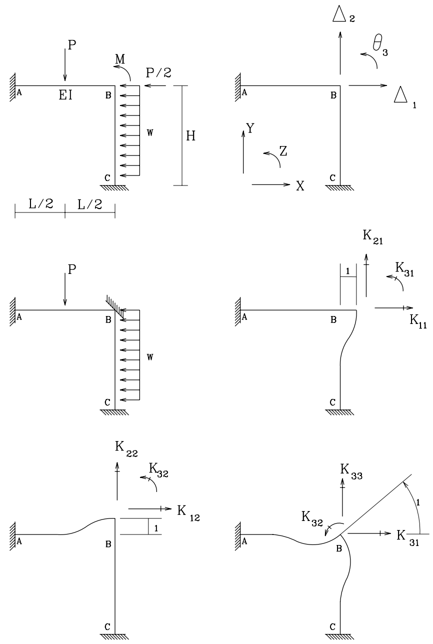

Figure 12.9: *Frame Example (correct K_{32} and K_{33})

We note that $\mathbf{K}_{tt}$ corresponds to the structural stiffness matrix.

47 The first equation enables the calculation of the unknown displacements.

$$\boxed{\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\left(\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u\right)} \tag{12.31}$$

48 The second equation enables the calculation of the reactions

$$\boxed{\mathbf{R}_u = \mathbf{K}_{ut}\boldsymbol{\Delta}_t + \mathbf{K}_{uu}\boldsymbol{\Delta}_u} \tag{12.32}$$

49 For internal book-keeping purpose, since we are assembling the *augmented* stiffness matrix, we proceed in two stages:

1. First number all the global unrestrained degrees of freedom
2. Then number all the global restrained degrees of freedom (i.e. those with known displacements, zero or otherwise) and multiply by -1[1].

12.3.4 Internal Forces

50 The element internal forces (axia and shear forces, and moment at each end of the member) are determined from

$$\boxed{p_{int}^{(e)} = \mathbf{k}^{(e)}\boldsymbol{\delta}^{(e)}} \tag{12.33}$$

[1] An alternative scheme is to separately number the restrained dof but assign a negative number. This will enable us later on to distinguish the restrained from unrestrained dof.

at the element level where $p_{int}^{(e)}$ is the six by six array of internal forces, $\mathbf{k}^{(e)}$ the element stiffness matrix in local coordinate systems, and $\boldsymbol{\delta}^{(e)}$ is the vector of nodal displacements in local coordinate system. Note that this last array is obrained by first identifying the displacements in global coordinate system, and then premultiplying it by the transofrmation matrix to obtain the displacements in local coordinate system.

12.3.5 Boundary Conditions, [ID] Matrix

51 Because of the boundary condition restraints, the total structure number of *active* degrees of freedom (i.e unconstrained) will be less than the number of nodes times the number of degrees of freedom per node.

52 To obtain the global degree of freedom for a given node, we need to define an [ID] matrix such that:

ID has dimensions $l \times k$ where l is the number of degree of freedom per node, and k is the number of nodes).

ID matrix is initialized to zero.

1. At input stage read `ID(idof,inod)` of each degree of freedom for every node such that:

$$\mathtt{ID}(\mathtt{idof},\mathtt{inod}) = \begin{cases} 0 & \text{if unrestrained d.o.f.} \\ 1 & \text{if restrained d.o.f.} \end{cases} \tag{12.34}$$

2. After all the node boundary conditions have been read, assign incrementally equation numbers

   (a) First to all the active dof

   (b) Then to the other (restrained) dof.

   (c) Multiply by -1 all the passive dof.

   Note that the total number of dof will be equal to the number of nodes times the number of dof/node NEQA.

3. The largest *positive* global degree of freedom number will be equal to NEQ (Number Of Equations), which is the size of the square matrix which will have to be decomposed.

53 For example, for the frame shown in Fig. 12.10:

1. The input data file may contain:

Node No.	$[\mathbf{ID}]^T$
1	0 0 0
2	1 1 0
3	0 0 0
4	1 0 0

2. At this stage, the [ID] matrix is equal to:

$$\mathbf{ID} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12.35}$$

3. After we determined the equation numbers, we would have:

$$\mathbf{ID} = \begin{bmatrix} 1 & -1 & 5 & -3 \\ 2 & -2 & 6 & 8 \\ 3 & 4 & 7 & 9 \end{bmatrix} \tag{12.36}$$

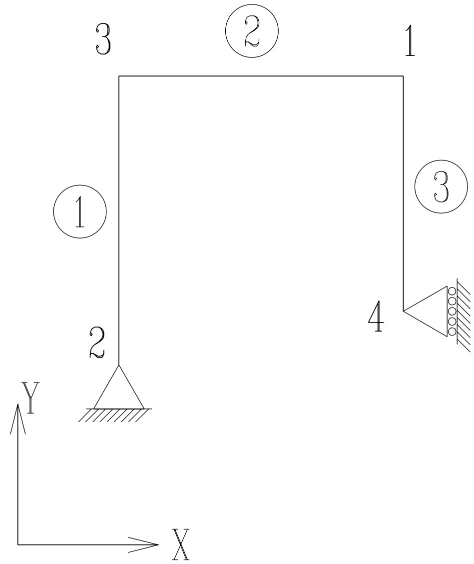

Figure 12.10: Example for [**ID**] Matrix Determination

12.3.6 `LM` Vector

54 The `LM` vector of a given element gives the global degree of freedom of each one of the element degree of freedom's. For the structure shown in Fig. 12.10, we would have:

$$\begin{array}{lcll}
\lfloor \mathtt{LM} \rfloor & = & \lfloor\ -1 \quad -2 \quad 4 \quad 5 \quad 6 \quad 7\ \rfloor & \text{element 1 } (2 \rightarrow 3) \\
\lfloor \mathtt{LM} \rfloor & = & \lfloor\ 5 \quad 6 \quad 7 \quad 1 \quad 2 \quad 3\ \rfloor & \text{element 2 } (3 \rightarrow 1) \\
\lfloor \mathtt{LM} \rfloor & = & \lfloor\ 1 \quad 2 \quad 3 \quad -3 \quad 8 \quad 9\ \rfloor & \text{element 3 } (1 \rightarrow 4)
\end{array}$$

12.3.7 Assembly of Global Stiffness Matrix

55 As for the element stiffness matrix, the global stiffness matrix [**K**] is such that K_{ij} is the force in degree of freedom i caused by a unit displacement at degree of freedom j.

56 Whereas this relationship was derived from basic analysis at the element level, at the structure level, this term can be obtained from the contribution of the element stiffness matrices $[\mathbf{K}^{(e)}]$ (written in global coordinate system).

57 For each K_{ij} term, we shall add the contribution of all the elements which can connect degree of freedom i to degree of freedom j, assuming that those forces are readily available from the individual element stiffness matrices written in global coordinate system.

58 K_{ij} is non-zero if and only if degree of freedom i and degree of freedom j are connected by an element or share a node.

59 There are usually more than one element connected to a dof. Hence, individual element stiffness matrices terms must be added up.

60 Because each term of all the element stiffness matrices must find its position inside the global stiffness matrix [**K**], it is found computationally most effective to initialize the global stiffness matrix $[\mathbf{K}^S]_{(NEQA \times NEQA)}$ to zero, and then loop through all the elements, and then through each entry of the respective element stiffness matrix $K_{ij}^{(e)}$.

61 The assignment of the element stiffness matrix term $K_{ij}^{(e)}$ (note that e, i, and j are all known since we are looping on e from 1 to the number of elements, and then looping on the rows and columns of the

element stiffness matrix i, j) into the global stiffness matrix K^S_{kl} is made through the `LM` vector (note that it is k and l which must be determined).

62 Since the global stiffness matrix is also symmetric, we would need to only assemble one side of it, usually the upper one.

63 Contrarily to the previous method, we will assemble the full *augmented* stiffness matrix.

Example 12-2: Assembly of the Global Stiffness Matrix

As an example, let us consider the frame shown in Fig. 12.11.

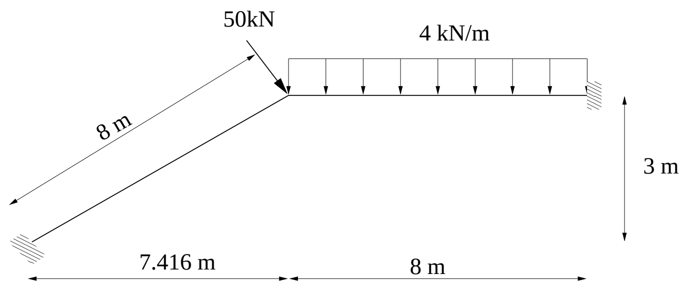

Figure 12.11: Simple Frame Anlysed with the MATLAB Code

The ID matrix is initially set to:

$$[ID] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{12.37}$$

We then modify it to generate the global degrees of freedom of each node:

$$[ID] = \begin{bmatrix} -4 & 1 & -7 \\ -5 & 2 & -8 \\ -6 & 3 & -9 \end{bmatrix} \tag{12.38}$$

Finally the LM vectors for the two elements (assuming that Element 1 is defined from node 1 to node 2, and element 2 from node 2 to node 3):

$$[LM] = \left[\begin{array}{ccc|ccc} -4 & -5 & -6 & 1 & 2 & 3 \\ 1 & 2 & 3 & -7 & -8 & -9 \end{array}\right] \tag{12.39}$$

Let us simplfy the operation by designating the element stiffness matrices in global coordinates as follows:

$$K^{(1)} = \begin{array}{c c} & \begin{array}{cccccc} -4 & -5 & -6 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} -4 \\ -5 \\ -6 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} \end{array} \tag{12.40-a}$$

$$K^{(2)} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & -7 & -8 & -9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ -7 \\ -8 \\ -9 \end{array} & \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} \end{bmatrix} \end{array} \tag{12.40-b}$$

We note that for each element we have shown the corresponding LM vector.

Now, we assemble the global stiffness matrix

$$K = \left[\begin{array}{ccc|cccccc}
A_{44}+B_{11} & A_{45}+B_{12} & A_{46}+B_{13} & A_{41} & A_{42} & A_{43} & B_{14} & B_{15} & B_{16} \\
A_{54}+B_{21} & A_{55}+B_{22} & A_{56}+B_{23} & A_{51} & A_{52} & A_{53} & B_{24} & B_{25} & B_{26} \\
A_{64}+B_{31} & A_{65}+B_{32} & A_{66}+B_{33} & A_{61} & A_{62} & A_{63} & B_{34} & B_{35} & B_{36} \\ \hline
A_{14} & A_{15} & A_{16} & A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\
A_{24} & A_{25} & A_{26} & A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\
A_{34} & A_{35} & A_{36} & A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\
B_{41} & B_{42} & B_{43} & 0 & 0 & 0 & B_{44} & B_{45} & B_{46} \\
B_{51} & B_{52} & B_{53} & 0 & 0 & 0 & B_{54} & B_{55} & B_{56} \\
B_{61} & B_{62} & B_{63} & 0 & 0 & 0 & B_{64} & B_{65} & B_{66}
\end{array}\right] \tag{12.41}$$

■

We note that some terms are equal to zero because we do not have a connection between the corresponding degrees of freedom (i.e. node 1 is not connected to node 3).

12.3.8 Algorithm

64 The direct stiffness method can be summarized as follows:

Preliminaries: First we shall

1. Identify type of structure (beam, truss, grid or frame) and determine the
   (a) Number of spatial coordinates (1D, 2D, or 3D)
   (b) Number of degree of freedom per node (local and global)
   (c) Number of cross-sectional and material properties
2. Determine the global unrestrained and restrained degree of freedom equation numbers for each node, Update the [**ID**] matrix (which included only 0's and 1's in the input data file).

Analysis :

1. For each element, determine
   (a) Vector **LM** relating local to global degree of freedoms.
   (b) Element stiffness matrix $[\mathbf{k}^{(e)}]$
   (c) Angle α between the local and global x axes.
   (d) Rotation matrix $[\mathbf{\Gamma}^{(e)}]$
   (e) Element stiffness matrix in global coordinates $[\mathbf{K}^{(e)}] = [\mathbf{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\mathbf{\Gamma}^{(e)}]$
2. Assemble the augmented stiffness matrix $[\mathbf{K}^{(S)}]$ of unconstrained and constrained degree of freedom's.
3. Extract $[\mathbf{K}_{tt}]$ from $[\mathbf{K}^{(S)}]$ and invert (or decompose into into $[\mathbf{K}_{tt}] = [\mathbf{L}][\mathbf{L}]^T$ where $[\mathbf{L}]$ is a lower triangle matrix.
4. Assemble load vector $\{\mathbf{P}\}$ in terms of nodal load and fixed end actions.
5. Backsubstitute and obtain nodal displacements in global coordinate system.
6. SOlve for the reactions.
7. For each element, transform its nodal displacement from global to local coordinates $\{\boldsymbol{\delta}\} = [\mathbf{\Gamma}^{(e)}]\{\mathbf{\Delta}\}$, and determine the internal forces $[\mathbf{p}] = [\mathbf{k}]\{\boldsymbol{\delta}\}$.

65 Some of the prescribed steps are further discussed in the next sections.

■ **Example 12-3: Direct Stiffness Analysis of a Truss**

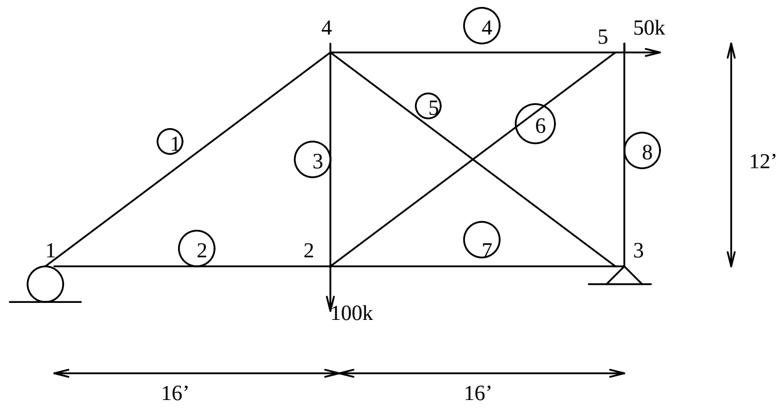

Figure 12.12:

Using the direct stiffness method, analyse the truss shown in Fig. 12.12.

Solution:

1. Determine the structure `ID` matrix

Node #	Bound. Cond. X	Y
1	0	1
2	0	0
3	1	1
4	0	0
5	0	0

$$ID = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{12.42-a}$$

$$= \begin{array}{c} Node \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 2 & -2 & 4 & 6 \\ -1 & 3 & -3 & 5 & 7 \end{bmatrix} \end{array} \tag{12.42-b}$$

2. The `LM` vector of each element is evaluated next

$$\lfloor LM \rfloor_1 = \lfloor\ 1 \quad -1 \quad 4 \quad 5\ \rfloor \tag{12.43-a}$$
$$\lfloor LM \rfloor_2 = \lfloor\ 1 \quad -1 \quad 2 \quad 3\ \rfloor \tag{12.43-b}$$
$$\lfloor LM \rfloor_3 = \lfloor\ 2 \quad 3 \quad 4 \quad 5\ \rfloor \tag{12.43-c}$$
$$\lfloor LM \rfloor_4 = \lfloor\ 4 \quad 5 \quad 6 \quad 7\ \rfloor \tag{12.43-d}$$
$$\lfloor LM \rfloor_5 = \lfloor\ -2 \quad -3 \quad 4 \quad 5\ \rfloor \tag{12.43-e}$$
$$\lfloor LM \rfloor_6 = \lfloor\ 2 \quad 3 \quad 6 \quad 7\ \rfloor \tag{12.43-f}$$
$$\lfloor LM \rfloor_7 = \lfloor\ 2 \quad 3 \quad 0 \quad 0\ \rfloor \tag{12.43-g}$$
$$\lfloor LM \rfloor_8 = \lfloor\ -2 \quad -3 \quad 6 \quad 7\ \rfloor \tag{12.43-h}$$

3. Determine the element stiffness matrix of each element in the global coordinate system noting that for a 2D truss element we have

$$[K^{(e)}] = [\mathbf{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\mathbf{\Gamma}^{(e)}] \tag{12.44-a}$$

$$= \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{12.44-b}$$

where $c = \cos\alpha = \frac{x_2-x_1}{L}$; $s = \sin\alpha = \frac{Y_2-Y_1}{L}$

Element 1 $L = 20'$, $c = \frac{16-0}{20} = 0.8$, $s = \frac{12-0}{20} = 0.6$, $\frac{EA}{L} = \frac{(30{,}000 \text{ ksi})(10 \text{ in}^2)}{20'} = 15{,}000$ k/ft.

$$[K_1] = \begin{array}{r} \\ 1 \\ -1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & -1 & 4 & 5 \end{array} \\ \left[\begin{array}{cccc} 9600 & 7200 & -9600 & -7200 \\ 7200 & 5400 & -7200 & -5400 \\ -9600 & -7200 & 9600 & 7200 \\ -7200 & -5400 & 7200 & 5400 \end{array}\right] \end{array} \tag{12.45}$$

Element 2 $L = 16'$, $c = 1$, $s = 0$, $\frac{EA}{L} = 18{,}750$ k/ft.

