Introduction to FEM

18

Shape Function Magic

'Magic' Means *Direct*

Do in 15 minutes what took smart people several months (and less gifted, several years)

But ... it looks like magic to the uninitiated

Shape Function Requirements

(A) Interpolation

(B) Local Support

(C) Continuity (Intra- & Inter-Element)

(D) Completeness

See Sec 18.1 for more detailed statement of (A) through (D). Implications of the last two requirements as regards *convergence* are discussed in Chapter 19.

Direct Construction of Shape Functions: Are Conditions Automatically Satisfied?

(A) ***Interpolation*** **Yes: by construction except scale factor**

(B) ***Local Support*** **Often yes, but not always possible**

(C) ***Continuity*** **No:** *a posteriori* **check necessary**

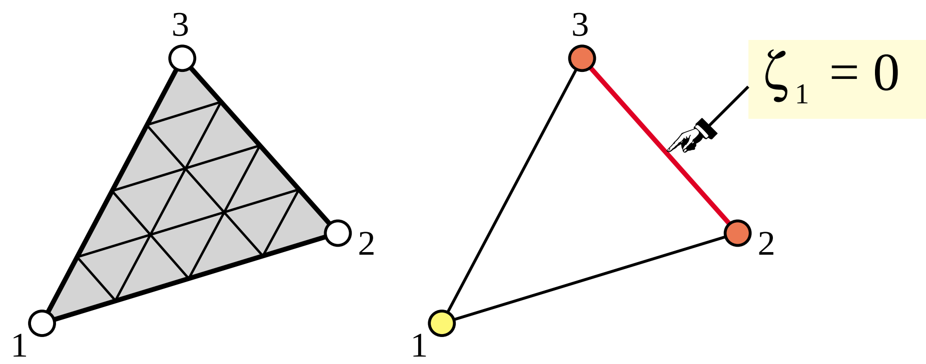
(D) ***Completeness*** **Satisfied if (B,C) are met and the sum of shape functions is identically one.** Sec 16.6 of Notes (advanced material) provides details for curious readers

Direct Construction of Shape Functions as "Line Products"

$$N_i^{(e)} \overset{\text{guess}}{=} c_i \, L_1 \, L_2 \, \ldots \, L_m$$

where $L_k = 0$ are equations of "lines" expressed in natural coordinates, that cross *all nodes* except i

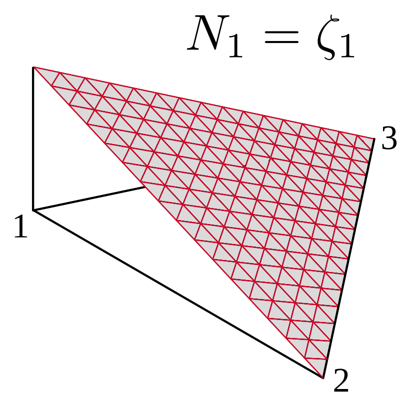
The Three Node Linear Triangle



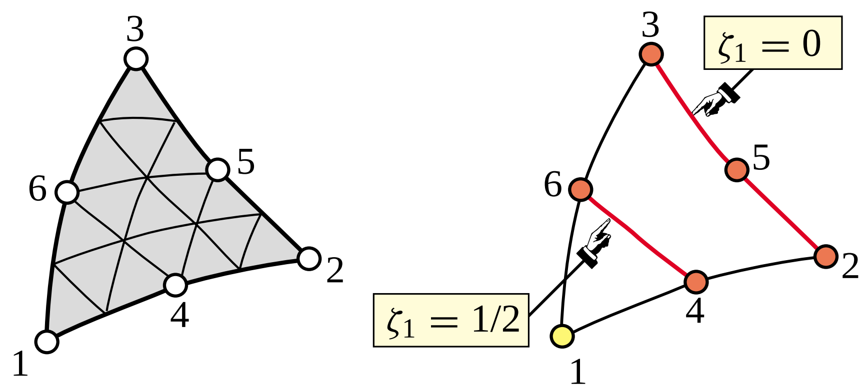
$$N_1 \overset{\text{guess}}{=} c_1 L_1 = c_1 L_{2\text{-}3}$$

