

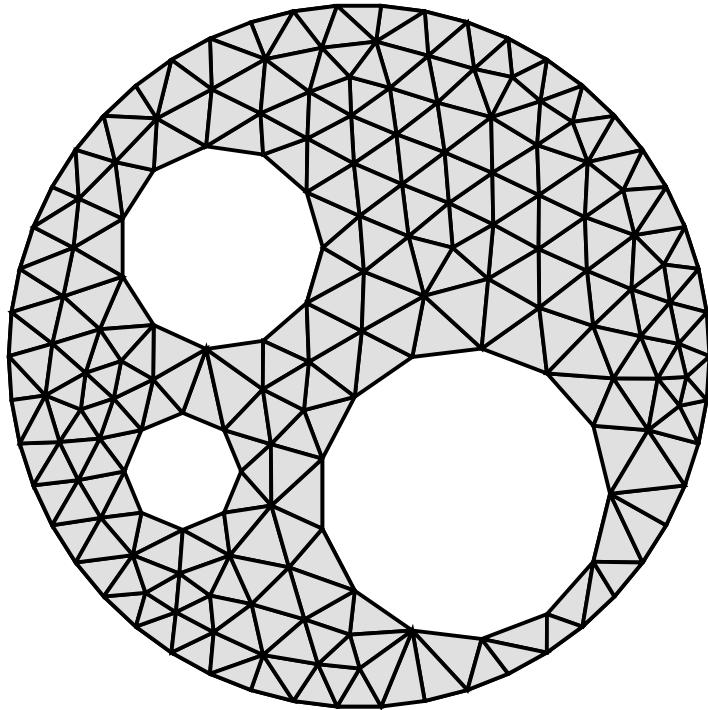
Introduction to FEM

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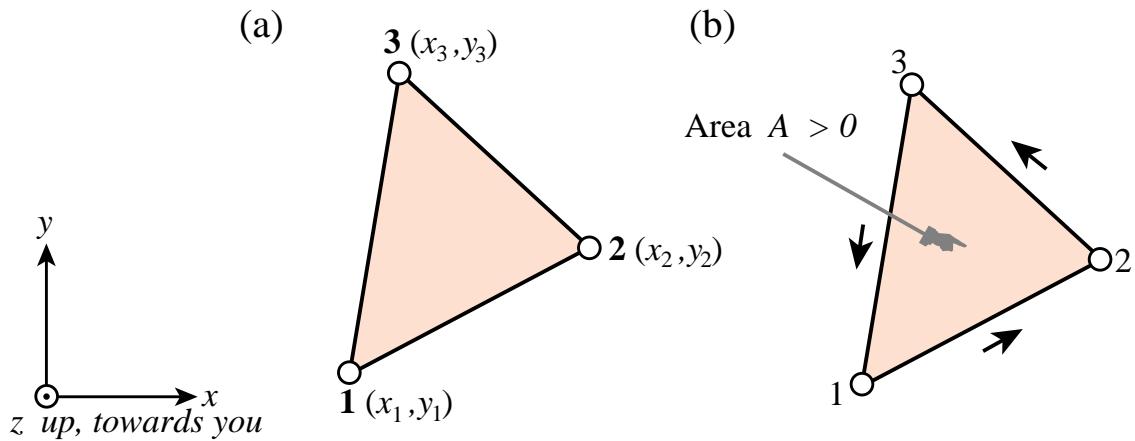
The Linear Plane Stress Triangle