$$[K_2] = \begin{array}{r} \\ 1 \\ -1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & -1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 18{,}750 & 0 & -18{,}750 & 0 \\ 0 & 0 & 0 & 0 \\ -18{,}750 & 0 & 18{,}750 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{12.46}$$

Element 3 $L = 12'$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25{,}000$ k/ft.

$$[K_3] = \begin{array}{r} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 25{,}000 & 0 & -25{,}000 \\ 0 & 0 & 0 & 0 \\ 0 & -25{,}000 & 0 & 25{,}000 \end{array}\right] \end{array} \tag{12.47}$$

Element 4 $L = 16'$, $c = 1$, $s = 0$, $\frac{EA}{L} = 18{,}750$ k/ft.

$$[K_4] = \begin{array}{r} \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc} 18{,}750 & 0 & -18{,}750 & 0 \\ 0 & 0 & 0 & 0 \\ -18{,}750 & 0 & 18{,}750 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{12.48}$$

Element 5 $L = 20'$, $c = \frac{-16-0}{20} = -0.8$, $s = 0.6$, $\frac{EA}{L} = 15{,}000$ k/ft.

$$[K_5] = \begin{array}{r} \\ -2 \\ -3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} -2 & -3 & 4 & 5 \end{array} \\ \left[\begin{array}{cccc} 9600 & -7200 & -9600 & 7200 \\ -7200 & 5400 & 7200 & -5400 \\ -9600 & 7200 & 9600 & -7200 \\ 7200 & -5400 & -7200 & 5400 \end{array}\right] \end{array} \tag{12.49}$$

Element 6 $L = 20'$, $c = 0.8$, $s = 0.6$, $\frac{EA}{L} = 15{,}000$ k/ft.

$$[K_6] = \begin{array}{r} \\ 2 \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc} 9600 & 7200 & -9600 & -7200 \\ 7200 & 5400 & -7200 & -5400 \\ -9600 & -7200 & 9600 & 7200 \\ -7200 & -5400 & 7200 & 5400 \end{array}\right] \end{array} \tag{12.50}$$

Element 7 $L = 16'$, $c = 1$, $s = 0$, $\frac{EA}{L} = 18{,}750$ k/ft.

$$[K_7] = \begin{array}{r} \\ 2 \\ 3 \\ 0 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 0 & 0 \end{array} \\ \left[\begin{array}{cccc} 18{,}750 & 0 & -18{,}750 & 0 \\ 0 & 0 & 0 & 0 \\ -18{,}750 & 0 & 18{,}750 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{12.51}$$

Element 8 $L = 12'$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25,000$ k/ft.

$$[K_8] = \begin{array}{c} \\ -2 \\ -3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} -2 & -3 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{array}\right] \end{array} \tag{12.52}$$

4. Assemble the global stiffness matrix in k/ft Note that we are not assembling the augmented stiffness matrix, but rather its submatrix $[\mathbf{K}_{tt}]$.

$$\left\{\begin{array}{c} 0 \\ 0 \\ -100k \\ 0 \\ 0 \\ 50k \\ 0 \end{array}\right\} = \left[\begin{array}{ccccccc} 9600+18,750 & -18,750 & 0 & -9600 & -7200 & 0 & 0 \\ & 9600+(2)\ 18,750 & 7200 & 0 & 0 & -9600 & -7200 \\ & & 5400+25,000 & 0 & -25,000 & -7200 & -5400 \\ & & & 18,750+(2)9600 & 7200-7200 & -18,750 & 0 \\ & \text{SYMMETRIC} & & & 25,000+5400(2) & 0 & 0 \\ & & & & & 18,750+9600 & 7200 \\ & & & & & & 25,000+5400 \end{array}\right] \left\{\begin{array}{c} u_1 \\ u_2 \\ v_3 \\ u_4 \\ v_5 \\ u_6 \\ v_7 \end{array}\right\} \tag{12.53}$$

5. convert to k/in and simplify

$$\left\{\begin{array}{c} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 50 \\ 0 \end{array}\right\} = \left[\begin{array}{ccccccc} 2362.5 & -1562.5 & 0 & -800 & -600 & 0 & 0 \\ & 3925.0 & 600 & 0 & 0 & -800 & -600 \\ & & 2533.33 & 0 & -2083.33 & -600 & -450 \\ & & & 3162.5 & 0 & -1562.5 & 0 \\ \text{SYMMETRIC} & & & & 2983.33 & 0 & 0 \\ & & & & & 2362.5 & 600 \\ & & & & & & 2533.33 \end{array}\right] \left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\} \tag{12.54}$$

6. Invert stiffness matrix and solve for displacements

$$\left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\} = \left\{\begin{array}{c} -0.0223 \text{ in} \\ 0.00433 \text{ in} \\ -0.116 \text{ in} \\ -0.0102 \text{ in} \\ -0.0856 \text{ in} \\ -0.00919 \text{ in} \\ -0.0174 \text{ in} \end{array}\right\} \tag{12.55}$$

7. Solve for member internal forces (in this case axial forces) in local coordinate systems

$$\left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\} = \left[\begin{array}{cccc} c & s & -c & -s \\ -c & -s & c & s \end{array}\right] \left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\} \tag{12.56}$$

Element 1

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^1 = (15,000 \text{ k/ft})(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \left[\begin{array}{cccc} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{array}\right] \left\{\begin{array}{c} -0.0223 \\ 0 \\ -0.0102 \\ -0.0856 \end{array}\right\} \tag{12.57-a}$$

$$= \left\{\begin{array}{c} 52.1 \text{ k} \\ -52.1 \text{ k} \end{array}\right\} \text{ Compression} \tag{12.57-b}$$

Element 2

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^2 = 18,750 \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{c} -0.0233 \\ 0 \\ 0.00433 \\ -0.116 \end{array}\right\} \tag{12.58-a}$$

$$= \begin{Bmatrix} -43.2 \text{ k} \\ 43.2 \text{ k} \end{Bmatrix} \text{ Tension} \tag{12.58-b}$$

Element 3

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^3 = 25,000 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.00433 \\ -0.116 \\ -0.0102 \\ -0.0856 \end{Bmatrix} \tag{12.59-a}$$

$$= \begin{Bmatrix} -63.3 \text{ k} \\ 63.3 \text{ k} \end{Bmatrix} \text{ Tension} \tag{12.59-b}$$

Element 4

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^4 = 18,750 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -0.0102 \\ -0.0856 \\ -0.00919 \\ -0.0174 \end{Bmatrix} \tag{12.60-a}$$

$$= \begin{Bmatrix} -1.58 \text{ k} \\ 1.58 \text{ k} \end{Bmatrix} \text{ Tension} \tag{12.60-b}$$

Element 5

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^5 = 15,000 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} -0.8 & 0.6 & 0.8 & -0.6 \\ 0.8 & -0.6 & -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} -0.0102 \\ -0.0856 \end{Bmatrix} \tag{12.61-a}$$

$$= \begin{Bmatrix} 54.0 \text{ k} \\ -54.0 \text{ k} \end{Bmatrix} \text{ Compression} \tag{12.61-b}$$

Element 6

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^6 = 15,000 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} -0.116 \\ -0.00919 \\ -0.0174 \end{Bmatrix} \tag{12.62-a}$$

$$= \begin{Bmatrix} -60.43 \text{ k} \\ 60.43 \text{ k} \end{Bmatrix} \text{ Tension} \tag{12.62-b}$$

Element 7

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^7 = 18,750 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -0.116 \\ 0 \\ 0 \end{Bmatrix} \tag{12.63-a}$$

$$= \begin{Bmatrix} 6.72 \text{ k} \\ -6.72 \text{ k} \end{Bmatrix} \text{ Compression} \tag{12.63-b}$$

Element 8

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^8 = 25,000 \ \text{ k/ft}(\frac{1}{12}\frac{\text{ft}}{\text{in}}) \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.00919 \\ -0.0174 \end{Bmatrix} \tag{12.64-a}$$

$$= \begin{Bmatrix} 36.3 \text{ k} \\ -36.3 \text{ k} \end{Bmatrix} \text{ Compression} \tag{12.64-b}$$

8. Determine the structure's `MAXA` vector

$$[\mathbf{K}] = \begin{bmatrix} 1 & 3 & & 9 & 14 & & \\ & 2 & 5 & 8 & 13 & 19 & 25 \\ & & 4 & 7 & 12 & 18 & 24 \\ & & & 6 & 11 & 17 & 23 \\ & & & & 10 & 16 & 22 \\ & & & & & 15 & 21 \\ & & & & & & 20 \end{bmatrix} \quad \mathtt{MAXA} = \begin{Bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 10 \\ 15 \\ 20 \end{Bmatrix} \tag{12.65}$$

Thus, 25 terms would have to be stored. ■

■ Example 12-4: Analysis of a Frame with MATLAB

The simple frame shown in Fig. 12.13 is to be analysed by the direct stiffness method. Assume: $E = 200,000$ MPa, $A = 6,000$ mm^2, and $I = 200 \times 10^6$ mm^4. The complete MATLAB solution is shown below along with the results.

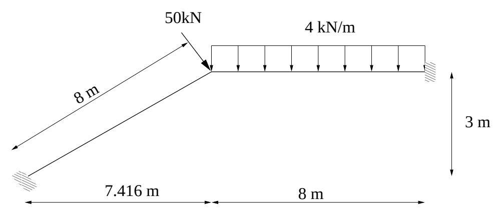

Figure 12.13: Simple Frame Anlysed with the MATLAB Code

```
% zero the matrices
k=zeros(6,6,2);
K=zeros(6,6,2);
Gamma=zeros(6,6,2);
% Structural properties units: mm^2, mm^4, and MPa(10^6 N/m)
A=6000;II=200*10^6;EE=200000;
% Convert units to meter and kN
A=A/10^6;II=II/10^12;EE=EE*1000;
% Element 1
i=[0,0];j=[7.416,3];
[k(:,:,1),K(:,:,1),Gamma(:,:,1)]=stiff(EE,II,A,i,j);
% Element 2
i=j;j=[15.416,3];
[k(:,:,2),K(:,:,2),Gamma(:,:,2)]=stiff(EE,II,A,i,j);
% Define ID matrix
ID=[
   -4 1 -7;
   -5 2 -8;
   -6 3 -9];
% Determine the LM matrix
LM=[
   -4 -5 -6  1  2  3;
    1  2  3 -7 -8 -9];
% Assemble augmented stiffness matrix
Kaug=zeros(9);
```

```
for elem=1:2
   for r=1:6
      lr=abs(LM(elem,r));
      for c=1:6
         lc=abs(LM(elem,c));
         Kaug(lr,lc)=Kaug(lr,lc)+K(r,c,elem);
      end
   end
end
% Extract the structures Stiffness Matrix
Ktt=Kaug(1:3,1:3);
% Determine the fixed end actions in local coordinate system
fea(1:6,1)=0;
fea(1:6,2)=[0,8*4/2,4*8^2/12,0,8*4/2,-4*8^2/12]';
% Determine the fixed end actions in global coordinate system
FEA(1:6,1)=Gamma(:,:,1)*fea(1:6,1);
FEA(1:6,2)=Gamma(:,:,2)*fea(1:6,2);
% FEA_Rest for all the restrained nodes
FEA_Rest=[0,0,0,FEA(4:6,2)'];
% Assemble the load vector for the unrestrained node
P(1)=50*3/8;P(2)=-50*7.416/8-fea(2,2);P(3)=-fea(3,2);
% Solve for the Displacements in meters and radians
Displacements=inv(Ktt)*P'
% Extract Kut
Kut=Kaug(4:9,1:3);
% Compute the Reactions and do not forget to add fixed end actions
Reactions=Kut*Displacements+FEA_Rest'
% Solve for the internal forces and do not forget to include the fixed end actions
dis_global(:,:,1)=[0,0,0,Displacements(1:3)'];
dis_global(:,:,2)=[Displacements(1:3)',0,0,0];
for elem=1:2
   dis_local=Gamma(:,:,elem)*dis_global(:,:,elem)';
   int_forces=k(:,:,elem)*dis_local+fea(1:6,elem)
end

function [k,K,Gamma]=stiff(EE,II,A,i,j)
% Determine the length
L=sqrt((j(2)-i(2))^2+(j(1)-i(1))^2);
% Compute the angle theta (carefull with vertical members!)
if(j(1)-i(1))~=0
   alpha=atan((j(2)-i(2))/(j(1)-i(1)));
else
   alpha=-pi/2;
end
% form rotation matrix Gamma
Gamma=[
cos(alpha)  sin(alpha)  0  0          0          0;
-sin(alpha) cos(alpha)  0  0          0          0;
0           0           1  0          0          0;
0           0           0  cos(alpha) sin(alpha) 0;
0           0           0 -sin(alpha) cos(alpha) 0;
0           0           0  0          0          1];
% form element stiffness matrix in local coordinate system
EI=EE*II;
EA=EE*A;
k=[EA/L,         0,          0, -EA/L,          0,         0;
   0,    12*EI/L^3,  6*EI/L^2,     0, -12*EI/L^3,  6*EI/L^2;
```

```
   0,     6*EI/L^2,    4*EI/L,     0,  -6*EI/L^2,    2*EI/L;
 -EA/L,          0,          0,  EA/L,          0,         0;
     0, -12*EI/L^3, -6*EI/L^2,     0,  12*EI/L^3, -6*EI/L^2;
     0,   6*EI/L^2,    2*EI/L,     0,  -6*EI/L^2,    4*EI/L];
% Element stiffness matrix in global coordinate system
K=Gamma'*k*Gamma;
```

This simple proigram will produce the following results:

```
Displacements =

    0.0010
   -0.0050
   -0.0005

Reactions =

  130.4973
   55.6766
   13.3742
 -149.2473
   22.6734
  -45.3557


int_forces =   int_forces =

  141.8530       149.2473
    2.6758         9.3266
   13.3742        -8.0315
 -141.8530      -149.2473
   -2.6758        22.6734
    8.0315       -45.3557
```

We note that the internal forces are consistent with the reactions (specially for the second node of element 2), and amongst themselves, i.e. the moment at node 2 is the same for both elements (8.0315). ■

■ Example 12-5: Analysis of a simple Beam with Initial Displacements

The full stiffness matrix of a beam element is given by

$$[\mathbf{K}] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad \theta_1 \quad & \quad v_2 \quad & \quad \theta_2 \quad \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array} \tag{12.66}$$

This matrix is singular, it has a rank 2 and order 4 (as it embodies also 2 rigid body motions).

66 We shall consider 3 different cases, Fig. 12.14

Cantilivered Beam/Point Load

1. The *element* stiffness matrix is

$$\mathbf{k} = \begin{array}{c} \\ -3 \\ -4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad -3 \quad & \quad -4 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

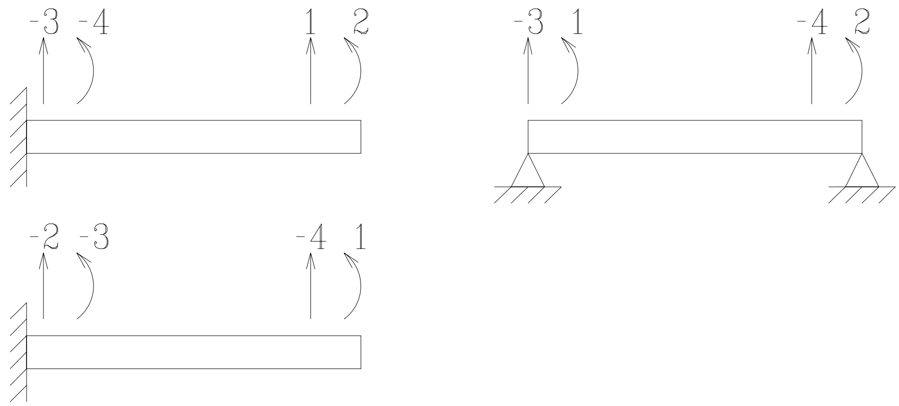

Figure 12.14: ID Values for Simple Beam

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{r} \\ 1 \\ 2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad -3 \quad & \quad -4 \quad \end{array} \\ \begin{bmatrix} 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

3. The global matrix can be rewritten as

$$\left\{ \begin{array}{c} -P\surd \\ 0\surd \\ \hline R_3? \\ R_4? \end{array} \right\} = \left[\begin{array}{cc|cc} 12EI/L^2 & -6EI/L^2 & -12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right] \left\{ \begin{array}{c} \Delta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \theta_4\surd \end{array} \right\}$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$\left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\ \boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right]$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\begin{aligned} \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u &= \left\{ \begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array} \right\} - \left[\begin{array}{cc|cc} L^3/3EI & L^2/2EI & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} \\ L^2/2EI & L/EI & \boxed{6EI/L^2} & \boxed{2EI/L} \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right] \left\{ \begin{array}{c} -P \\ 0 \\ \hline \boxed{0} \\ \boxed{0} \end{array} \right\} \\ &= \left\{ \begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array} \right\} \end{aligned}$$