At node 1, $N_1 = 1$ whence $c_1 = 1$
and $N_1 = \zeta_1$ Likewise for N_2 and N_3

Three Node Triangle Shape Function Plot



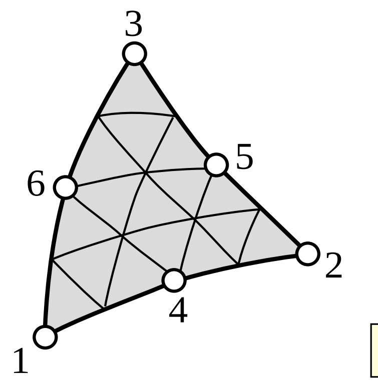
The Six Node Triangle - Corner Node

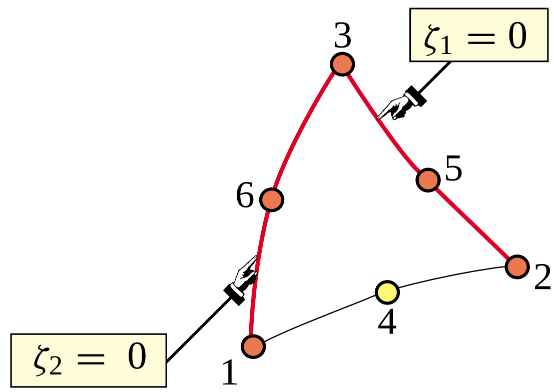


$$N_1^{(e)} \stackrel{\text{guess}}{=} c_1\, L_{2\text{-}3}\, L_{4\text{-}6}$$

For rest of derivation, see Notes

The Six Node Triangle - Midside Node

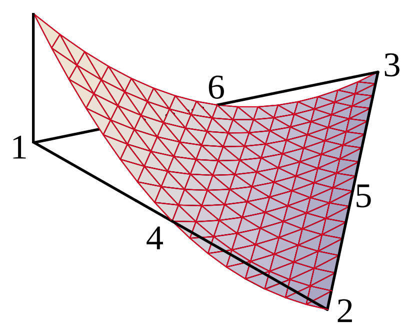


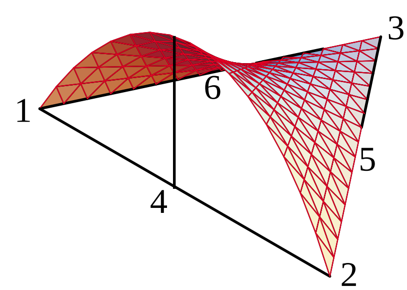


$$N_1^{(e)} \overset{\text{guess}}{=} c_1 \, L_{2\text{-}3} \, L_{4\text{-}6}$$

For rest of derivation, see Notes

The Six Node Triangle: Shape Function Plots





$$N_1^{(e)} = \zeta_1(2\zeta_1 - 1) \qquad N_4^{(e)} = 4\zeta_1\zeta_2$$

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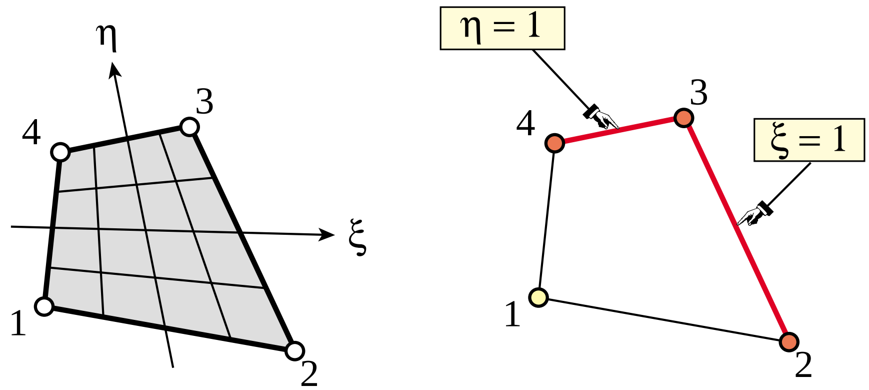
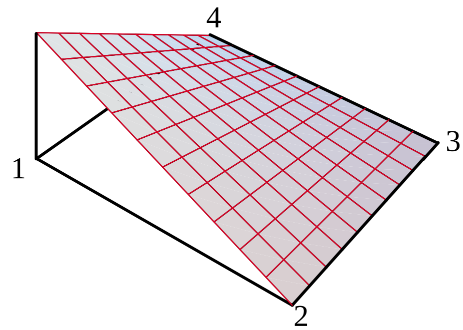
The Four Node Bilinear Quad



