

Introduction to FEM

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Isoparametric Representation

Introduction to FEM

Isoparametric Representation of Finite Elements

*Geometry and displacements
are represented by
same set of shape functions (iso = same)*

Advantages

*Unification: same steps for all elements
No need to distinguish straight vs. curved side elements
Quick construction of shape functions*

Before Isoparametric Concept was Discovered, FEM Developers Did "SuperParametric" Elems

*Element shape functions refined,
more nodes and DOFs added*

*But element geometry was kept simple
with straight sides*

For the 3 Node Triangle

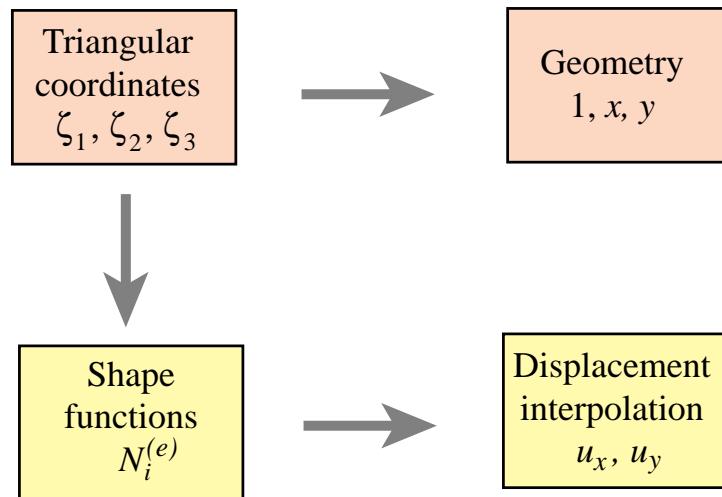
Geometric Description

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

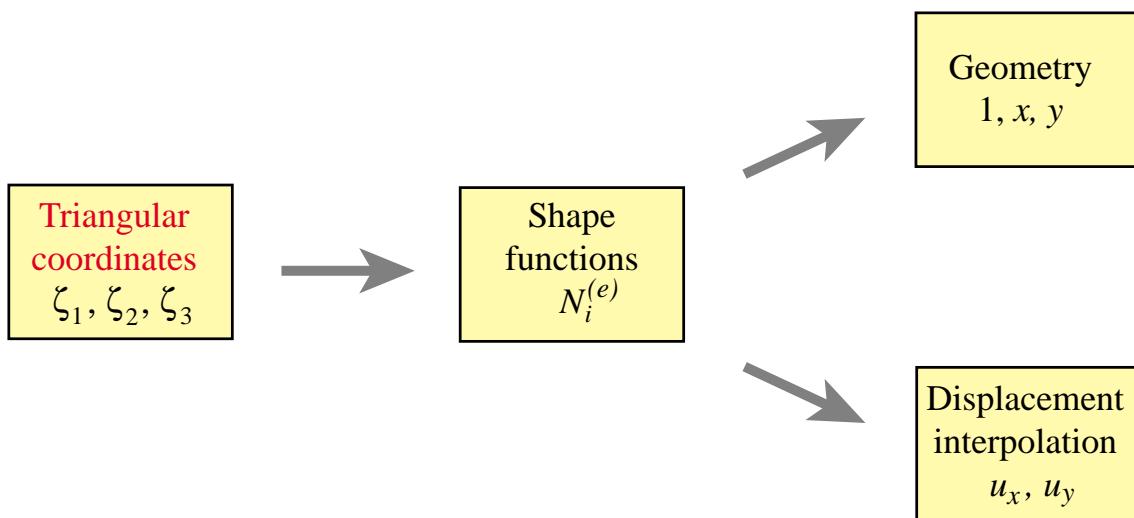
Displacement Interpolation

$$\begin{aligned} u_x &= u_{x1}N_1^{(e)} + u_{x2}N_2^{(e)} + u_{x3}N_3^{(e)} = u_{x1}\xi_1 + u_{x2}\xi_2 + u_{x3}\xi_3 \\ u_y &= u_{y1}N_1^{(e)} + u_{y2}N_2^{(e)} + u_{y3}N_3^{(e)} = u_{y1}\xi_1 + u_{y2}\xi_2 + u_{y3}\xi_3 \end{aligned}$$