6. Now we solve for the displacement $\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\boldsymbol{\Delta}_t$

$$\begin{aligned} \left\{ \begin{array}{c} \boxed{\Delta_1} \\ \boxed{\theta_2} \\ \hline 0 \\ 0 \end{array} \right\} &= \left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{L^2/2EI} & -12EI/L^3 & -6EI/L^2 \\ \boxed{L^2/2EI} & \boxed{L/EI} & 6EI/L^2 & 2EI/L \\ \hline -12EI/L^3 & 6EI/L^2 & 12EI/L^3 & 6EI/L^2 \\ -6EI/L^2 & 2EI/L & 6EI/L^2 & 4EI/L \end{array} \right] \left\{ \begin{array}{c} \boxed{-P} \\ \boxed{0} \\ \hline 0 \\ 0 \end{array} \right\} \\ &= \left\{ \begin{array}{c} \boxed{-PL^3/3EI} \\ \boxed{-PL^2/2EI} \\ \hline 0 \\ 0 \end{array} \right\} \end{aligned}$$

7. Finally, we solve for the reactions, $\mathbf{R}_u = \mathbf{K}_{ut}\mathbf{\Delta}_{tt} + \mathbf{K}_{uu}\mathbf{\Delta}_u$, and overwrite $\mathbf{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} -PL^3/3EI \\ -PL^2/2EI \\ \hline \boxed{R_3} \\ \boxed{R_4} \end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & L^2/2EI & -12EI/L^3 & -6EI/L^2 \\ L^2/2EI & L/EI & 6EI/L^2 & 2EI/L \\ \hline \boxed{-12EI/L^3} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} \\ \boxed{-6EI/L^2} & \boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} \end{array}\right] \left\{\begin{array}{c} \boxed{-PL^3/3EI} \\ \boxed{-PL^2/2EI} \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -PL^3/3EI \\ -PL^2/2EI \\ \hline \boxed{P} \\ \boxed{PL} \end{array}\right\}$$

Simply Supported Beam/End Moment

1. The *element* stiffness matrix is

$$\mathbf{k} = \begin{array}{c} \\ -3 \\ 1 \\ -4 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} -3 & 1 & -4 & 2 \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & -3 & -4 \end{array} \\ \left[\begin{array}{cccc} 4EI/L & 2EI/L & 6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & 6EI/L^2 & -6EI/L^2 \\ 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \end{array}$$

3. The global stiffness matrix can be rewritten as

$$\left\{\begin{array}{c} 0\surd \\ M\surd \\ \hline R_3? \\ R_4? \end{array}\right\} = \left[\begin{array}{cc|cc} 4EI/L & 2EI/L & 6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1? \\ \theta_2? \\ \hline \Delta_3\surd \\ \Delta_4\surd \end{array}\right\}$$

4. $\mathbf{K}_{tt}$ is inverted

$$\left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2 \\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right]$$

5. We compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u = \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\} - \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ -L/6EI & L/3EI & \boxed{6EI/L^2} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} 0 \\ M \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\}$$

6. Solve for the displacements, $\mathbf{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\mathbf{\Delta}_t$

$$\left\{\begin{array}{c} \boxed{\theta_1} \\ \boxed{\theta_2} \\ \hline 0 \\ 0 \end{array}\right\} = \left[\begin{array}{cc|cc} \boxed{L^3/3EI} & \boxed{-L/6EI} & 6EI/L^2 & -6EI/L^2 \\ \boxed{-L/6EI} & \boxed{L/3EI} & 6EI/L^2 & -6EI/L^2 \\ \hline 6EI/L^2 & 6EI/L^2 & 12EI/L^3 & -12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & -12EI/L^3 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \boxed{0} \\ \boxed{M} \\ \hline 0 \\ 0 \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{-ML/6EI} \\ \boxed{ML/3EI} \\ \hline 0 \\ 0 \end{array}\right\}$$

7. Solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\boldsymbol{\Delta}_{tt} + \mathbf{K}_{uu}\boldsymbol{\Delta}_u$, and overwrite $\boldsymbol{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} -ML/6EI \\ ML/3EI \\ \hline \boxed{R_1} \\ \boxed{R_2} \end{array}\right\} = \left[\begin{array}{cc|cc} L^3/3EI & -L/6EI & 6EI/L^2 & -6EI/L^2 \\ -L/6EI & L/3EI & 6EI/L^2 & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{-12EI/L^3} \\ \boxed{-6EI/L^2} & \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{12EI/L^3} \end{array}\right] \left\{\begin{array}{c} \boxed{-ML/6EI} \\ \boxed{ML/3EI} \\ \hline \boxed{0} \\ \boxed{0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -ML/6EI \\ ML/3EI \\ \hline \boxed{M/L} \\ \boxed{-M/L} \end{array}\right\}$$

Cantilivered Beam/Initial Displacement and Concentrated Moment

1. The *element* stiffness matrix is

$$\mathbf{k} = \begin{array}{c} \\ -2 \\ -3 \\ -4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} -2 & -3 & -4 & 1 \end{array} \\ \left[\begin{array}{cccc} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}\right] \end{array}$$

2. The *structure* stiffness matrix is assembled

$$\mathbf{K} = \begin{array}{c} \\ 1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & -2 & -3 & -4 \end{array} \\ \left[\begin{array}{cccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \end{array}$$

3. The global matrix can be rewritten as

$$\left\{\begin{array}{c} M\surd \\ \hline R_2? \\ R_3? \\ R_4? \end{array}\right\} = \left[\begin{array}{c|ccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1? \\ \hline \Delta_2\surd \\ \theta_3\surd \\ \Delta_4\surd \end{array}\right\}$$

4. $\mathbf{K}_{tt}$ is inverted (or actually decomposed) and stored in the same global matrix

$$\left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right]$$

5. Next we compute the equivalent load, $\mathbf{P}'_t = \mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u$, and overwrite $\mathbf{P}_t$ by $\mathbf{P}'_t$

$$\mathbf{P}_t - \mathbf{K}_{tu}\boldsymbol{\Delta}_u = \left\{\begin{array}{c} \boxed{M} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\} - \left[\begin{array}{c|ccc} L/4EI & \boxed{6EI/L^2} & \boxed{2EI/L} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1 \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta^0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{M + 6EI\Delta^0/L^2} \\ 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

6. Now we solve for the displacements, $\boldsymbol{\Delta}_t = \mathbf{K}_{tt}^{-1}\mathbf{P}'_t$, and overwrite $\mathbf{P}_t$ by $\boldsymbol{\Delta}_t$

$$\left\{\begin{array}{c} \boxed{\theta_1} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\} = \left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \boxed{M + 6EI\Delta^0/L^2} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{ML/4EI + 3\Delta^0/2L} \\ \hline 0 \\ 0 \\ \Delta^0 \end{array}\right\}$$

7. Finally, we solve for the reactions, $\mathbf{R}_t = \mathbf{K}_{ut}\boldsymbol{\Delta}_{tt} + \mathbf{K}_{uu}\boldsymbol{\Delta}_u$, and overwrite $\boldsymbol{\Delta}_u$ by $\mathbf{R}_u$

$$\left\{\begin{array}{c} ML/4EI + 3\Delta^0/2L \\ \hline \boxed{R_2} \\ \boxed{R_3} \\ \boxed{R_4} \end{array}\right\} = \left[\begin{array}{cc|cc} L/4EI & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} & \boxed{-12EI/L^3} \\ \boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} & \boxed{-6EI/L^2} \\ \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} & \boxed{12EI/L^3} \end{array}\right] \left\{\begin{array}{c} \boxed{ML/4EI + 3\Delta^0/2L} \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta^0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} ML/4EI + 3\Delta^0/2L \\ \hline \boxed{3M/2L - 3EI\Delta^0/L^3} \\ \boxed{M/2 - 3EI\Delta^0/L^2} \\ \boxed{-3M/2L + 3EI\Delta^0/L^3} \end{array}\right\}$$

■

12.4 Computer Program Organization

66 The main program should,

1. Read
   (a) title card
   (b) control card which should include:
      i. Number of nodes
      ii. Number of elements
      iii. Type of structure: beam, grid, truss, or frame; (2D or 3D)
      iv. Number of different element properties
      v. Number of load cases
2. Determine:
   (a) Number of spatial coordinates for the structure
   (b) Number of local and global degrees of freedom per node
3. For each node read:
   (a) Node number
   (b) Boundary conditions of each global degree of freedom [**ID**]
   (c) Spatial coordinates

   Note that all the above are usually written on the same "data card"
4. For each element, read:
   (a) Element number
   (b) First and second node
   (c) Element Property number
5. For each element property group read the associated elastic and cross sectional characteristics. Note these variables will depend on the structure type.
6. Determine the vector $\boldsymbol{\Delta}_u$ which stores the initial displacements.
7. Loop over all the elements and for each one:

(a) Retrieve its properties

(b) Determine the length

(c) Call the appropriate subroutines which will determine:

i. The stiffness matrix in local coordinate systems $[\mathbf{k}^{(e)}]$.

ii. The angle α and the transformation matrix $[\mathbf{\Gamma}^{(e)}]$.

8. Assembly of the global stiffness matrix

(a) Initialize the global stiffness matrix to zero

(b) Loop through each element, e, and for each element:

i. Retrieve its stiffness matrix (in local coordinates) $[\mathbf{k}^{(e)}]$ and transformation matrix $[\mathbf{\Gamma}^{(e)}]$.

ii. Compute the element stiffness matrix in global coordinates from $[\mathbf{K}^{(e)}] = [\mathbf{\Gamma}^{(e)}]^T[\mathbf{k}^{(e)}][\mathbf{\Gamma}^{(e)}]$.

iii. Define the $\{\mathbf{LM}\}$ array of the element

iv. Loop through each row i and column j of the element stiffness matrix, and for those degree of freedom not equal to zero, add the contributions of the element to the structure's stiffness matrix $K^S[LM(i), LM(j)] = K^S[LM(i), LM(j)] + K^{(e)}[i, j]$

9. Extract the structure's stiffness matrix $[\mathbf{K}_{tt}]$ from the augmented stiffness matrix.

10. Invert the structure's stiffness matrix (or decompose it).

11. For each load case:

(a) Determine the nodal equivalent loads (fixed end actions), if any.

(b) Assemble the load vector

(c) Load assembly (once for each load cae) once the stiffness matrix has been decomposed, than the main program should loop through each load case and,

i. Initialize the load vector (of length NEQ) to zero.

ii. Read number of loaded nodes. For each loaded node store the non-zero values inside the load vector (using the $[\mathbf{ID}]$ matrix for determining storage location).

iii. Loop on all loaded elements:

A. Read element number, and load value

B. Compute the fixed end actions and rotate them from local to global coordinates.

C. Using the **LM** vector, add the fixed end actions to the nodal load vector (unless the corresponding equation number is zero, ie. restrained degree of freedom).

D. Store the fixed end actions for future use.

(d) Apply Eq. 12.31 to determine the nodal displacements $\mathbf{\Delta}_t = \mathbf{K}_{tt}^{-1}(\mathbf{P}_t - \mathbf{K}_{tu}\mathbf{\Delta}_u)$

(e) Apply Eq. 12.32 to determine the nodal reactions $\mathbf{R}_t = \mathbf{K}_{ut}\mathbf{\Delta}_t + \mathbf{K}_{uu}\mathbf{\Delta}_u$

(f) Determine the internal forces (axial, shear and moment)

i. For each element retrieve:

A. nodal coordinates

B. rotation matrix $[\mathbf{\Gamma}^{(e)}]$.

C. element stiffness matrix $[\mathbf{k}^{(e)}]$.

ii. Compute nodal displacements in local coordinate system from $\left\{\boldsymbol{\delta}^{(e)}\right\} = [\mathbf{\Gamma}^{(e)}]\{\mathbf{\Delta}\}$

iii. Compute element internal forces from $\{\mathbf{p}\} = [\mathbf{k}^{(e)}]\left\{\boldsymbol{\delta}^{(e)}\right\}$

iv. If the element is loaded, add corresponding fixed end actions

v. print the interior forces

12.5 Computer Implementation with MATLAB

67 You will be required, as part of your term project, to write a simple MATLAB (or whatever other language you choose) program for the analysis of two dimensional frames with nodal load and initial displacement, as well as element load.

68 To facilitate the task, your instructor has taken the liberty of taking a program written by Mr. Dean Frank (as part of his term project with this instructor in the Advanced Structural Analysis course, Fall 1995), modified it with the aid of Mr. Pawel Smolarki, and is making available most, but not all of it to you. Hence, you will be expected to first familiarize yourself with the code made available to you, and then complete it by essentially filling up the missing parts.

12.5.1 Program Input

`From Dean Frank's User's Manual`

69 It is essential that the structure be idealized such that it can be discretized. This discretization should define each node and element uniquely. In order to decrease the required amount of computer storage and computation it is best to number the nodes in a manner that minimizes the numerical separation of the node numbers on each element. For instance, an element connecting nodes 1 and 4, could be better defined by nodes 1 and 2, and so on. As it was noted previously, the user is required to have a decent understanding of structural analysis and structural mechanics. As such, it will be necessary for the user to generate or modify an input file input.m using the following directions. Open the file called input.m and set the existing variables in the file to the appropriate values. The input file has additional helpful directions given as comments for each variable. After setting the variables to the correct values, be sure to save the file. Please note that the program is case-sensitive.

70 In order for the program to be run, the user must supply the required data by setting certain variables in the file called `indat.m` equal to the appropriate values. All the user has to do is open the text file called indat.txt, fill in the required values and save the file as `indat.m` in a directory within MATAB's path. There are helpful hints within this file. It is especially important that the user keep track of units for all of the variables in the input data file. All of the units MUST be consistent. It is suggested that one always use the same units for all problems. For example, always use kips and inches, or kilo-newtons and millimeters.

12.5.1.1 Input Variable Descriptions

71 A brief description of each of the variables to be used in the input file is given below:

`npoin` This variable should be set equal to the number of nodes that comprise the structure. A node is defined as any point where two or more elements are joined.

`nelem` This variable should be set equal to the number of elements in the structure. Elements are the members which span between nodes.

`istrtp` This variable should be set equal to the type of structure. There are six types of structures which this program will analyze: beams, 2-D trusses, 2-D frames, grids, 3-D trusses, and 3-D frames. Set this to 1 for beams, 2 for 2D-trusses, 3 for 2D- frames, 4 for grids, 5 for 3D-trusses, and 6 for 3D-frames. An error will occur if it is not set to a number between 1 and 6. Note only `istrp=3` was kept.

`nload` This variable should be set equal to the number of different load cases to be analyzed. A load case is a specific manner in which the structure is loaded.

`ID` (matrix) The ID matrix contains information concerning the boundary conditions for each node. The number of rows in the matrix correspond with the number of nodes in the structure and the number of columns corresponds with the number of degrees of freedom for each node for that type of structure type. The matrix is composed of ones and zeros. A one indicates that the degree of freedom is restrained and a zero means it is unrestrained.

`nodecoor` (matrix) This matrix contains the coordinates (in the global coordinate system) of the nodes in the structure. The rows correspond with the node number and the columns correspond with the global coordinates x, y, and z, respectively. It is important to always include all three coordinates for each node even if the structure is only two- dimensional. In the case of a two-dimensional structure, the z-coordinate would be equal to zero.

`lnods` (matrix) This matrix contains the nodal connectivity information. The rows correspond with the element number and the columns correspond with the node numbers which the element is connected from and to, respectively.

`E,A,Iy` (arrays) These are the material and cross-sectional properties for the elements. They are arrays with the number of terms equal to the number of elements in the structure. The index number of each term corresponds with the element number. For example, the value of A(3) is the area of element 3, and so on. E is the modulus of elasticity, A is the cross-sectional area, Iy is the moment of inertia about the y axes

`Pnods` This is an array of nodal loads in global degrees of freedom. Only put in the loads in the global degrees of freedom and if there is no load in a particular degree of freedom, then put a zero in its place. The index number corresponds with the global degree of freedom.

`Pelem` This an array of element loads, or loads which are applied between nodes. Only one load between elements can be analyzed. If there are more than one element loads on the structure, the equivalent nodal load can be added to the nodal loads. The index number corresponds with the element number. If there is not a load on a particular member, put a zero in its place. These should be in local coordinates.

`a` This is an array of distances from the left end of an element to the element load. The index number corresponds to the element number. If there is not a load on a particular member, put a zero in its place. This should be in local coordinates.

`w` This is an array of distributed loads on the structure. The index number corresponds with the element number. If there is not a load on a particular member, put a zero in its place. This should be in local coordinates

`dispflag` Set this variable to 1 if there are initial displacements and 0 if there are none.

`initial_displ` This is an array of initial displacements in all structural degrees of freedom. This means that you must enter in values for all structure degrees of freedom, not just those restrained. For example, if the structure is a 2D truss with 3 members and 3 node, there would be 6 structural degrees of freedom, etc. If there are no initial displacements, then set the values equal to zero.

`angle` This is an array of angles which the x-axis has possibly been rotated. This angle is taken as positive if the element has been rotated towards the z-axis. The index number corresponds to the element number.

`drawflag` Set this variable equal to 1 if you want the program to draw the structure and 0 if you do not.