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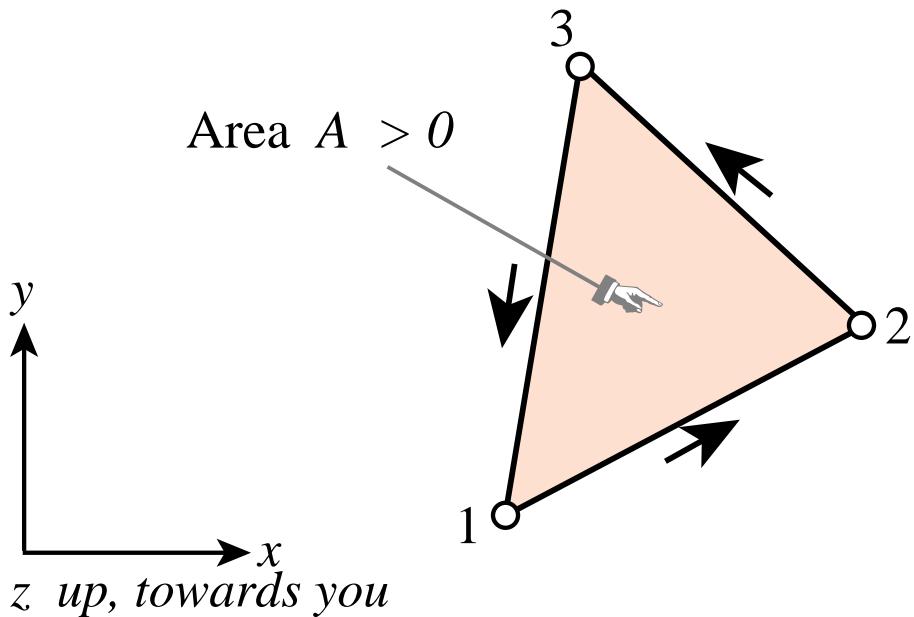
Triangles are Still Popular Because of Geometric Versatility and Ease of Automated Mesh Generation



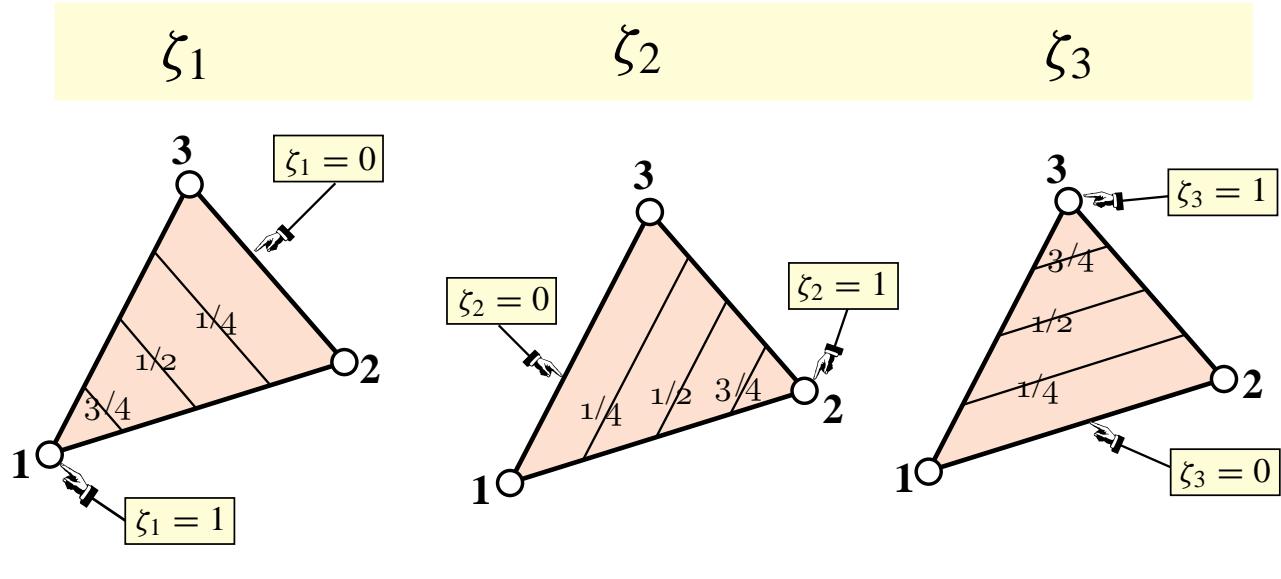
The Triangle Geometry and its Nodal Configuration



Positive Element Boundary Traversal Convention



Triangular Coordinates



$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

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Using Triangular Coordinates to Formulate Linear Interpolation

Cartesian form

$$f(x, y) = a_0 + a_1x + a_2y$$

Variation defined by 3 corner values f_1, f_2, f_3

Natural form

$$\begin{aligned} f(\xi_1, \xi_2, \xi_3) &= f_1\xi_1 + f_2\xi_2 + f_3\xi_3 = [f_1 \quad f_2 \quad f_3] \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \\ &= [\xi_1 \quad \xi_2 \quad \xi_3] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{aligned}$$