$$N_1^{(e)} \overset{\text{guess}}{=} c_1\, L_{2\text{-}3}\, L_{3\text{-}4}$$

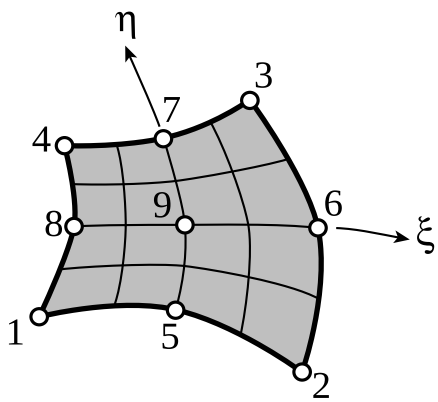
For rest of derivation, see Notes

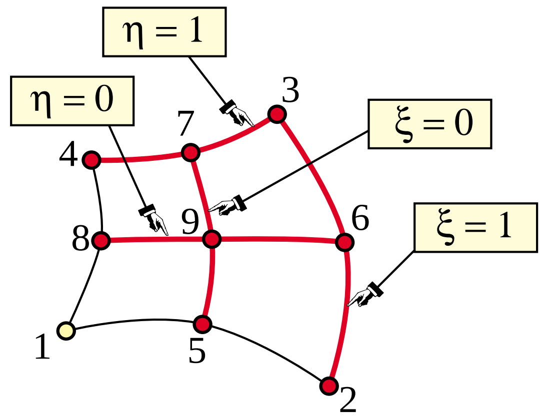
The Four Node Bilinear Quad: Shape Function Plot



$$N_1^{(e)} = \frac{1}{4}(1-\xi)(1-\eta)$$

The Nine Node Biquadratic Quad
Corner Node Shape Function

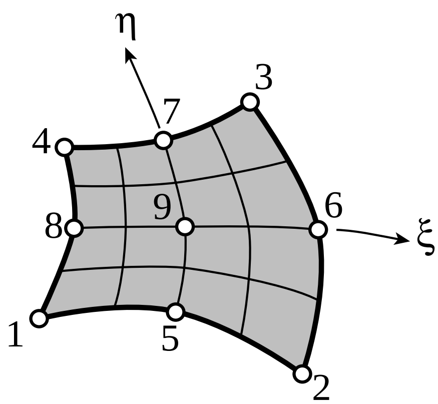


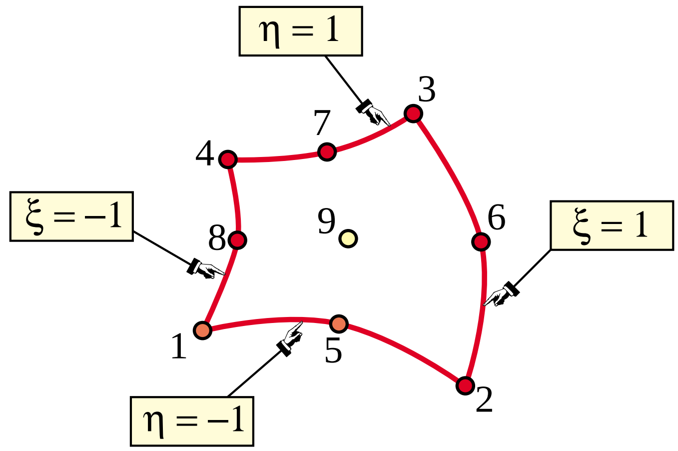


$$N_1^{(e)} \overset{\text{guess}}{=} c_1 L_{2\text{-}3} L_{3\text{-}4} L_{5\text{-}7} L_{6\text{-}8} = c_1(\xi - 1)(\eta - 1)\xi\eta$$