12.5.1.2 Sample Input Data File

The contents of the input.m file which the user is to fill out is given below:

```
%*********************************************************************************************
% Scriptfile name: indat.m (EXAMPLE 2D-FRAME INPUT DATA)
%
% Main Program: casap.m
%
% This is the main data input file for the computer aided
% structural analysis program CASAP.  The user must supply
% the required numeric values for the variables found in
% this file (see user's manual for instructions).
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% HELPFUL INSTRUCTION COMMENTS IN ALL CAPITALS

% SET NPOIN EQUAL TO THE NUMBER OF NODES IN THE STRUCTURE

npoin=3;

% SET NELEM EQUAL TO THE NUMBER OF ELEMENTS IN THE STRUCTURE
```

```
nelem=2;

% SET NLOAD EQUAL TO THE NUMBER OF LOAD CASES

nload=1;

% INPUT THE ID MATRIX CONTAINING THE NODAL BOUNDARY CONDITIONS (ROW # = NODE #)

ID=[1 1 1;
   0 0 0;
   1 1 1];

% INPUT THE NODE COORDINATE (X,Y) MATRIX, NODECOOR (ROW # = NODE #)

nodecoor=[
   0 0;
   7416 3000;
   15416 3000
];

% INPUT THE ELEMENT CONNECTIVITY MATRIX, LNODS (ROW # = ELEMENT #)

lnods=[
   1 2;
   2 3
];

% INPUT THE MATERIAL PROPERTIES ASSOCIATED WITH THIS TYPE OF STRUCTURE
% PUT INTO ARRAYS WHERE THE INDEX NUMBER IS EQUAL TO THE CORRESPONDING ELEMENT NUMBER.
% COMMENT OUT VARIABLES THAT WILL NOT BE USED

E=[200 200];
A=[6000 6000];
Iz=[200000000 200000000];

% INPUT THE LOAD DATA.  NODAL LOADS, PNODS SHOULD BE IN MATRIX FORM.  THE COLUMNS CORRESPOND
% TO THE GLOBAL DEGREE OF FREEDOM IN WHICH THE LOAD IS ACTING AND THE THE ROW NUMBER CORRESPONDS
% WITH THE LOAD CASE NUMBER.  PELEM IS THE ELEMENT LOAD, GIVEN IN A MATRIX, WITH COLUMNS
% CORRESPONDING TO THE ELEMENT NUMBER AND ROW THE LOAD CASE.  ARRAY "A" IS THE DISTANCE FROM
% THE LEFT END OF THE ELEMENT TO THE LOAD, IN ARRAY FORM.  THE DISTRIBUTED LOAD, W SHOULD BE
% IN MATRIX FORM ALSO WITH COLUMNS = ELEMENT NUMBER UPON WHICH W IS ACTING AND ROWS = LOAD CASE.
% ZEROS SHOULD BE USED IN THE MATRICES WHEN THERE IS NO LOAD PRESENT.  NODAL LOADS SHOULD
% BE GIVEN IN GLOBAL COORDINATES, WHEREAS THE ELEMENT LOADS AND DISTRIBUTED LOADS SHOULD BE
% GIVEN IN LOCAL COORDINATES.

Pnods=[18.75 -46.35 0];
Pelem=[0 0];
a=[0 0];
w=[0 4/1000];

% IF YOU WANT THE PROGRAM TO DRAW THE STUCTURE SET DRAWFLAG=1, IF NOT SET IT EQUAL TO 0.
% THIS IS USEFUL FOR CHECKING THE INPUT DATA.

drawflag=1;

% END OF INPUT DATA FILE
```

12.5.1.3 Program Implementation

In order to ”run” the program, open a new MATLAB Notebook. On the first line, type the name of the main program `CASAP` and evaluate that line by typing ctrl-enter. At this point, the main program reads the input file you have just created and calls the appropriate subroutines to analyze your structure. In doing so, your input data is echoed into your MATLAB notebook and the program results are also displayed. As a note, the program can also be executed directly from the MATAB workspace window, without Microsoft Word.

12.5.2 Program Listing

12.5.2.1 Main Program

```
%*********************************************************************************************
%Main Program: casap.m
%
% This is the main program, Computer Aided Structural Analysis Program
% CASAP.  This program primarily contains logic for calling scriptfiles and does not
% perform calculations.
%
% All variables are global, but are defined in the scriptfiles in which they are used.
%
% Associated scriptfiles:
%
% (for all stuctures)
% indat.m (input data file)
% idrasmbl.m
% elmcoord.m
% draw.m
%
% (3 - for 2D-frames)
% length3.m
% stiffl3.m
% trans3.m
% assembl3.m
% loads3.m
% disp3.m
% react3.m
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% COMMENT CARDS ARE IN ALL CAPITALS

% SET NUMERIC FORMAT

format short e

% CLEAR MEMORY OF ALL VARIABLES

clear

%  INITIALIZE OUTPUT FILE
fid = fopen('casap.out', 'wt');

% SET ISTRTP EQUAL TO THE NUMBER CORRESPONDING TO THE TYPE OF STRUCTURE:
% 3 = 2DFRAME

istrtp=3;

% READ INPUT DATA SUPPLIED BY THE USER

indat

% REASSAMBLE THE ID MATRIX AND CALCULATE THE LM VECTORS
% CALL SCRIPTFILE IDRASMBL

idrasmbl

% ASSEMBLE THE ELEMENT COORDINATE MATRIX

elmcoord

% 2DFRAME CALCULATIONS
```

```
% CALCULATE THE LENGTH AND ORIENTATION ANGLE, ALPHA FOR EACH ELEMENT
% CALL SCRIPTFILE LENGTH3.M

length3

% CALCULATE THE 2DFRAME ELEMENT STIFFNESS MATRIX IN LOCAL COORDINATES
% CALL SCRIPTFILE STIFFL3.M

stiffl3

% CALCULATE THE 2DFRAME ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
% CALL SCRIPTFILE TRANS3.M

trans3

% ASSEMBLE THE GLOBAL STRUCTURAL STIFFNESS MATRIX
% CALL SCRIPTFILE ASSEMBL3.M

assembl3

%  PRINT STRUCTURAL INFO

print_general_info

% LOOP TO PERFORM ANALYSIS FOR EACH LOAD CASE
for iload=1:nload

   print_loads

   % DETERMINE THE LOAD VECTOR IN GLOBAL COORDINATES
   % CALL SCRIPTFILE LOADS3.M

   loads3

   % CALCULATE THE DISPLACEMENTS
   % CALL SCRIPTFILE DISP3.M

   disp3

   % CALCULATE THE REACTIONS AT THE RESTRAINED DEGREES OF FREEDOM
   % CALL SCRIPTFILE REACT3.M

   react3

   % CALCULATE THE INTERNAL FORCES FOR EACH ELEMENT

   intern3

   % END LOOP FOR EACH LOAD CASE

end

% DRAW THE STRUCTURE, IF USER HAS REQUESTED (DRAWFLAG=1)
% CALL SCRIPTFILE DRAW.M

draw

st=fclose('all');
% END OF MAIN PROGRAM (CASAP.M)
disp('Program completed! - See "casap.out" for complete output');
```

12.5.2.2 Assembly of ID Matrix

```
%***********************************************************************************************
%SCRIPTFILE NAME: IDRASMBL.M
%
%MAIN FILE : CASAP
%
%Description : This file re-assambles the ID matrix such that the restrained
% degrees of freedom are given negative values and the unrestrained
```

```
% degrees of freedom are given incremental values begining with one
% and ending with the total number of unrestrained degrees of freedom.
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% TAKE CARE OF SOME INITIAL BUSINESS: TRANSPOSE THE PNODS ARRAY

Pnods=Pnods.';

% SET THE COUNTER TO ZERO

count=1;
negcount=-1;

% REASSEMBLE THE ID MATRIX

if istrtp==3
ndofpn=3;
nterm=6;
else
error('Incorrect structure type specified')
end

% SET THE ORIGINAL ID MATRIX TO TEMP MATRIX

orig_ID=ID;

% REASSEMBLE THE ID MATRIX, SUBSTITUTING RESTRAINED DEGREES OF FREEDOM WITH NEGATIVES,
%  AND NUMBERING GLOBAL DEGREES OF FREEDOM

for inode=1:npoin
for icoord=1:ndofpn
if ID(inode,icoord)==0
ID(inode,icoord)=count;
count=count+1;
elseif ID(inode,icoord)==1
         ID(inode,icoord)=negcount;
         negcount=negcount-1;
else
error('ID input matrix incorrect')
end
end
end

% CREATE THE LM VECTORS FOR EACH ELEMENT

for ielem=1:nelem
LM(ielem,1:ndofpn)=ID(lnods(ielem,1),1:ndofpn);
LM(ielem,(ndofpn+1):(2*ndofpn))=ID(lnods(ielem,2),1:ndofpn);
end

% END OF IDRASMBL.M SCRIPTFILE
```

12.5.2.3 Element Nodal Coordinates

```
%*******************************************************************************************
%SCRIPTFILE NAME: ELEMCOORD.M
%
%MAIN FILE : CASAP
%
%Description : This file assembles a matrix, elemcoor which contains the coordinates
% of the first and second nodes on each element, respectively.
%
```

```
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% ASSEMBLE THE ELEMENT COORDINATE MATRIX, ELEMCOOR FROM NODECOOR AND LNODS

for ielem=1:nelem

elemcoor(ielem,1)=nodecoor(lnods(ielem,1),1);
elemcoor(ielem,2)=nodecoor(lnods(ielem,1),2);
%elemcoor(ielem,3)=nodecoor(lnods(ielem,1),3);
elemcoor(ielem,3)=nodecoor(lnods(ielem,2),1);
elemcoor(ielem,4)=nodecoor(lnods(ielem,2),2);
%elemcoor(ielem,6)=nodecoor(lnods(ielem,2),3);
end
% END OF ELMCOORD.M SCRIPTFILE
```

12.5.2.4 Element Lengths

```
%*********************************************************************************************
% Scriptfile name : length3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it computes the length of each element and the
% angle alpha between the local and global x-axes.  This file can be used
% for 2-dimensional elements such as 2-D truss, 2-D frame, and grid elements.
% This information will be useful for transformation between local and global
% variables.
%
% Variable descriptions: (in the order in which they appear)
%
% nelem =  number of elements in the structure
% ielem =  counter for loop
% L(ielem) =  length of element ielem
% elemcoor(ielem,4) =  xj-coordinate of element ielem
% elemcoor(ielem,1) =  xi-coordinate of element ielem
% elemcoor(ielem,5) =  yj-coordinate of element ielem
% elemcoor(ielem,2) =  yi-coordinate of element ielem
% alpha(ielem) =  angle between local and global x-axes
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% COMPUTE THE LENGTH AND ANGLE BETWEEN LOCAL AND GLOBAL X-AXES FOR EACH ELEMENT

for ielem=1:nelem
L(ielem)=
XXXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
alpha(ielem)=
XXXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

% END OF LENGTH3.M SCRIPTFILE
```

12.5.2.5 Element Stiffness Matrices

```
%*********************************************************************************************
% Scriptfile name: stiffl3.m (for 2d-frame structures)
%
```

```
% Main program: casap.m
%
% When this file is called, it computes the element stiffenss matrix
% of a 2-D frame element in local coordinates.  The element stiffness
% matrix is calculated for each element in the structure.
%
% The matrices are stored in a single matrix of dimensions 6x6*i and
% can be recalled individually later in the program.
%
% Variable descriptions: (in the order in which the appear)
%
% ielem = counter for loop
% nelem = number of element in the structure
% k(ielem,6,6)= element stiffness matrix in local coordinates
% E(ielem) = modulus of elasticity of element ielem
% A(ielem) = cross-sectional area of element ielem
% L(ielem) = lenght of element ielem
% Iz(ielem) = moment of inertia with respect to the local z-axis of element ielem
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%**********************************************************************************************

for ielem=1:nelem
  k(1:6,1:6,ielem)=...
  XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end

% END OF STIFFL3.M SCRIPTFILE
```

12.5.2.6 Transformation Matrices

```
%**********************************************************************************************
% Scriptfile name : trans3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% This file calculates the rotation matrix and the element stiffness
% matrices for each element in a 2D frame.
%
% Variable descriptions: (in the order in which they appear)
%
% ielem = counter for the loop
% nelem = number of elements in the structure
% rotation = rotation matrix containing all elements info
% Rot = rotational matrix for 2d-frame element
% alpha(ielem) = angle between local and global x-axes
% K = element stiffness matrix in global coordinates
% k = element stiffness matrix in local coordinates
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%**********************************************************************************************

% CALCULATE THE ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
% FOR EACH ELEMENT IN THE STRUCTURE

for ielem=1:nelem

% SET UP THE ROTATION MATRIX, ROTATAION

rotation(1:6,1:6,ielem)=...
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
   ktemp=k(1:6,1:6,ielem);
% CALCULATE THE ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES
Rot=rotation(1:6,1:6,ielem);
K(1:6,1:6,ielem)=
```

```
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end
% END OF TRANS3.M SCRIPTFILE
```

12.5.2.7 Assembly of the Augmented Stiffness Matrix

```
%*********************************************************************************************
% Scriptfile name : assembl3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% This file assembles the global structural stiffness matrix from the
% element stiffness matrices in global coordinates using the LM vectors.
% In addition, this file assembles the augmented stiffness matrix.
%
% Variable Descritpions (in order of appearance):
%
% ielem = Row counter for element number
% nelem = Number of elements in the structure
% iterm = Counter for term number in LM matrix
% LM(a,b) = LM matrix
% jterm = Column counter for element number
% temp1 = Temporary variable
% temp2 = Temporary variable
% temp3 = Temporary variable
% temp4 = Temporary variable
% number_gdofs = Number of global dofs
% new_LM = LM matrix used in assembling the augmented stiffness matrix
% aug_total_dofs = Total number of structure dofs
% K_aug        = Augmented structural stiffness matrix
% Ktt          = Structural Stiffness Matrix (Upper left part of Augmented structural stiffness matrix)
% Ktu = Upper right part of Augmented structural stiffness matrix
% Kut = Lower left part of Augmented structural stiffness matrix
% Kuu = Lower rigth part of Augmented structural stiffness matrix
%
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************


%  RENUMBER DOF INCLUDE ALL DOF, FREE DOF FIRST, RESTRAINED NEXT
new_LM=LM;
number_gdofs=max(LM(:));
new_LM(find(LM<0))=number_gdofs-LM(find(LM<0));
aug_total_dofs=max(new_LM(:));

% ASSEMBLE THE AUGMENTED STRUCTURAL STIFFNESS MATRIX
K_aug=zeros(aug_total_dofs);
for ielem=1:nelem
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Tough one!
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
end

% SET UP SUBMATRICES FROM THE AUGMENTED STIFFNESS MATRIX

Ktt=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Ktu=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Kut=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
Kuu=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
% END OF ASSEMBL3.M SCRIPTFILE
```

12.5.2.8 Print General Information

```
%*********************************************************************************************
% Scriptfile name : print_general_info.m
%
% Main program : casap.m
%
% Prints the general structure info to the output file
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

fprintf(fid,'\n\nNumber of Nodes: %d\n',npoin);
fprintf(fid,'Number of Elements: %d\n',nelem);
fprintf(fid,'Number of Load Cases: %d\n',nload);
fprintf(fid,'Number of Restrained dofs: %d\n',abs(min(LM(:))));
fprintf(fid,'Number of Free dofs: %d\n',max(LM(:)));

fprintf(fid,'\nNode Info:\n');
for inode=1:npoin
   fprintf(fid,'     Node %d (%d,%d)\n',inode,nodecoor(inode,1),nodecoor(inode,2));
   freedof=' ';
   if(ID(inode,1))>0
      freedof=strcat(freedof,' X ');
   end
   if(ID(inode,2))>0
      freedof=strcat(freedof,' Y ');
   end
   if(ID(inode,3))>0
      freedof=strcat(freedof,' Rot');
   end
   if freedof==' '
      freedof=' none; node is fixed';
   end
   fprintf(fid,'        Free dofs:%s\n',freedof);
end

fprintf(fid,'\nElement Info:\n');
for ielem=1:nelem
   fprintf(fid,'     Element %d (%d->%d)',ielem,lnods(ielem,1),lnods(ielem,2));
   fprintf(fid,'   E=%d A=%d Iz=%d \n',E(ielem),A(ielem),Iz(ielem));
end
```

12.5.2.9 Print Load

```
%*********************************************************************************************
% Scriptfile name : print_loads.m
%
% Main program : casap.m
%
% Prints the current load case data to the output file
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

Load_case=iload
if iload==1
   fprintf(fid,'\n_____________________________________________________________________________\n\n');
end

fprintf(fid,'Load Case: %d\n\n',iload);
fprintf(fid,'   Nodal Loads:\n');
for k=1:max(LM(:));
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==k);
   elem=fix(LM_spot(1)/(nterm+1))+1;
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
```

```
   switch(dof)
   case {1,4}, dof='Fx';
   case {2,5}, dof='Fy';
   otherwise, dof=' M';
   end
   %PRINT THE DISPLACEMENTS
   if Pnods(k)~=0
      fprintf(fid,'     Node: %2d %s = %14d\n',node, dof, Pnods(k));
   end
end

fprintf(fid,'\n   Elemental Loads:\n');
for k=1:nelem
   fprintf(fid,'     Element: %d   Point load = %d at %d from left\n',k,Pelem(k),a(k));
   fprintf(fid,'                  Distributed load = %d\n',w(k));
end
fprintf(fid,'\n');
```