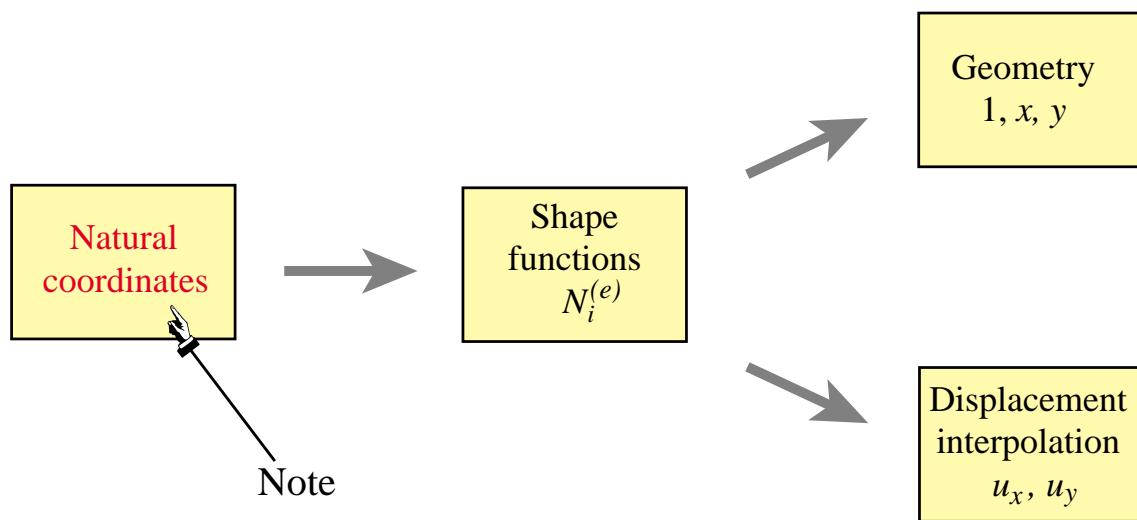
SuperParametric Representation (Triangles)



Isoparametric Representation (Iso=Equal) for Triangular Elements



Isoparametric Representation for *any* 2D Element



Iso-P Representation of 2D Plane Stress Elements with n Nodes

Element Geometry:

$$1 = \sum_{i=1}^n N_i^{(e)}, \quad x = \sum_{i=1}^n x_i N_i^{(e)}, \quad y = \sum_{i=1}^n y_i N_i^{(e)}$$

Displacement Interpolation

$$u_x = \sum_{i=1}^n u_{xi} N_i^{(e)}, \quad u_y = \sum_{i=1}^n u_{yi} N_i^{(e)}$$

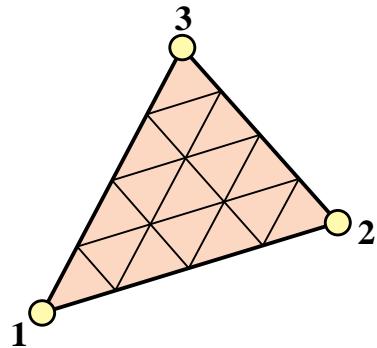
Matrix Form of Above

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_n^{(e)} \end{bmatrix}$$

More Rows May be Added to Interpolate other Quantities from Node Values

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ \text{thickness } h \\ \text{temperature } T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_n^{(e)} \end{bmatrix}.$$

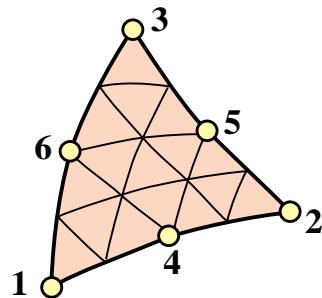
The Linear Triangle



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \zeta_1, \quad N_2^{(e)} = \zeta_2, \quad N_3^{(e)} = \zeta_3$$