For rest of derivation, see Notes

The Nine Node Biquadratic Quad Internal Node Shape Function





$$N_9^{(e)} = c_9\, L_{1\text{-}2} L_{2\text{-}3} L_{3\text{-}4} L_{4\text{-}1} = c_9\, (\xi - 1)(\eta - 1)(\xi + 1)(\eta + 1)$$

For rest of derivation, see Notes

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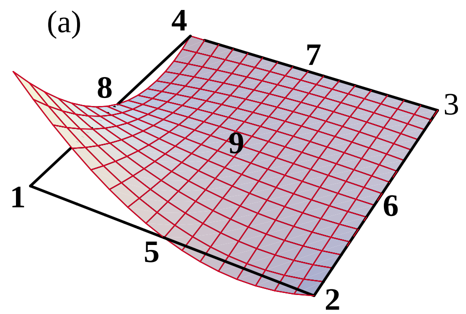
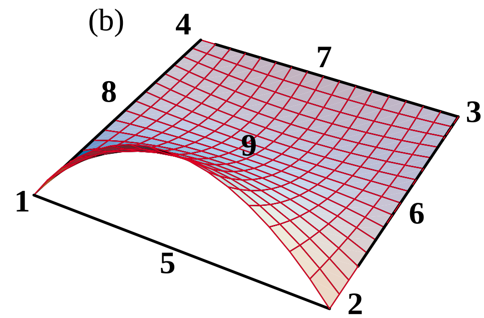
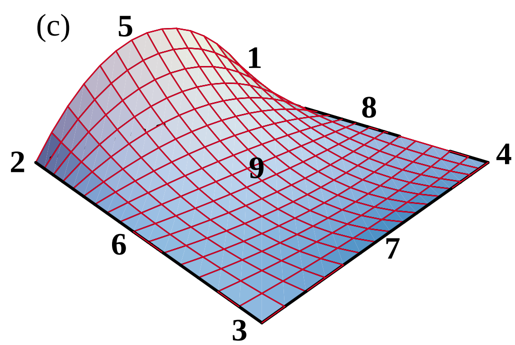
The Nine-Node Biquadratic Quad: Shape Function Plots



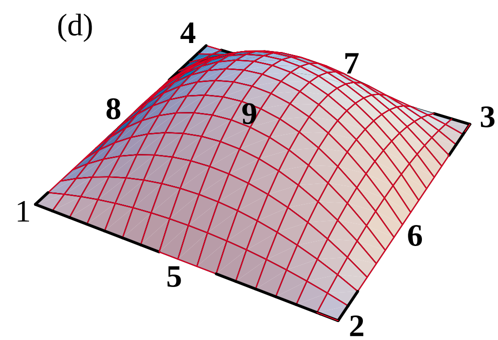
$N_1^{(e)} = \frac{1}{4}(\xi - 1)(\eta - 1)\xi\eta$