12.5.2.10 Load Vector

```
%*********************************************************************************************
% Scriptfile name: loads3.m (for 2d-frame structures)
%
% Main program: casap.m
%
% When this file is called, it computes the fixed end actions for elements which
% carry distributed loads for a 2-D frame.
%
% Variable descriptions: (in the order in which they appear)
%
% ielem = counter for loop
% nelem = number of elements in the structure
% b(ielem) = distance from the right end of the element to the point load
% L(ielem) = length of the element
% a(ielem) = distance from the left end of the element to the point load
% Ffl = fixed end force (reaction) at the left end due to the point load
% w(ielem) = distributed load on element ielem
% L(ielem) = length of element ielem
% Pelem(ielem) = element point load on element ielem
% Mfl = fixed end moment (reaction) at the left end due to the point load
% Ffr = fixed end force (reaction) at the right end due to the point load
% Mfr = fixed end moment (reaction) at the right end due to the point load
% feamatrix_local = matrix containing resulting fixed end actions in local coordinates
% feamatrix_global = matrix containing resulting fixed end actions in global coordinates
% fea_vector = vector of fea's in global dofs, used to calc displacements
% fea_vector_abs = vector of fea's in every structure dof
% dispflag = flag indicating initial displacements
% Ffld = fea (vert force) on left end of element due to initial disp
% Mfld = fea (moment) on left end of element due to initial disp
% Ffrd = fea (vert force) on right end of element due to initial disp
% Mfrd = fea (moment) on right end of element due to initial disp
% fea_vector_disp = vector of fea's due to initial disp, used to calc displacements
% fea_vector_react = vector of fea's due to initial disp, used to calc reactions
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%*********************************************************************************************

% CALCULATE THE FIXED END ACTIONS AND INSERT INTO A MATRIX IN WHICH THE ROWS CORRESPOND
% WITH THE ELEMENT NUMBER AND THE COLUMNS CORRESPOND WITH THE ELEMENT LOCAL DEGREES
% OF FREEDOM

for ielem=1:nelem

b(ielem)=L(ielem)-a(ielem);

Ffl=((w(ielem)*L(ielem))/2)+((Pelem(ielem)*(b(ielem))^2)/(L(ielem))^3)*(3*a(ielem)+b(ielem));
Mfl=((w(ielem)*(L(ielem))^2))/12+(Pelem(ielem)*a(ielem)*(b(ielem))^2)/(L(ielem))^2;
Ffr=((w(ielem)*L(ielem))/2)+((Pelem(ielem)*(a(ielem))^2)/(L(ielem))^3)*(a(ielem)+3*b(ielem));
```

```
Mfr=-((w(ielem)*(L(ielem))^2))/12+(Pelem(ielem)*a(ielem)*(b(ielem))^2)/(L(ielem))^2;

feamatrix_local(ielem,1:6)=[0 Ffl Mfl 0 Ffr Mfr];

% ROTATE THE LOCAL FEA MATRIX TO GLOBAL

feamatrix_global=...
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

end

% CREATE A LOAD VECTOR USING THE LM MATRIX

% INITIALIZE FEA VECTOR TO ALL ZEROS

for idofpn=1:ndofpn
fea_vector(idofpn,1)=0;
end

for ielem=1:nelem
for idof=1:6
if ielem==1
if LM(ielem,idof)>0
fea_vector(LM(ielem,idof),1)=feamatrix_global(idof,ielem);
end

elseif ielem>1
if LM(ielem,idof)>0
fea_vector(LM(ielem,idof),1)=fea_vector(LM(ielem,1))+feamatrix_global(idof,ielem);
end
end
end
end

for ielem=1:nelem
for iterm=1:nterm
if feamatrix_global(iterm,ielem)==0
else
if new_LM(ielem,iterm)>number_gdofs
fea_vector_react(iterm,1)=feamatrix_global(iterm,ielem);
end
end
end
end

% END OF LOADS3.M SCRIPTFILE
```

12.5.2.11 Nodal Displacements

```
%*********************************************************************************************
% Scriptfile name : disp3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it computes the displacements in the global
% degrees of freedom.
%
% Variable descriptions: (in the order in which they appear)
%
% Ksinv = inverse of the structural stiffness matrix
% Ktt = structural stiffness matrix
% Delta = vector of displacements for the global degrees of freedom
% Pnods = vector of nodal loads in the global degrees of freedom
% fea_vector = vector of fixed end actions in the global degrees of freedom
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
```

```
%
%*********************************************************************************************

% CREATE A TEMPORARY VARIABLE EQUAL TO THE INVERSE OF THE STRUCTURAL STIFFNESS MATRIX

Ksinv=inv(Ktt);

% CALCULATE THE DISPLACEMENTS IN GLOBAL COORDINATES

Delta=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

%  PRINT DISPLACEMENTS WITH NODE INFO

fprintf(fid,'   Displacements:\n');
for k=1:size(Delta,1)
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==k);
   elem=fix(LM_spot(1)/(nterm+1))+1;
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
   switch(dof)
   case {1,4}, dof='delta X';
   case {2,5}, dof='delta Y';
   otherwise, dof='rotate ';
   end
   %PRINT THE DISPLACEMENTS
   fprintf(fid,'     (Node: %2d %s) %14d\n',node, dof, Delta(k));
end
fprintf(fid,'\n');

% END OF DISP3.M SCRIPTFILE
```

12.5.2.12 Reactions

```
%*********************************************************************************************
% Scriptfile name : react3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it calculates the reactions at the restrained degrees of
% freedom.
%
% Variable Descriptions:
%
% Reactions = Reactions at restrained degrees of freedom
% Kut = Upper left part of aug stiffness matrix, normal structure stiff matrix
% Delta = vector of displacements
% fea_vector_react = vector of fea's in restrained dofs
%
%
% By Dean A. Frank
% CVEN 5525 - Term Project
% Fall 1995
%
%  Edited by Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

% CALCULATE THE REACTIONS FROM THE AUGMENTED STIFFNESS MATRIX

Reactions=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

fprintf(fid,'   Reactions:\n');
for k=1:size(Reactions,1)
   %WORK BACKWARDS WITH LM MATRIX TO FIND NODE# AND DOF
   LM_spot=find(LM'==-k);
   elem=fix(LM_spot(1)/(nterm+1))+1;
```

```
   dof=mod(LM_spot(1)-1,nterm)+1;
   node=lnods(elem,fix(dof/4)+1);
   switch(dof)
   case {1,4}, dof='Fx';
   case {2,5}, dof='Fy';
   otherwise, dof='M ';
   end
   %PRINT THE REACTIONS
   fprintf(fid,'     (Node: %2d %s) %14d\n',node, dof, Reactions(k));
end
fprintf(fid,'\n');

% END OF REACT3.M SCRIPTFILE
```

12.5.2.13 Internal Forces

```
%*********************************************************************************************
% Scriptfile name : intern3.m (for 2d-frame structures)
%
% Main program : casap.m
%
% When this file is called, it calculates the internal forces in all elements
% freedom.
%
%     By Pawel Smolarkiewicz, 3/16/99
% Simplified for 2D Frame Case only
%
%*********************************************************************************************

Pglobe=zeros(6,nelem);
Plocal=Pglobe;

fprintf(fid,'   Internal Forces:');
%LOOP FOR EACH ELEMENT
for ielem=1:nelem
   %FIND ALL 6 LOCAL DISPLACEMENTS
   elem_delta=zeros(6,1);
   for idof=1:6
      gdof=LM(ielem,idof);
      if gdof<0
         elem_delta(idof)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      else
         elem_delta(idof)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      end
   end

   %SOLVE FOR ELEMENT FORCES (GLOBAL)
   Pglobe(:,ielem)=K(:,:,ielem)*elem_delta+feamatrix_global(:,ielem);
   %ROTATE FORCES FROM GLOBAL TO LOCAL COORDINATES

   %ROTATE FORCES TO LOCAL COORDINATES
   Plocal(:,ielem)=
XXXXXXXXXXXXXXXXXXXXXX COMPLETE XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

   %PRINT RESULTS
   fprintf(fid,'\n      Element: %2d\n',ielem);
   for idof=1:6
      if idof==1
         fprintf(fid,'       At Node: %d\n',lnods(ielem,1));
      end
      if idof==4
         fprintf(fid,'       At Node: %d\n',lnods(ielem,2));
      end
      switch(idof)
      case {1,4}, dof='Fx';
      case {2,5}, dof='Fy';
      otherwise, dof='M ';
      end
      fprintf(fid,'          (Global : %s ) %14d',dof, Pglobe(idof,ielem));
```

```
      fprintf(fid,'    (Local : %s ) %14d\n',dof, Plocal(idof,ielem));
   end
end
fprintf(fid,'\n________________________________________________________________________\n\n');
```

12.5.2.14 Plotting

```
%*********************************************************************************************
% SCRIPTFILE NAME : DRAW.M
%
% MAIN FILE : CASAP
%
% Description : This file will draw a 2D or 3D structure (1D structures are generally
% boring to draw).
%
% Input Variables : nodecoor  - nodal coordinates
% ID - connectivity matrix
% drawflag - flag for performing drawing routine
%
% By Dean A. Frank
% CVEN 5525
% Advanced Structural Analysis - Term Project
% Fall 1995
%
% (with thanks to Brian Rose for help with this file)
%
%*********************************************************************************************

% PERFORM OPERATIONS IN THIS FILE IF DRAWFLAG = 1

if drawflag==1
if istrtp==1
       drawtype=2;
elseif istrtp==2
       drawtype=2;
elseif istrtp==3
       drawtype=2;
elseif istrtp==4
       drawtype=2;
elseif istrtp==5
       drawtype=3;
elseif istrtp==6
       drawtype=3;
else
       error('Incorrect structure type in indat.m')
end


ID=orig_ID.';

% DRAW 2D STRUCTURE IF DRAWTYPE=2

if drawtype==2

       % RETREIVE NODAL COORDINATES

       x=nodecoor(:,1);
       y=nodecoor(:,2);

       %IF 2D-TRUSS, MODIFY ID MATRIX

       if istrtp==2

        for ipoin=1:npoin
        if ID(1:2,ipoin)==[0;0]
        ID(1:3,ipoin)=[0;0;0]
        elseif ID(1:2,ipoin)==[0;1]
        ID(1:3,ipoin)=[0;1;0]
        elseif ID(1:2,ipoin)==[1;1]
        ID(1:3,ipoin)=[1;1;0]
        end
```

```
 end
end

%if size(ID,1)==2
% ID=[ID;zeros(1,size(ID,2))];
%end

% IF GRID, SET ID=ZEROS

if istrtp==4
ID=ID*0;
end

% SET UP FIGURE

handle=figure;
margin=max(max(x)-min(x),max(y)-min(y))/10;
axis([min(x)-margin, max(x)+margin, min(y)-margin, max(y)+margin])
axis('equal')
hold on

% CALC NUMBER OF NODES, ETC.

number_nodes=length(x);
number_elements=size(lnods,1);
number_fixities=size(ID,2);
axislimits=axis;
circlesize=max(axislimits(2)-axislimits(1),axislimits(4)-axislimits(3))/40;

% DRAW SUPPORTS

for i=1:number_fixities

 % DRAW HORIZ. ROLLER

 if ID(:,i)==[0 1 0]'
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i),cos(0:0.1:pi*2)*circlesize/2-circlesize/2+y(i),'r')

 % DRAW PIN SUPPORT

 elseif ID(:,i)==[1 1 0]'
 plot([x(i),x(i)-circlesize,x(i)+circlesize,x(i)],[y(i),y(i)-circlesize,y(i)-circlesize,y(i)],'r')

 % DRAW HOERIZ. ROLLER SUPPORT

 elseif ID(:,i)==[0 1 1]'
 plot([x(i)+circlesize*2,x(i)-circlesize*2],[y(i),y(i)],'r');
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i),cos(0:0.1:pi*2)*circlesize/2-circlesize/2+y(i),'r')
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)+circlesize,cos(0:0.1:pi*2)*circlesize/2-circlesize/2+y(i),'r')
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)-circlesize,cos(0:0.1:pi*2)*circlesize/2-circlesize/2+y(i),'r')

 % DRAW VERT. ROLLER SUPPORT

 elseif ID(:,i)==[1 0 0]'
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)-circlesize*.5,cos(0:0.1:pi*2)*circlesize/2,'r')

 % DRAW ROLLER SUPPORT WITH NO ROTATION

 elseif ID(:,i)==[1 0 1]'
 plot([x(i),x(i)],[y(i)+circlesize*2,y(i)-circlesize*2],'r');
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)-circlesize*.5,cos(0:0.1:pi*2)*circlesize/2,'r')
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)-circlesize*.5,cos(0:0.1:pi*2)*circlesize/2+circlesize,'r')
 plot(sin(0:0.1:pi*2)*circlesize/2+x(i)-circlesize*.5,cos(0:0.1:pi*2)*circlesize/2-circlesize,'r')
 end

 % DRAW HORIZ. PLATFORM

 if min(ID(:,i)==[0 1 0]') |  min(ID(:,i)==[1 1 0]')  | min(ID(:,i)==[0 1 1]')
plot([x(i)-circlesize*2,x(i)+circlesize*2],[y(i)-circlesize,y(i)-circlesize],'r')
 plot([x(i)-circlesize*1.5,x(i)-circlesize*2],[y(i)-circlesize,y(i)-circlesize*1.5],'r')
 plot([x(i)-circlesize*1,x(i)-circlesize*2],[y(i)-circlesize,y(i)-circlesize*2],'r')
 plot([x(i)-circlesize*.5,x(i)-circlesize*1.5],[y(i)-circlesize,y(i)-circlesize*2],'r')
```

```
    plot([x(i)+circlesize*0,x(i)-circlesize*1],[y(i)-circlesize,y(i)-circlesize*2],'r')
    plot([x(i)+circlesize*.5,x(i)-circlesize*(0.5)],[y(i)-circlesize,y(i)-circlesize*2],'r')
    plot([x(i)+circlesize*1,x(i)+circlesize*0],[y(i)-circlesize,y(i)-circlesize*2],'r')
    plot([x(i)+circlesize*1.5,x(i)+circlesize*.5],[y(i)-circlesize,y(i)-circlesize*2],'r')
    plot([x(i)+circlesize*2,x(i)+circlesize*1],[y(i)-circlesize,y(i)-circlesize*2],'r')
    plot([x(i)+circlesize*2,x(i)+circlesize*1.5],[y(i)-circlesize*1.5,y(i)-circlesize*2],'r')

    % DRAW FIXED SUPPORT

    elseif ID(:,i)==[1 1 1]'
    plot([x(i)-circlesize*2,x(i)+circlesize*2],[y(i)-circlesize,y(i)-circlesize]+circlesize,'r')
    plot([x(i)-circlesize*1.5,x(i)-circlesize*2],[y(i)-circlesize,y(i)-circlesize*1.5]+circlesize,'r')
    plot([x(i)-circlesize*1,x(i)-circlesize*2],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)-circlesize*.5,x(i)-circlesize*1.5],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*0,x(i)-circlesize*1],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*.5,x(i)-circlesize*(0.5)],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*1,x(i)+circlesize*0],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*1.5,x(i)+circlesize*.5],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*2,x(i)+circlesize*1],[y(i)-circlesize,y(i)-circlesize*2]+circlesize,'r')
 plot([x(i)+circlesize*2,x(i)+circlesize*1.5],[y(i)-circlesize*1.5,y(i)-circlesize*2]+circlesize,'r')

    % DRAW VERT. PLATFORM

    elseif  min(ID(:,i)==[1 0 0]') | min(ID(:,i)==[1 0 1]')
    xf=[x(i)-circlesize,x(i)-circlesize*2];
 yf=[y(i),y(i)- circlesize];
 plot(xf,yf,'r')
  plot(xf,yf+circlesize*.5,'r')
 plot(xf,yf+circlesize*1,'r')
 plot(xf,yf+circlesize*1.5,'r')
 plot(xf,yf+circlesize*2,'r')
 plot([x(i)-circlesize*1.5,x(i)-circlesize*2],[y(i)+circlesize*2,y(i)+ circlesize*1.5],'r')
 plot(xf,yf-circlesize*.5,'r')
 plot(xf,yf-circlesize*1,'r')
plot([x(i)-circlesize,x(i)-circlesize*1.5],[y(i)-circlesize*1.5,y(i)- circlesize*2],'r')
 plot([xf(1),xf(1)],[y(i)+circlesize*2,y(i)-circlesize*2],'r');
    end
   end

   % DRAW ELEMENTS

   for i=1:number_elements
    plot([x(lnods(i,1)),x(lnods(i,2))],[y(lnods(i,1)),y(lnods(i,2))],'b');
    if i==1
    end
   end

   % DRAW JOINTS

   for i=1:number_nodes
    if ~max(ID(:,i))
    plot(x(i),y(i),'mo')
    end
   end

   % DRAW ELEMENT NUMBERS

   for i=1:number_elements
    set(handle,'DefaultTextColor','blue')
    text(  (x(lnods(i,1))+x(lnods(i,2)))/2+circlesize,(y(lnods(i,1))+y(lnods(i,2)))/2+circlesize,int2str(i))
   end

   % DRAW JOINT NUMBERS

   for i=1:number_nodes
    set(handle,'DefaultTextColor','magenta')
    text(x(i)+circlesize,y(i)+circlesize,int2str(i))
   end

   if exist('filename')
    title(filename)
   end
```

```
        hold off
        set(handle,'DefaultTextColor','white')

% DRAW 3D STRUCTURE IF DRAWTYPE=3

elseif drawtype==3

        % RETREIVE NODE COORIDINATES

        x=nodecoor(:,1);
        y=nodecoor(:,2);
        z=nodecoor(:,3);

        % SET UP FIGURE

        handle=figure;
        margin=max([max(x)-min(x),max(y)-min(y),max(z)-min(z)])/10;
        axis([min(x)-margin, max(x)+margin, min(y)-margin, max(y)+margin, min(z)-margin, max(z)+margin])
        axis('equal')
        hold on

        % RETREIVE NUMBER OF NODES, ETC.

        number_nodes=length(x);
        number_elements=size(lnods,1);
        axislimits=axis;
        circlesize=max([axislimits(2)-axislimits(1),axislimits(4)-axislimits(3),axislimits(6)-axislimits(5)])/40;

        % DRAW ELEMENTS

        for i=1:number_elements
         plot3([x(lnods(i,1)),x(lnods(i,2))],[y(lnods(i,1)),y(lnods(i,2))],[z(lnods(i,1)),z(lnods(i,2))],'b');
        end

        % DRAW JOINTS

        for i=1:number_nodes
         plot3(x(i),y(i),z(i),'mo')
        end

        % DRAW ELEMENT NUMBERS

        for i=1:number_elements
         set(handle,'DefaultTextColor','blue')
         text(  (x(lnods(i,1))+x(lnods(i,2)))/2+circlesize,(y(lnods(i,1))+y(lnods(i,2)))/2+circlesize,(z(lnods(i,1))+z(lnods(i,2)
        end

        % DRAW JOINT NUMBERS

        for i=1:number_nodes
         set(handle,'DefaultTextColor','magenta')
         text(x(i)+circlesize,y(i)+circlesize,z(i)+circlesize,int2str(i))
        end

        if exist('filename')
         title(filename)
        end

        xlabel('x')
        ylabel('y')
        zlabel('z')

        % DRAW GROUND

        X=x;
        Y=y;
        Z=z;
        X=axislimits(1)-margin:margin:axislimits(2)+margin;
        Y=axislimits(3)-margin:margin:axislimits(4)+margin;
        Z=zeros(length(X),length(Y));
        mesh(X,Y,Z)
```

```
        size(X)
        size(Y)
        size(Z)

        hold off
        set(handle,'DefaultTextColor','white')

        hAZ=uicontrol('style','slider','position',[.7 .95 .3 .05],'units','normalized','min',0,'max',360,...
        'callback','[az,el]=view; az=get(gco,''val''); view(az,el);');

        hEL=uicontrol('style','slider','position',[.7 .89 .3 .05],'units','normalized','min',0,'max',180,...
        'callback','[az,el]=view; el=get(gco,''val''); view(az,el);');

        hdef=uicontrol('style','pushbutton','callback','view(-37.5, 30)','position',[.88 .83 .12 .05],'units','normalized','Strin

        %set(handle,'units','normalized')
        %text(.68,.95,'azimuth')
        %text(.68,.89,'elevation')
end
end

% END OF DRAW.M SCRIPTFILE
```