Relating Triangular & Cartesian Coordinates

Triangular to Cartesian:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Cartesian to Triangular:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Here $x_{jk} = x_j - x_k$ $y_{jk} = y_j - y_k$
 A_{jk} denotes the area subtended by corners j, k
and the origin of the x - y system

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Partial Derivatives for Cartesian-Triangular Coordinate Transformations

$$\frac{\partial x}{\partial \xi_i} = x_i \quad \frac{\partial y}{\partial \xi_i} = y_i$$

$$2A \frac{\partial \xi_i}{\partial x} = y_{jk} \quad 2A \frac{\partial \xi_i}{\partial y} = x_{kj}$$

$$i = 1, 2, 3 \quad j = 2, 3, 1 \quad k = 3, 1, 2$$

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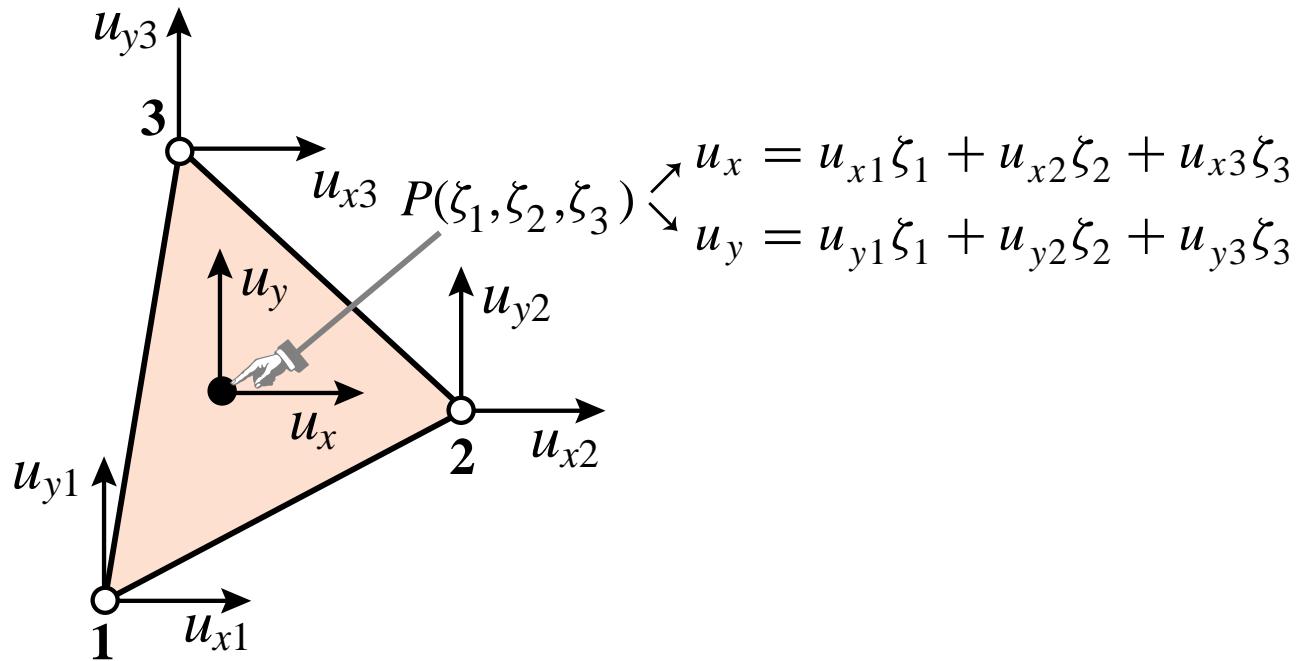
Cartesian Partial Derivatives of a General Triangular-Coordinates Function

$$f = f(\xi_1, \xi_2, \xi_3)$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{2A} \left(\frac{\partial f}{\partial \xi_1} y_{23} + \frac{\partial f}{\partial \xi_2} y_{31} + \frac{\partial f}{\partial \xi_3} y_{12} \right) \\ \frac{\partial f}{\partial y} &= \frac{1}{2A} \left(\frac{\partial f}{\partial \xi_1} x_{32} + \frac{\partial f}{\partial \xi_2} x_{13} + \frac{\partial f}{\partial \xi_3} x_{21} \right)\end{aligned}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \xi_1} \\ \frac{\partial f}{\partial \xi_2} \\ \frac{\partial f}{\partial \xi_3} \end{bmatrix}$$

