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The Plane Stress Problem

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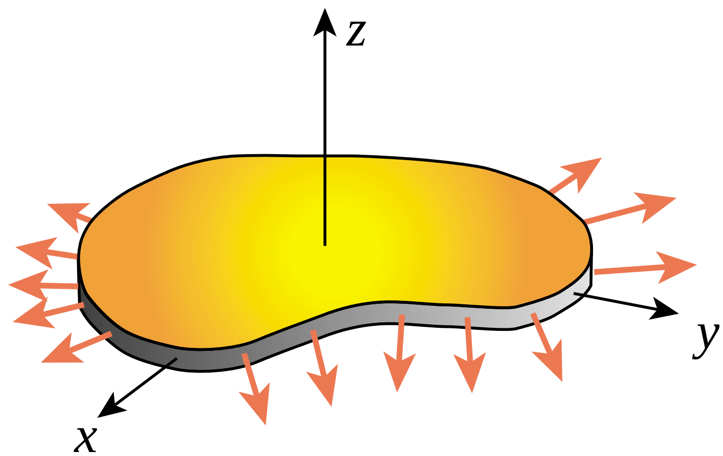
Plate in Plane Stress

Thickness dimension or transverse dimension



Inplane dimensions: in *x,y* plane

Plane Stress Physical Assumptions

Plate is flat and has a symmetry plane (the midplane)

All loads and support conditions are midplane symmetric

Thickness dimension is much smaller than inplane dimensions

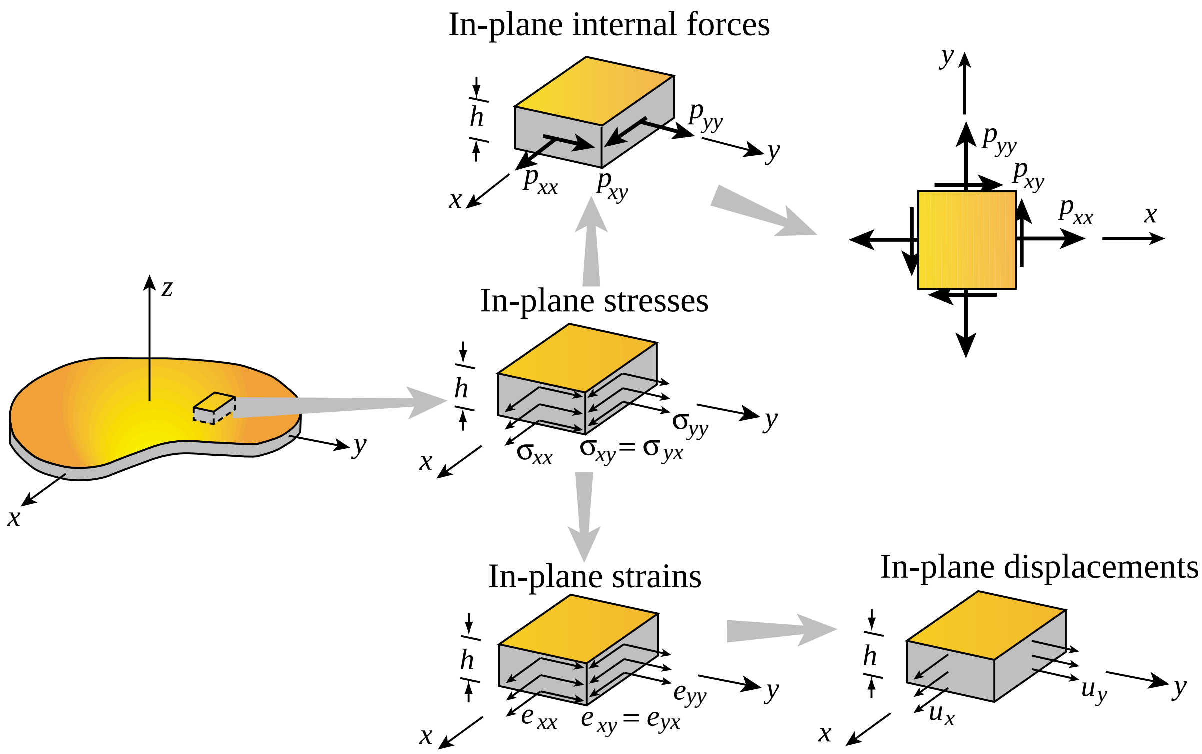
Inplane displacements, strains and stresses uniform through thickness