12.5.2.15 Sample Output File

CASAP will display figure 12.15.

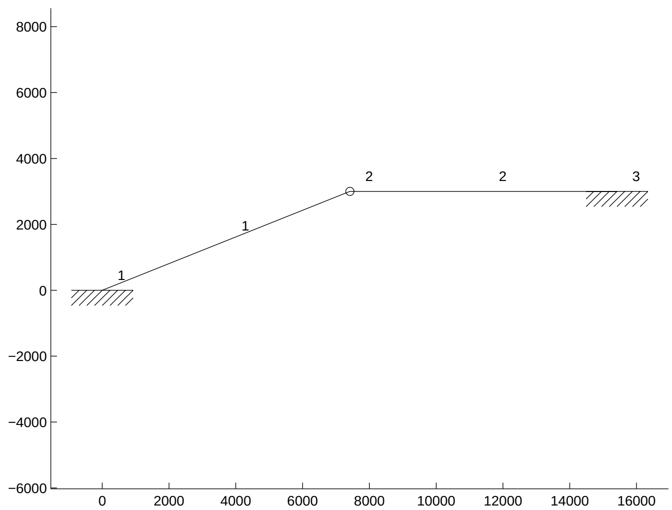

Figure 12.15: Structure Plotted with CASAP

```
Number of Nodes: 3
Number of Elements: 2
Number of Load Cases: 1
Number of Restrained dofs: 6
Number of Free dofs: 3

Node Info:
     Node 1 (0,0)
        Free dofs: none; node is fixed
     Node 2 (7416,3000)
        Free dofs: X Y Rot
     Node 3 (15416,3000)
        Free dofs: none; node is fixed

Element Info:
     Element 1 (1->2)   E=200 A=6000 Iz=200000000
     Element 2 (2->3)   E=200 A=6000 Iz=200000000
```

```
------------------------------------------------------------------------

Load Case: 1

   Nodal Loads:
     Node:  2 Fx =  1.875000e+001
     Node:  2 Fy = -4.635000e+001

   Elemental Loads:
     Element: 1   Point load = 0 at 0 from left
                  Distributed load = 0
     Element: 2   Point load = 0 at 0 from left
                  Distributed load = 4.000000e-003

   Displacements:
     (Node:  2 delta X)  9.949820e-001
     (Node:  2 delta Y) -4.981310e+000
     (Node:  2 rotate ) -5.342485e-004

   Reactions:
     (Node:  1 Fx)  1.304973e+002
     (Node:  1 Fy)  5.567659e+001
     (Node:  1 M )  1.337416e+004
     (Node:  3 Fx) -1.492473e+002
     (Node:  3 Fy)  2.267341e+001
     (Node:  3 M ) -4.535573e+004

   Internal Forces:
      Element:  1
       At Node: 1
          (Global : Fx )  1.304973e+002    (Local : Fx )  1.418530e+002
          (Global : Fy )  5.567659e+001    (Local : Fy )  2.675775e+000
          (Global : M  )  1.337416e+004    (Local : M  )  1.337416e+004
       At Node: 2
          (Global : Fx ) -1.304973e+002    (Local : Fx ) -1.418530e+002
          (Global : Fy ) -5.567659e+001    (Local : Fy ) -2.675775e+000
          (Global : M  )  8.031549e+003    (Local : M  )  8.031549e+003

      Element:  2
       At Node: 2
          (Global : Fx )  1.492473e+002    (Local : Fx )  1.492473e+002
          (Global : Fy )  9.326590e+000    (Local : Fy )  9.326590e+000
          (Global : M  ) -8.031549e+003    (Local : M  ) -8.031549e+003
       At Node: 3
          (Global : Fx ) -1.492473e+002    (Local : Fx ) -1.492473e+002
          (Global : Fy )  2.267341e+001    (Local : Fy )  2.267341e+001
          (Global : M  ) -4.535573e+004    (Local : M  ) -4.535573e+004

------------------------------------------------------------------------
```

Chapter 13

INFLUENCE LINES (unedited)

UNEDITED

An influence line is a diagram whose ordinates are the values of some function of the structure (reaction, shear, moment, etc.) as a unit load moves across the structure.

The shape of the influence line for a function (moment, shear, reaction, etc.) can be obtained by removing the resistance of the structure to that function, at the section where the influence line is desired, and applying an internal force associated with that function at the section so as to produce a unit deformation at the section. The deformed shape that the structure will take represents the shape of the influence line.

Chapter 14

COLUMN STABILITY

14.1 Introduction; Discrete Rigid Bars

14.1.1 Single Bar System

1 Let us begin by considering a rigid bar connected to the support by a spring and axially loaded at the other end, Fig. 14.1. Taking moments about point A:

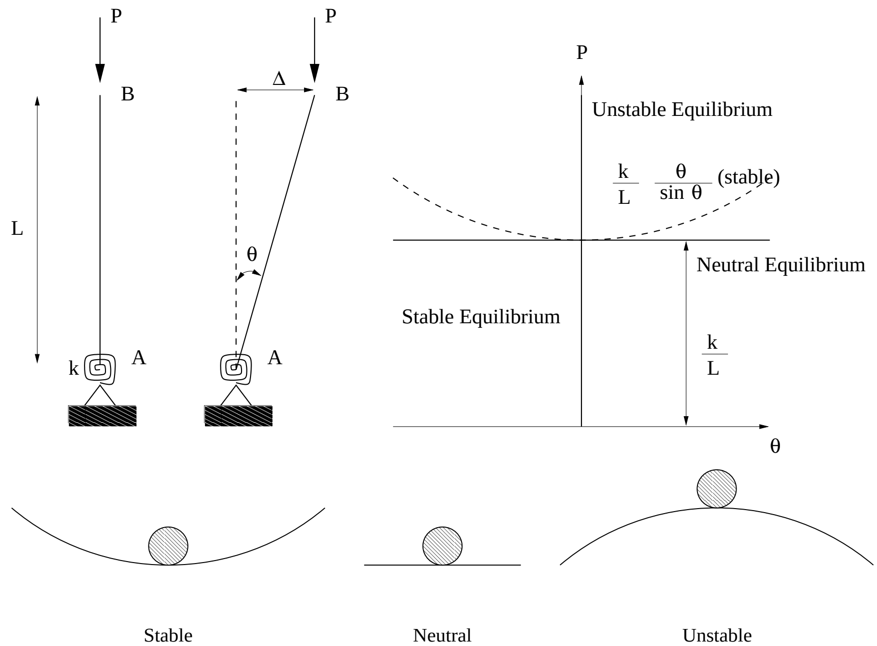

Figure 14.1: Stability of a Rigid Bar

$$\begin{aligned}
\Sigma M_A &= P\Delta - k\theta = 0 && \text{(14.1-a)}\\
\Delta &= L\theta \;\; \text{for small rotation} && \text{(14.1-b)}\\
P\theta L - k\theta &= 0 && \text{(14.1-c)}\\
(P - \frac{k}{L}) &= 0 && \text{(14.1-d)}
\end{aligned}$$

hence we have three possibilities:

Stable Equilibrium: for $P < k/L$, $\theta = 0$

Neutral Equilibrium: for $P = k/L$, and θ can take any value

Unstable equilibrium: for $P > k/L$, $\theta = 0$

2 Thus we introduce a critical load defined by

$$\boxed{P_{cr} = \frac{k}{L}} \tag{14.2}$$

3 For large rotation, we would have

$$\Sigma M_A = P\Delta - k\theta = 0 \tag{14.3-a}$$
$$\Delta = L\sin(\theta)P\theta L - k\theta = 0 \tag{14.3-b}$$
$$P_{cr} = \frac{k}{L}\frac{\theta}{\sin(\theta)} \tag{14.3-c}$$

4 Because there is more than one possible path (θ) when $P = P_{cr}$, we call this point a **biffurcation point**.

5 If we now assume that there is an initial imperfection in the column, i.e. the column is initially "crooked", Fig. 14.2, then

$$\Sigma M_A = PL\theta - k(\theta - \theta_0) = 0 \tag{14.4-a}$$
$$P_{cr} = \frac{k}{L}(1 - \frac{\theta_0}{\theta}) \tag{14.4-b}$$

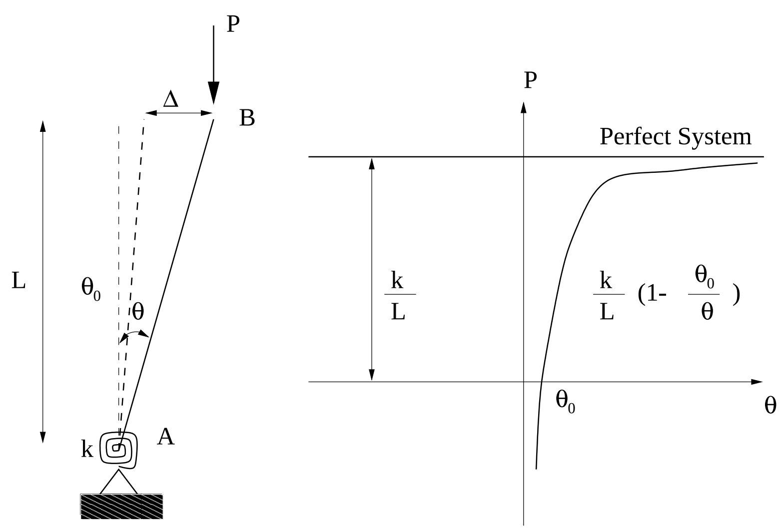

Figure 14.2: Stability of a Rigid Bar with Initial Imperfection

14.1.2 Two Bars System

6 Next we consider the two rigid bar problem illustrated in Fig. 14.3.

$$\Sigma M_B = PL\theta_2 - k(\theta_2 - \theta_1) = 0 \tag{14.5-a}$$
$$\Rightarrow -\theta_1 + \theta_2 = \frac{PL}{k}\theta_2 \tag{14.5-b}$$
$$\Sigma M_A = PL\theta_1 + k(\theta_2 - \theta_1) - k\theta_1 = 0 \tag{14.5-c}$$
$$\Rightarrow 2\theta_1 - \theta_2 = \frac{PL}{k}\theta_1 \tag{14.5-d}$$

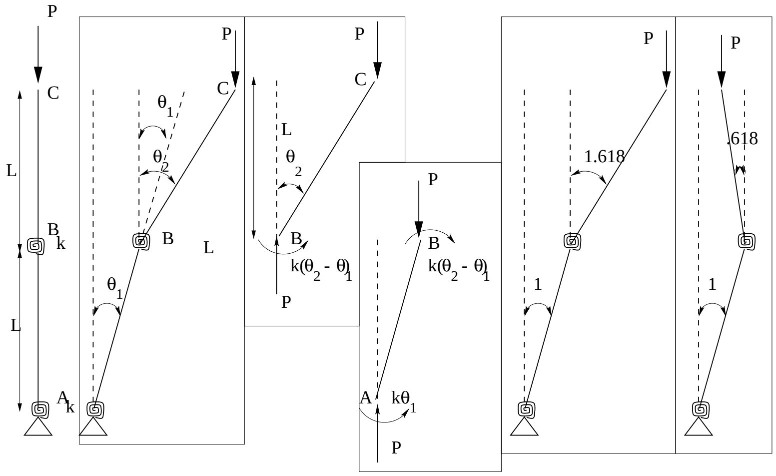

Figure 14.3: Stability of a Two Rigid Bars System

7 Those two equations can be cast in matrix form

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \lambda \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \tag{14.6}$$

where $\lambda = PL/k$, this is an **eigenvalue** formulation and can be rewritten as

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \lambda \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.7}$$

This is a homogeneous system of equation, and it can have a non-zero solution only if its determinant is equal to zero

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \tag{14.8-a}$$

$$2-\lambda-2\lambda+\lambda^2-1 = 0 \tag{14.8-b}$$

$$\lambda^2-3\lambda+1 = 0 \tag{14.8-c}$$

$$\lambda_{1,2} = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2} \tag{14.8-d}$$

8 Hence we now have two critical loads:

$$\boxed{\begin{aligned} P_{cr_1} &= \frac{3-\sqrt{5}}{2}\frac{k}{L} = 0.382\frac{k}{L} && (14.9) \\ P_{cr_2} &= \frac{3+\sqrt{5}}{2}\frac{k}{L} = 2.618\frac{k}{L} && (14.10) \end{aligned}}$$

9 We now seek to determine the deformed shape for each of the first critical loads. It should be noted that since the column will be failing at the critical buckling load, we can not determine the absolute values of the deformations, but rather the shape of the buckling column.

$$\lambda_1 = \frac{3-\sqrt{5}}{2} \tag{14.11-a}$$

$$\begin{bmatrix} 2-\frac{3-\sqrt{5}}{2} & -1 \\ -1 & 1-\frac{3-\sqrt{5}}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.11-b}$$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & 1-\frac{-1+\sqrt{5}}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.11-c}$$

$$\begin{bmatrix} 1.618 & -1 \\ -1 & 0.618 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.11-d}$$

we now arbitrarily set $\theta_1 = 1$, then $\theta_2 = 1/0.618 = 1.618$, thus the first eigenmode is

$$\boxed{\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}} \tag{14.12}$$

10 Note that we can determine the deformed shape upon buckling but not the geometry.

11 Finally, we examine the second mode shape loads

$$\lambda_2 = \frac{3+\sqrt{5}}{2} \tag{14.13-a}$$

$$\begin{bmatrix} 2-\frac{3+\sqrt{5}}{2} & -1 \\ -1 & 1-\frac{3+\sqrt{5}}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.13-b}$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -1 & 1-\frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.13-c}$$

$$\begin{bmatrix} -0.618 & -1 \\ -1 & -1.618 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14.13-d}$$

we now arbitrarily set $\theta_1 = 1$, then $\theta_2 = -1/1.618 = -0.618$, thus the second eigenmode is

$$\boxed{\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}} \tag{14.14}$$

14.1.3 ‡Analogy with Free Vibration

12 The problem just considered bears great ressemblence with the vibration of a two degree of freedom mass spring system, Fig. 14.4.