Displacement Interpolation over Triangle



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Displacement Interpolation (Cont'd)

$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$

$$u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N} \mathbf{u}^{(e)}$$

The shape functions are $N_i = \zeta_i$, $i=1,2,3$

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Strain-Displacement Relations

$$\mathbf{e} = \mathbf{DN} \mathbf{u}^{(e)} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B} \mathbf{u}^{(e)}$$

Stress-Strain Relations

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E} \mathbf{e}$$

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Element Stiffness Matrix

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega^{(e)}$$

Because \mathbf{B} and \mathbf{E} are constant over the triangle area:

$$\begin{aligned} \mathbf{K}^{(e)} &= \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^{(e)}} h d\Omega^{(e)} \\ &= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^{(e)}} h d\Omega^{(e)} \end{aligned}$$

If the Plate Thickness h is Constant

$$\mathbf{K}^{(e)} = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Consistent Node Force Vector for Body Forces

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h (\mathbf{N}^{(e)})^T \mathbf{b} d\Omega^{(e)} = \int_{\Omega^{(e)}} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \mathbf{b} d\Omega^{(e)}$$

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If Body Forces are Constant over Element

Using *

$$\int_{\Omega^{(e)}} \zeta_1 \, d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_2 \, d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_3 \, d\Omega^{(e)} = \frac{1}{3} A$$

we get

$$\mathbf{f}^{(e)} = \frac{Ah}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$

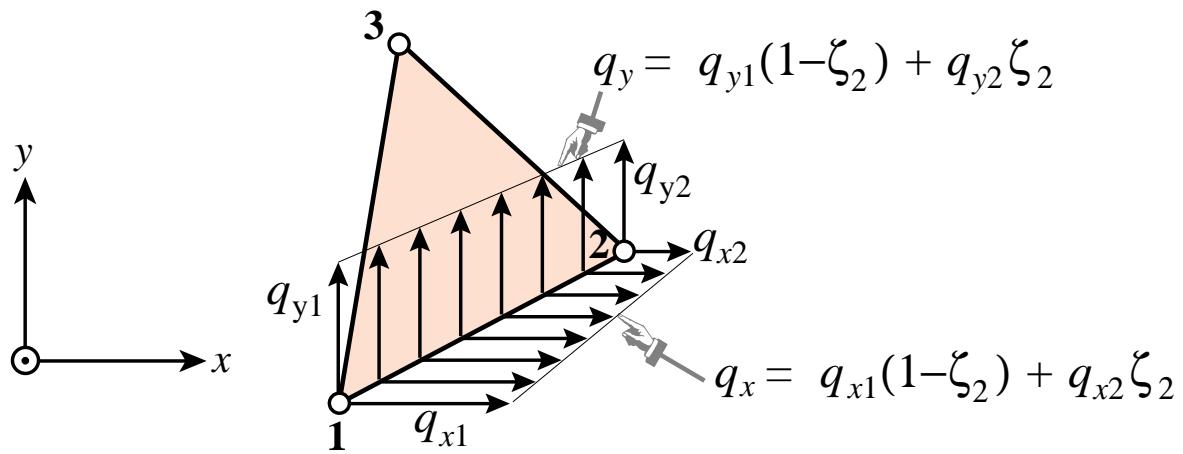
same result as
straightforward
"load lumping"

* Above integrals are instances of general formula

$$\frac{1}{2A} \int_{\Omega^{(e)}} \zeta_1^i \zeta_2^j \zeta_3^k \, d\Omega^{(e)} = \frac{i! j! k!}{(i + j + k + 2)!}, \quad i \geq 0, j \geq 0, k \geq 0$$

valid for triangles with *straight* sides

Line Forces on Triangle Side



Lumping to node forces given as Exercise 15.4