$N_5^{(e)} = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1)$

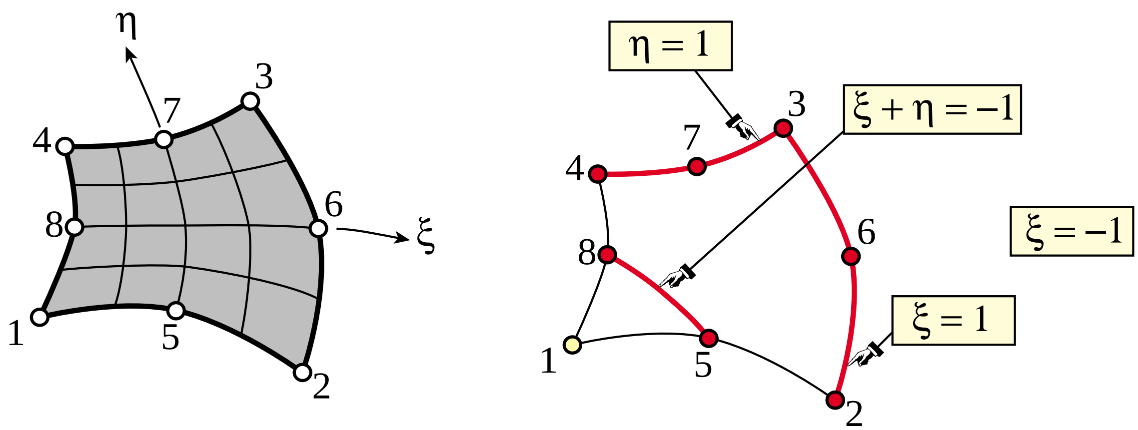


$N_5^{(e)} = \frac{1}{2}(1 - \xi^2)\eta(\eta - 1)$ (back view)



$N_9^{(e)} = (1 - \xi^2)(1 - \eta^2)$

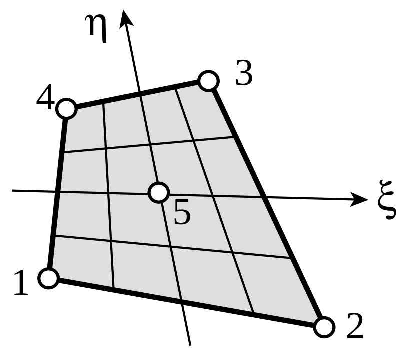
The Eight-Node "Serendipity" Quad Corner Node Shape Function



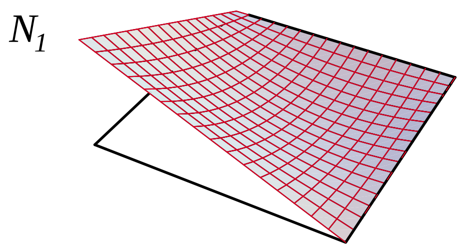
$$N_1^{(e)} = c_1 L_{2\text{-}3} L_{3\text{-}4} L_{5\text{-}8} = c_1(\xi - 1)(\eta - 1)(1 + \xi + \eta)$$

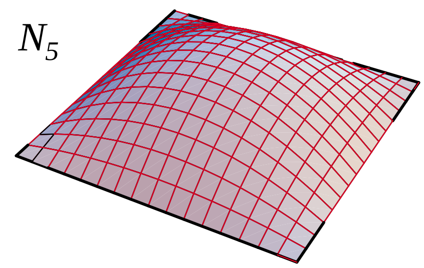
For rest of derivation, see Notes

Can the Magic Wand Fail? Yes



(Exercise 18.6)





Also method needs additional steps for transition elements.
These tougher cases are discussed in Section 18.6

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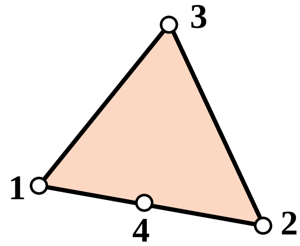
Transition Element Example



For N_1 try the magic wand: product of side 2-3 ($\zeta_1 = 0$) and median 3-4 ($\zeta_1 = \zeta_2$):

$$N_1^{(e)} \stackrel{\text{guess}}{=} c_1\zeta_1(\zeta_1 - \zeta_2), \quad N_1(1,0,0) = 1 = c_1$$ ***fails (C)***

Next, try the shape function of the linear 3-node triangle plus a correction: $N_1^{(e)} \stackrel{\text{guess}}{=} \zeta_1 + c_1\zeta_1\zeta_2$

Coefficient c_1 is determined by requiring this shape function vanish at midside node 4: $N_1^{(e)}(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{2} + c_1\frac{1}{4} = 0$, whence $c_1 = -2$ and

$$\boxed{N_1^{(e)} = \zeta_1 - 2\zeta_1\zeta_2}$$ ***works***