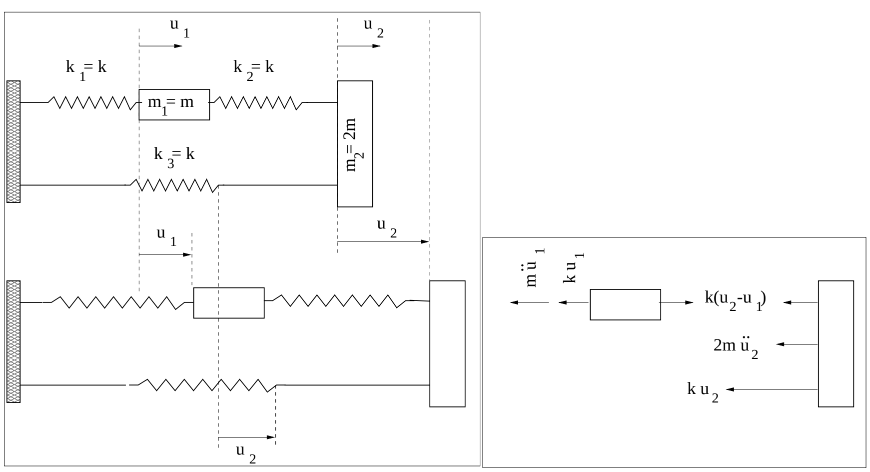

Figure 14.4: Two DOF Dynamic System

13 Each mass is subjected to an inertial force equals to the mass times the acceleration, and the spring force:

$$2m\ddot{\mathbf{u}}_2 + k\mathbf{u}_2 + k(\mathbf{u}_2 - \mathbf{u}_1) = 0 \tag{14.15-a}$$

$$m\ddot{\mathbf{u}}_1 + k\mathbf{u}_1 + k9\mathbf{u}_2 - \mathbf{u}_10 \quad = \quad 0 \tag{14.15-b}$$

or in matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{Bmatrix} \ddot{\mathbf{u}}_1 \\ \ddot{\mathbf{u}}_2 \end{Bmatrix}}_{\ddot{\mathbf{U}}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix}}_{\mathbf{U}} \tag{14.16}$$

14 The characteristic equation is $|\mathbf{K} - \lambda\mathbf{M}|$ where $\lambda = \omega^2$, and ω is the natural frequency.

$$\begin{vmatrix} 2h - \lambda & -h \\ -h & 2h - 2\lambda \end{vmatrix} = 0 \tag{14.17}$$

where $h = k/m$. This reduces to the polynomial

$$\lambda^2 - 3h\lambda + \frac{3}{2}h^2 = 0 \tag{14.18}$$

Solving, $\lambda = (3 \pm -\sqrt{3})h/2$ or

$$\omega_1 \quad = \quad 0.796\sqrt{k/m} \tag{14.19-a}$$
$$\omega_1 \quad = \quad 1.538\sqrt{k/m} \tag{14.19-b}$$

15 To find the mode shapes ϕ_1 and ϕ_2 (relative magnitudes of the DOF) we substitute in the characteristic equation and set the first element equal to 1:

$$\phi_1 = \begin{Bmatrix} 1.000 \\ 1.3660 \end{Bmatrix} \quad \text{and} \quad \phi_1 = \begin{Bmatrix} 1.000 \\ -0.3660 \end{Bmatrix} \tag{14.20}$$

14.2 Continuous Linear Elastic Systems

16 Column buckling theory originated with Leonhard Euler in 1744.

17 An initially straight member is concentrically loaded, and all fibers remain elastic until buckling occur.

18 For buckling to occur, it must be assumed that the column is slightly bent as shown in Fig. 14.5. Note, in reality no column is either perfectly straight, and in all cases a minor imperfection is present.

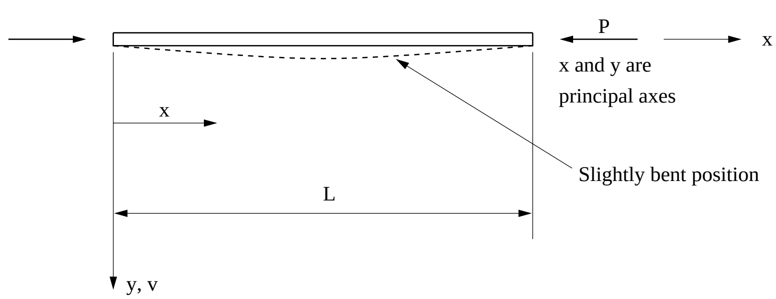

Figure 14.5: Euler Column

19 Two sets of solutions will be presented, in both cases the **equation of equilibrium is written for the deformed element**.

14.2.1 Lower Order Differential Equation

20 At any location x along the column, the imperfection in the column compounded by the concentric load P, gives rise to a moment

$$M_z = Pv \tag{14.21}$$

Note that the value of y is irrelevant.

21 Recalling that

$$\frac{d^2v}{dx^2} = \frac{M_z}{EI} \tag{14.22}$$

upon substitution, we obtain the following differential equation

$$\frac{d^2v}{dx^2} - \frac{P}{EI}v = 0 \tag{14.23}$$

22 Letting $k^2 = \frac{P}{EI}$, the solution to this second-order linear differential equation is

$$v = -A\sin kx - B\cos kx \tag{14.24}$$

23 The two constants are determined by applying the boundary conditions

1. $v = 0$ at $x = 0$, thus $B = 0$
2. $v = 0$ at $x = L$, thus

$$A\sin kL = 0 \tag{14.25}$$

24 This last equation can be satisfied if: 1) $A = 0$, that is there is no deflection; 2) $kL = 0$, that is no applied load; or 3)

$$kL = n\pi \tag{14.26}$$

Thus buckling will occur if $\frac{P_{cr}}{EI} = \left(\frac{n\pi}{L}\right)^2$ or

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}$$

25 The fundamental buckling mode, i.e. a single curvature deflection, will occur for $n = 1$; Thus Euler critical load for a pinned column is

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{14.27}$$

26 The corresponding critical stress is

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r_{min}}\right)^2}} \tag{14.28}$$

where $I = Ar^2_{min}$.

27 Note that buckling will take place with respect to the weakest of the two axis.

14.2.2 Higher Order Differential Equation

14.2.2.1 Derivation

28 In the preceding approach, the buckling loads were obtained for a column with specified boundary conditons. A second order differential equation, valid specifically for the member being analyzed was used.

29 In the next approach, we derive a single fourth order equation which will be applicable to any column regardelss of the boundary conditions.

30 Considering a beam-column subjected to axial and shear forces as well as a moment, Fig. 14.6 (note analogy with cable structure, Fig. 4.1), taking the moment about i for the beam segment and assuming

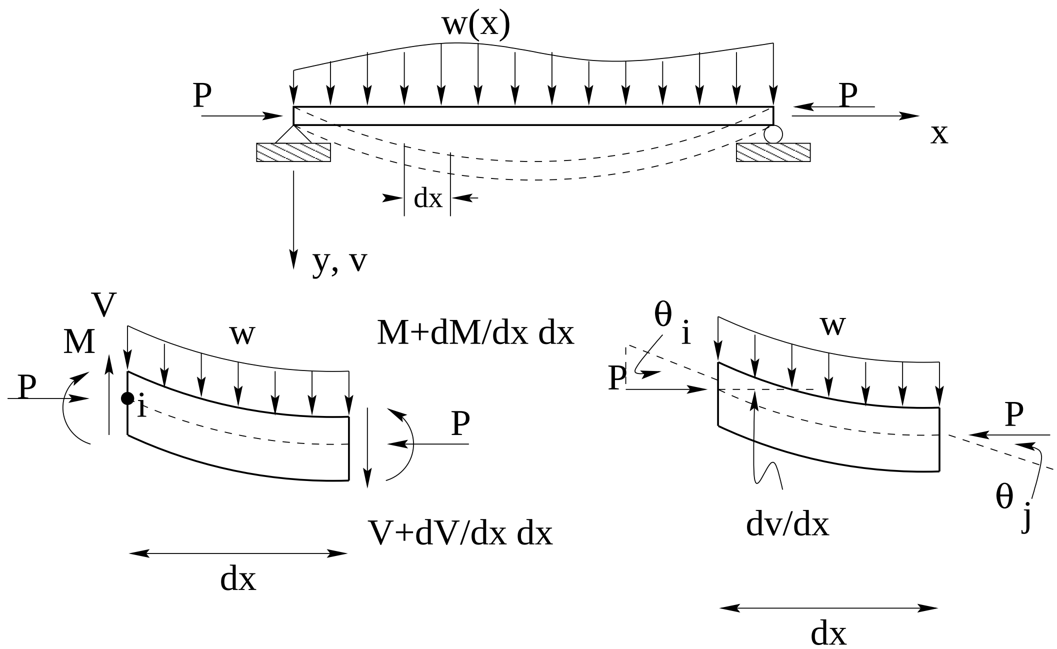

Figure 14.6: Simply Supported Beam Column; Differential Segment; Effect of Axial Force P

the angle $\frac{dv}{dx}$ between the axis of the beam and the horizontal axis is small, leads to

$$\underbrace{M - \left(M + \frac{dM}{dx}dx\right) + w\frac{(dx)^2}{2} + \left(V + \frac{dV}{dx}dx\right)} - \underbrace{P\left(\frac{dv}{dx}\right)dx} = 0 \tag{14.29}$$

Note that the first underbraced term is identical to the one used in earlier derivation of the beam's differential equation.

31 neglecting the terms in dx^2 which are small, and then differentiating each term with respect to x, we obtain

$$\frac{d^2M}{dx^2} - \frac{dV}{dx} - P\frac{d^2v}{dx^2} = 0 \tag{14.30}$$

32 However, considering equilibrium in the y direction gives

$$\frac{dV}{dx} = -w \tag{14.31}$$

33 From beam theory, neglecting axial and shear deformations, we have

$$M = -EI\frac{d^2v}{dx^2} \tag{14.32}$$

34 Substituting Eq. 14.31 and 14.32 into 14.30, and assuming a beam of uniform cross section, we obtain

$$EI\frac{d^4v}{dx^4} - P\frac{d^2v}{dx^2} = w \tag{14.33}$$

substituting $\lambda = \sqrt{\frac{P}{EI}}$, and $w = 0$ we finally obtain

$$\boxed{\frac{d^4v}{dx^4} + \lambda^2 \frac{d^2v}{dx^2} = 0} \tag{14.34}$$

35 Again, we note that by considering equilibrium in the deformed state we have introduced the second term to what would otherwise be the governing differential equation for flexural members (beams). Finally, we note the analogy between this equation, and the governing differential equation for a cable structure, Eq. 4.12.

36 The general solution of this fourth order differential equation to any set of boundary conditions is

$$v = C_1 + C_2x + C_3 \sin \lambda x + C_4 \cos \lambda x \tag{14.35}$$

37 The constants $C's$ are obtained from the boundary condtions. For columns, those are shown in Table **??** The essential boundary conditions are associated with displacement, and slope, the natural ones with shear and moments (through their respective relationships with the displacement).

Essential (Dirichlet)	Natural (Neumann)
v	$\frac{d^3v}{dx^3}$ (V)
$\frac{dv}{dx}$	$\frac{d^2v}{dx^2}$ (M)

Table 14.1: Essential and Natural Boundary Conditions for Columns

38 We note that at each node, we should have two boundary conditions, all combinations are possible except pairs from the same raw (i.e. we can not have known displacement and shear, or known slope and moment).

14.2.2.2 Hinged-Hinged Column

39 If we consider again the stability of a hinged-hinged column, the boundary conditions are displacement (v) and moment ($\frac{d^2v}{dx^2}EI$) equal to zero at both ends[1], or

$$\begin{array}{rclrclrcl} v &=& 0, & \frac{d^2v}{dx^2} &=& 0 & \text{at} \quad x &=& 0 \\ v &=& 0, & \frac{d^2v}{dx^2} &=& 0 & \text{at} \quad x &=& L \end{array} \tag{14.36}$$

substitution of the two conditions at $x = 0$ leads to $C_1 = C_4 = 0$. From the remaining conditions, we obtain

$$C_3 \sin \lambda L + C_2 L = 0 \tag{14.37-a}$$

$$-C_3k^2 \sin \lambda L = 0 \tag{14.37-b}$$

these relations are satisfied either if $C_2 = C_3 = 0$ or if $\sin \lambda L = C_2 = 0$. The first alternative leads to the trivial solution of equilibrium at all loads, and the second to $\lambda L = n\pi$ for $n = 1, 2, 3 \cdots$. The critical load is

$$\boxed{P_{cr} = \frac{n^2\pi^2 EI}{L^2}} \tag{14.38}$$

which was derived earlier using the lower order differential equation.

[1] It will be shown in subsequent courses, that the former BC is an **essential B.C.**, and the later a **natural B.C.**

40 The shape of the buckled column is

$$y = C_3 \sin \frac{n\pi x}{L} \tag{14.39}$$

and only the shape (but not the geometry) can be determined. In general, we will assume $C_3 = 1$ and plot y.

14.2.2.3 Fixed-Fixed Column

41 We now consider a column which is restrained against rotation at both ends, the boundary conditions are given by:

$$\begin{aligned}
v(0) &= 0 = C_1 + C_4 && \text{(14.40-a)}\\
v'(0) &= 0 = C_2 + C_3\lambda && \text{(14.40-b)}\\
v(L) &= 0 = C_1 + C_2 L + C_3 \sin\lambda L + C_4 \cos\lambda L && \text{(14.40-c)}\\
v'(L) &= 0 = C_2 + C_3\lambda\cos(\lambda L) - C_4 \sin(\lambda L) && \text{(14.40-d)}
\end{aligned}$$

those equations can be set in matrix form

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & \lambda & 0 \\ 1 & L & \sin\lambda L & \cos\lambda L \\ 0 & 1 & \lambda\cos\lambda L & -\lambda\sin\lambda L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} &= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} && \text{(14.41-a)}\\
\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & \lambda & 0 \\ 1 & L & \sin\lambda L & \cos\lambda L \\ 0 & 1 & \lambda\cos\lambda L & -\lambda\sin\lambda L \end{vmatrix} &= 0 && \text{(14.41-b)}
\end{aligned}$$

The determinant is obtain from

$$\begin{aligned}
\begin{vmatrix} 1 & \lambda & 0 \\ L & \sin\lambda L & \cos\lambda L \\ 1 & \lambda\cos\lambda L & -\lambda\sin\lambda L \end{vmatrix} - \begin{vmatrix} 1 & 1 & \lambda \\ 1 & L & \sin\lambda L \\ 0 & 1 & \lambda\cos\lambda L \end{vmatrix} &= 0 && \text{(14.42-a)}\\
\begin{vmatrix} \sin\lambda L & \cos\lambda L \\ \lambda\cos\lambda L & -\lambda\sin\lambda L \end{vmatrix} - \lambda \begin{vmatrix} L & \cos\lambda L \\ 1 & -\lambda\sin\lambda L \end{vmatrix} + \begin{vmatrix} 1 & \sin\lambda L \\ 0 & \lambda\cos\lambda L \end{vmatrix} - \lambda \begin{vmatrix} 1 & L \\ 0 & 1 \end{vmatrix} &= 0 && \text{(14.42-b)}\\
-2\lambda + \lambda^2 L \sin\lambda L + 2\lambda\cos\lambda L &= 0 && \text{(14.42-c)}
\end{aligned}$$

The first solution, $\lambda = 0$ is a trivial one, and the next one

$$\begin{aligned}
\lambda L \sin\lambda L + 2\cos\lambda L - 2 &= 0 && \text{(14.43-a)}\\
\lambda L \sin\lambda L &= 2(1 - \cos\lambda L) && \text{(14.43-b)}
\end{aligned}$$

The solution to this transcendental equation is

$$\begin{aligned}
\lambda L &= 2n\pi && \text{(14.44-a)}\\
\sqrt{\frac{P}{EI}} &= \frac{2n\pi}{L} && \text{(14.44-b)}
\end{aligned}$$

thus the critical load and stresses are given by

$$\boxed{\begin{aligned}
P_{cr} &= \frac{4n^2\pi^2 EI}{L^2} && \text{(14.45)}\\
\sigma_{cr} &= \frac{4n^2\pi^2 E}{(L/r)^2} && \text{(14.46)}
\end{aligned}}$$

42 The deflected shape (or eigenmodes) can be obtained by substituting the value of λ into the $c's$.

14.2.2.4 Fixed-Hinged Column

43 Next we consider a column with one end fixed (at $x = L$), and one end hinged (at $x = 0$). The boundary conditions are

$$\begin{array}{lllllll} v & = & 0, & \frac{d^2v}{dx^2} & = & 0 & \text{at } x = 0 \\ v & = & 0, & \frac{dv}{dx} & = & 0 & \text{at } x = L \end{array} \tag{14.47}$$

The first two B.C. yield $C_1 = C_4 = 0$, and the other two

$$\sin \lambda L - \lambda L \cos \lambda L = 0 \tag{14.48}$$

But since $\cos \lambda L$ can not possibly be equal to zero, the preceding equation can be reduced to

$$\tan \lambda L = \lambda L \tag{14.49}$$

which is a transcendental algebraic equation and can only be solved numerically. We are essentially looking at the intersection of $y = x$ and $y = \tan x$, Fig. 14.7 and the smallest positive root is $\lambda L = 4.4934$,

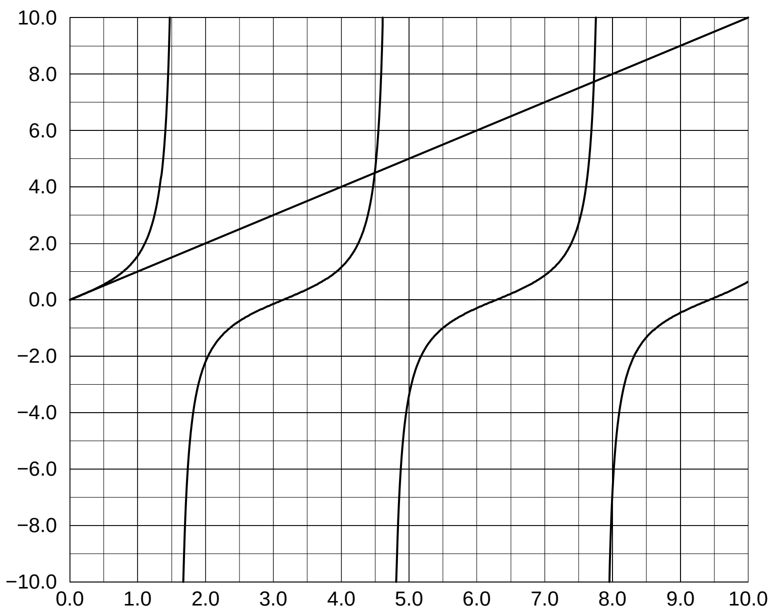

Figure 14.7: Solution of the Tanscendental Equation for the Buckling Load of a Fixed-Hinged Column

since $k^2 = \frac{P}{EI}$, the smallest critical load is

$$\boxed{P_{cr} = \frac{(4.4934)^2}{L^2} EI = \frac{\pi^2}{(0.699L)^2} EI} \tag{14.50}$$

Note that if we were to solve for x such that $v_{,xx} = 0$ (i.e. an inflection point), then $x = 0.699L$.