The Quadratic Triangle



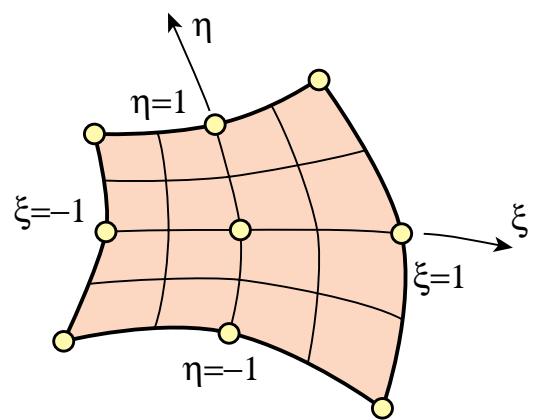
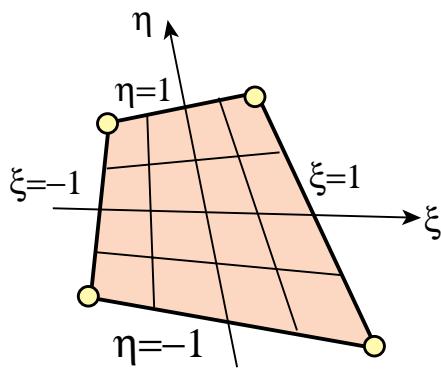
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \zeta_1(2\zeta_1 - 1) \quad N_4^{(e)} = 4\zeta_1\zeta_2$$

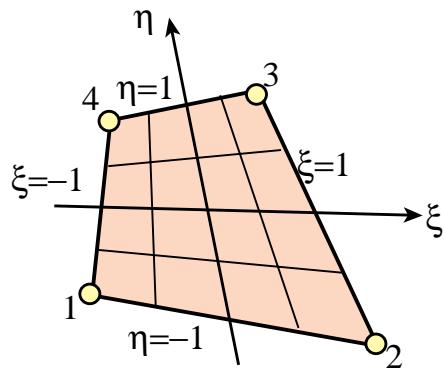
$$N_2^{(e)} = \zeta_2(2\zeta_2 - 1) \quad N_5^{(e)} = 4\zeta_2\zeta_3$$

$$N_3^{(e)} = \zeta_3(2\zeta_3 - 1) \quad N_6^{(e)} = 4\zeta_3\zeta_1$$

Quadrilateral Coordinates ξ, η

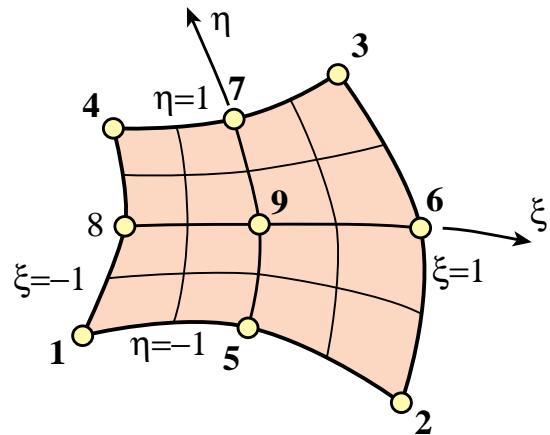


4-Node Bilinear Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \end{bmatrix} \quad \begin{aligned} N_1^{(e)} &= \frac{1}{4}(1 - \xi)(1 - \eta) \\ N_2^{(e)} &= \frac{1}{4}(1 + \xi)(1 - \eta) \\ N_3^{(e)} &= \frac{1}{4}(1 + \xi)(1 + \eta) \\ N_4^{(e)} &= \frac{1}{4}(1 - \xi)(1 + \eta) \end{aligned}$$

9 Node Biquadratic Quadrilateral

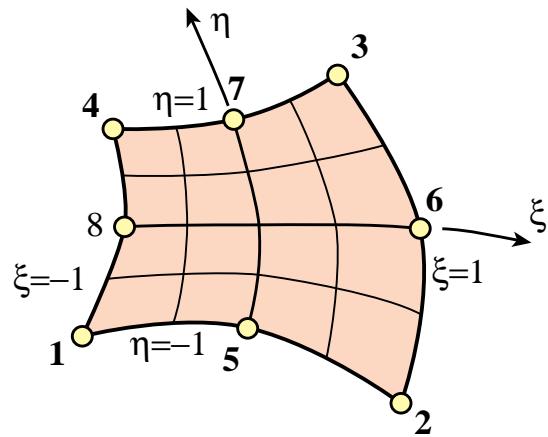


$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_9^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \frac{1}{4}(1-\xi)(1-\eta)\xi\eta \quad N_5^{(e)} = -\frac{1}{2}(1-\xi^2)(1-\eta)\eta \\ N_2^{(e)} = -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta \quad N_6^{(e)} = \frac{1}{2}(1+\xi)(1-\eta^2)\xi \quad N_9^{(e)} = (1-\xi^2)(1-\eta^2)$$

•••

8 Node "Serendipity" Quadrilateral



Derivation of shape functions is an
Exercise in Chapter 18