14.2.3 Effective Length Factors K

44 Recall that the Euler buckling load was derived for a pinned column. In many cases, a column will have different boundary conditions. It can be shown that in all cases, the buckling load would be given by

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(KL)^2}} \tag{14.51}$$

where K is called **effective length factor**, and KL is the **effective length**. and

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{KL}{r_{min}}\right)^2}} \tag{14.52}$$

The ratio $\frac{KL}{r_{min}}$ is termed the **slenderness ratio**. Note that r_{min} should be the smallest radius of gyration in the unbraced direction(s).

45 The effective length, can only be determined by numerical or approximate methods, and is the distance between two adjacent (real or virtual) inflection points, Fig. 14.8, 14.9

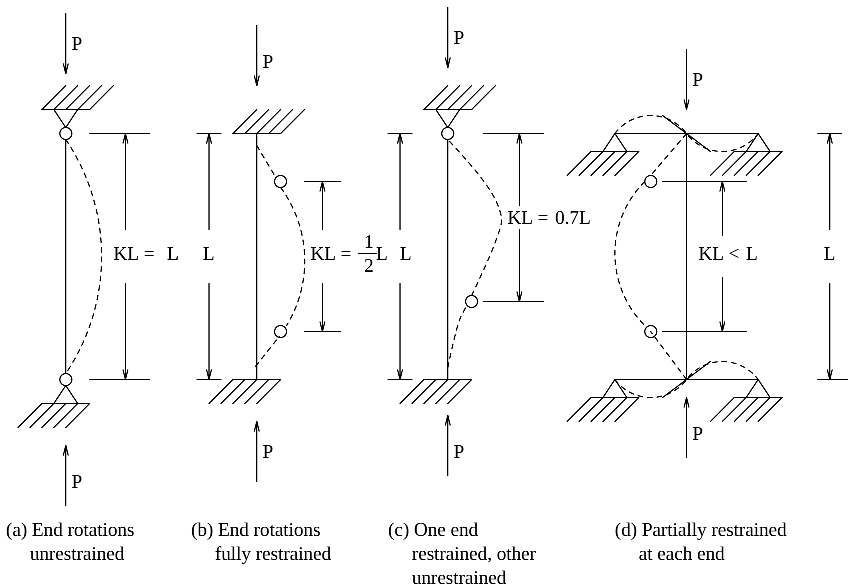

Figure 14.8: Column Effective Lengths

46 The most widely used charts for the effective length determination are those produced by the Structural Stability Research Council. The alignment chart, for an individual column, Fig. 14.10 is shown in Fig. 14.11. It should be noted that this chart assumes that all members are still in the **elastic** range.

47 The use of the alignment chart involces computing G at each end of the column using the following formula

$$\boxed{G_a = \frac{\sum \frac{I_c}{L_c}}{\sum \frac{I_g}{L_g}}} \tag{14.53}$$

where G_a is the stiffness at end a of the column, I_c, I_g are the moment of inertias of the columns and girders respectively. The summation must include only those members which are **rigidly** connected to that joint and lying in the plane for which buckling is being considered.

48 Hence, once G_a and G_b are determined, those values are connected by a straight line in the appropriate chart, and k is the point of intersection of that line with the midle axis.

49 Alternatively :-)

$$\boxed{\frac{G_A G_B}{4}\left(\frac{\pi}{K}\right)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan \pi/K}\right) + \frac{2\tan \pi/2K}{\pi/K} = 1} \tag{14.54}$$

(Ref. McGuire P. 467).

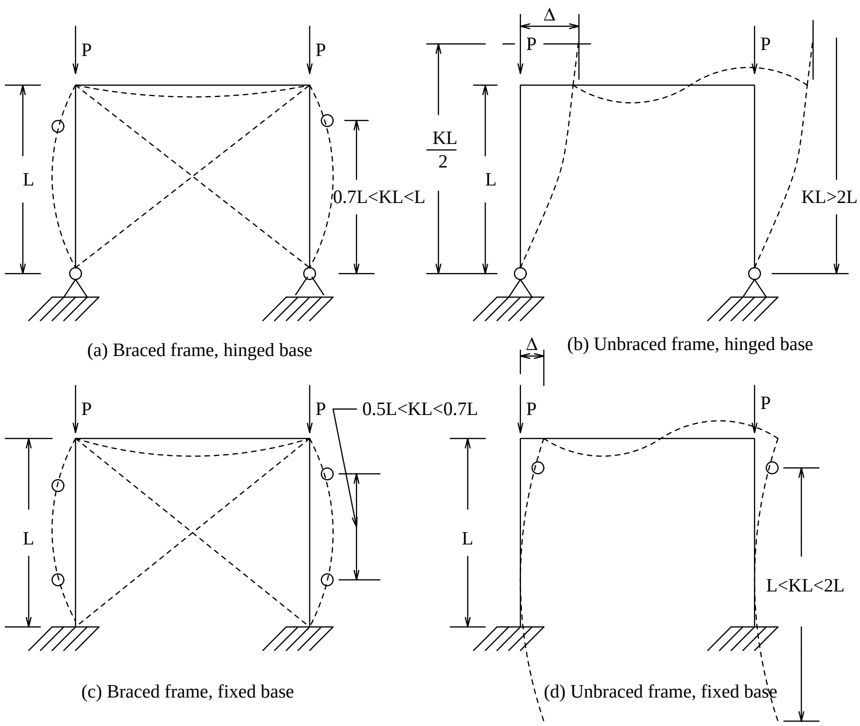

Figure 14.9: Frame Effective Lengths

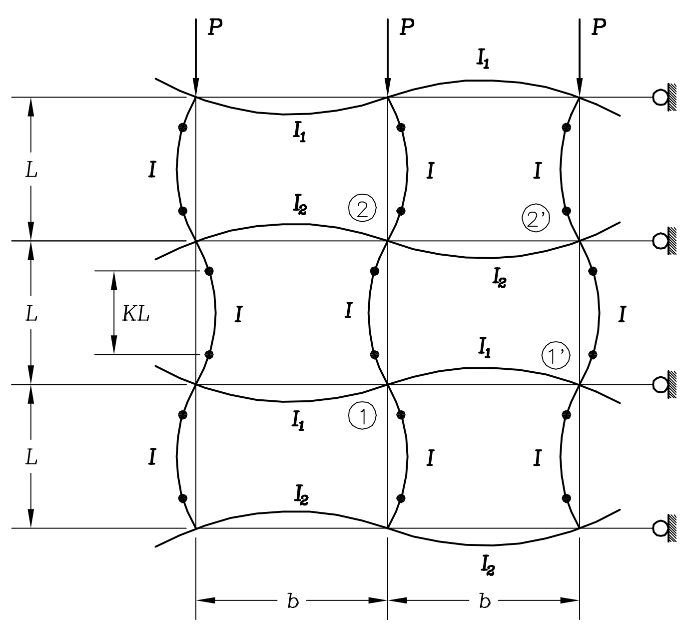

Figure 14.10: Column Effective Length in a Frame

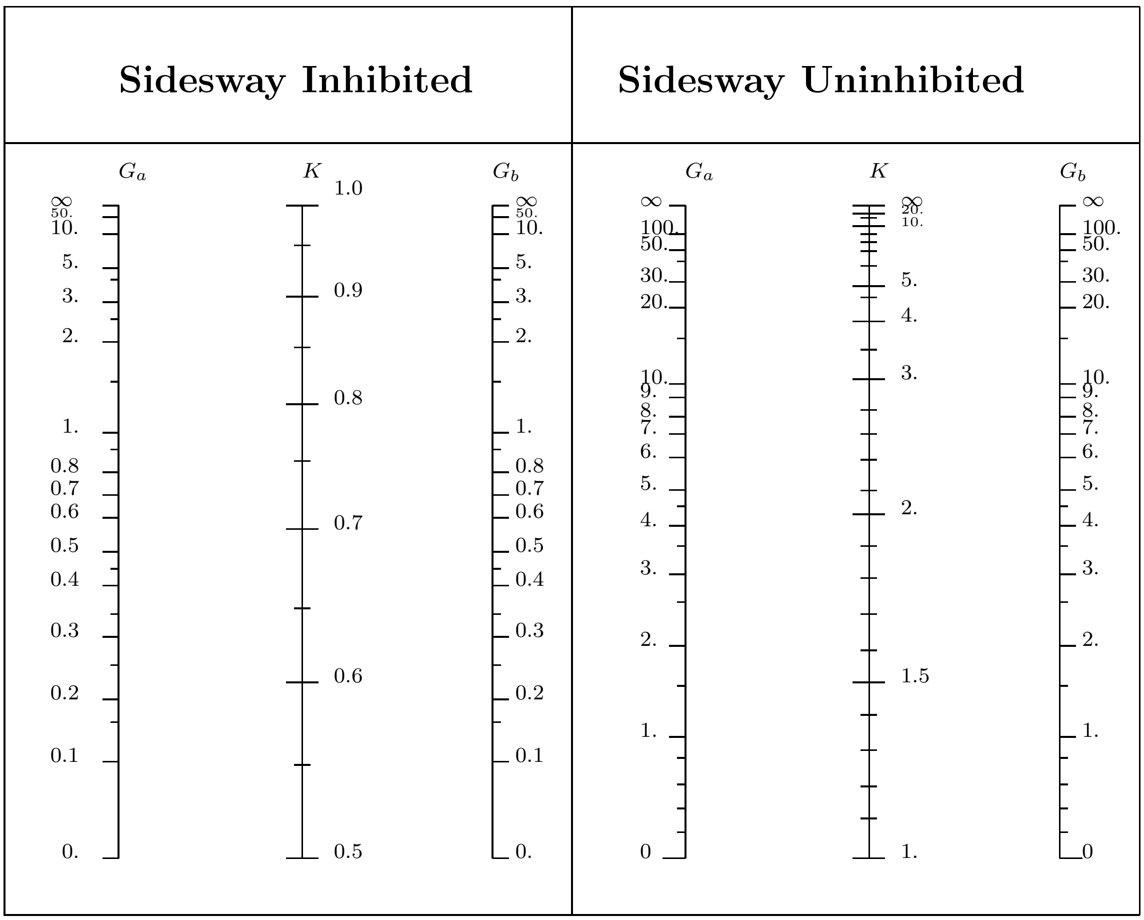

Figure 14.11: Standard Alignment Chart (AISC)

14.3 Inelastic Columns

50 There are two limiting loads for a column

1. Yielding of the gross section $P_{cr} = F_y A_g$, which occirs in short stiff columns
2. Elastic (Euler) buckling, Eq. 14.51 $P_{cr} = \frac{\pi^2 EI}{(KL)^2}$, in long slender columns.

51 Those two expression are asymptotic values for actual column buckling. Intermediary failure loads are caused by the presence of residual stresses which in turn give rise to **inelastic buckling**.

52 Inelastic buckling buckling occurs when the stresses (average) have not yet reached the yield stress, and is based on the tangent modulus E_t which is lower than the initial modulus E.

53 Residual stresses (caused by uneven cooling) will initiate inelastic buckling, Fig. 14.12.

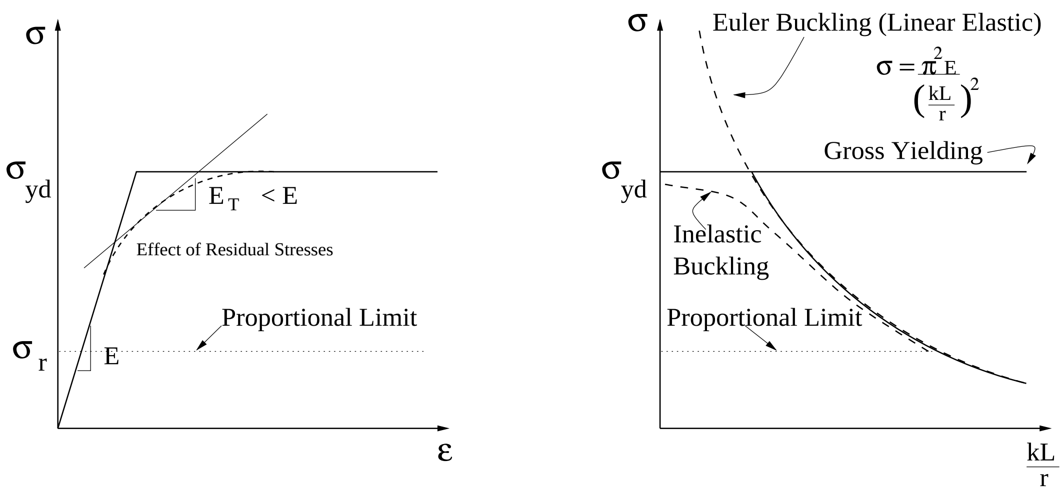

Figure 14.12: Inelastic Buckling

54 The inelastic buckling curve has to be asymptotic to both the Euler elastic buckling (for slender columns), and to the yield stress (for stiff columns).

55 The Structural Stability Research Council (SSRC) has proposed a parabolic curve which provides a transition between elastic buckling and yielding, thus accounting for the presence of residual stresses and the resulting inelastic buckling, Fig. 14.13.

$$\boxed{\sigma_{cr} = \sigma_y \left[1 - \frac{\sigma_y}{4\pi^2 E}\left(\frac{KL}{r_{min}}\right)^2\right]} \tag{14.55}$$

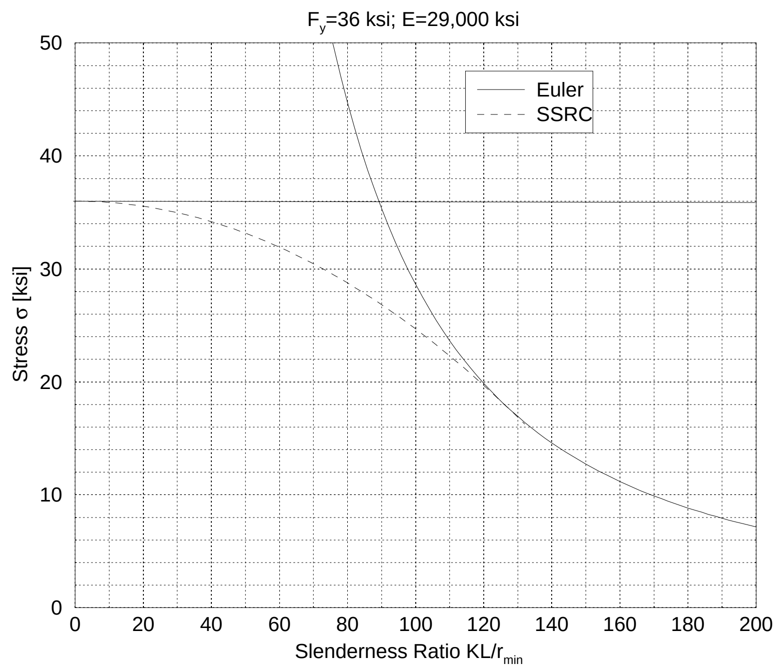

Figure 14.13: Euler Buckling, and SSRC Column Curve

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