Transverse stresses σ_{zz}, σ_{xz} **and** σ_{yz} **negligible**

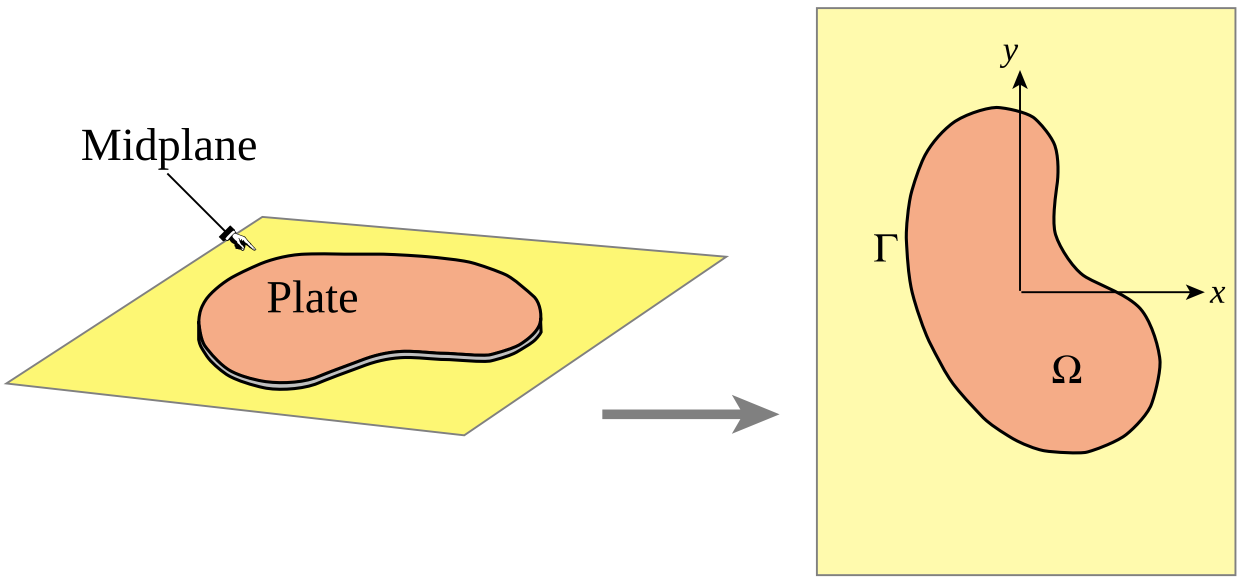
Unessential but used in this course:

Plate fabricated of homogeneous material through thickness

Notation for stresses, strains, forces, displacements



Mathematical Idealization as a Two Dimensional Problem



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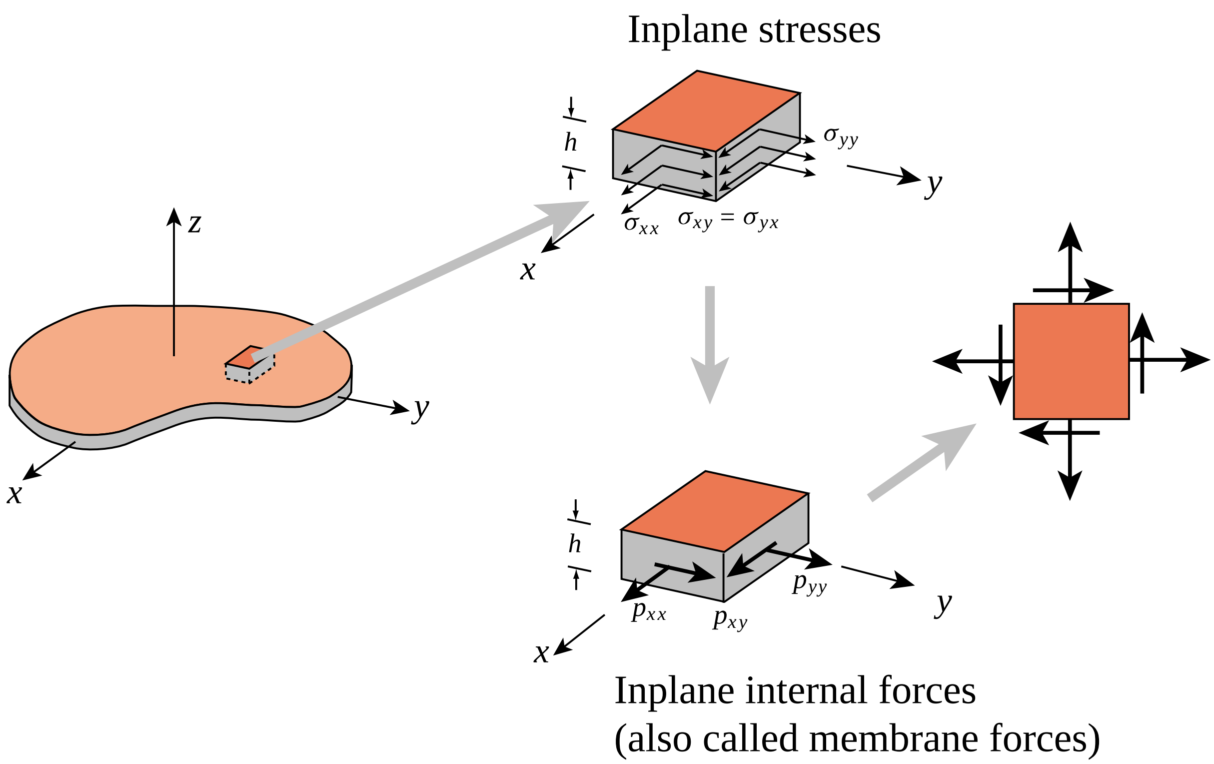
Inplane Forces are Obtained by Stress Integration Through Thickness



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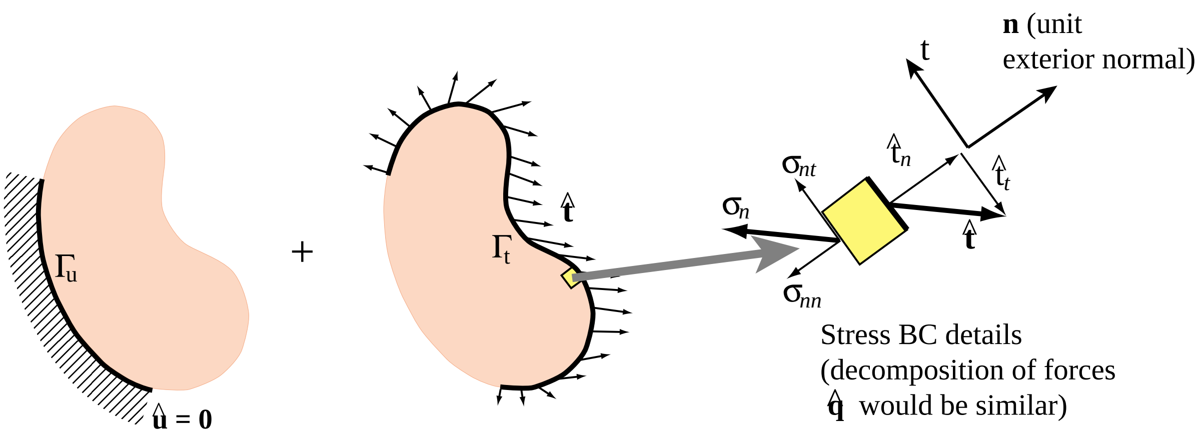
Plane Stress Boundary Conditions



Stress BC details (decomposition of forces $\hat{\mathbf{q}}$ would be similar)

Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on Γ_t

The Plane Stress Problem

Given:

geometry
material properties
wall fabrication (thickness only for homogeneous plates)
applied body forces
boundary conditions:
- ***prescribed boundary forces or tractions***
- ***prescribed displacements***

Find:

inplane displacements
inplane strains
inplane stresses and/or internal forces

Matrix Notation for Internal Fields

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}$$ ***displacements***

$$\mathbf{e}(x,y) = \begin{bmatrix} e_{xx}(x,y) \\ e_{yy}(x,y) \\ 2e_{xy}(x,y) \end{bmatrix}$$ ***strains***

$$\boldsymbol{\sigma}(x,y) = \begin{bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{bmatrix}$$ ***stresses***

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Governing Plane Stress Elasticity Equations in Matrix Form

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\mathbf{e} = \mathbf{D}\mathbf{u} \qquad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \qquad \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

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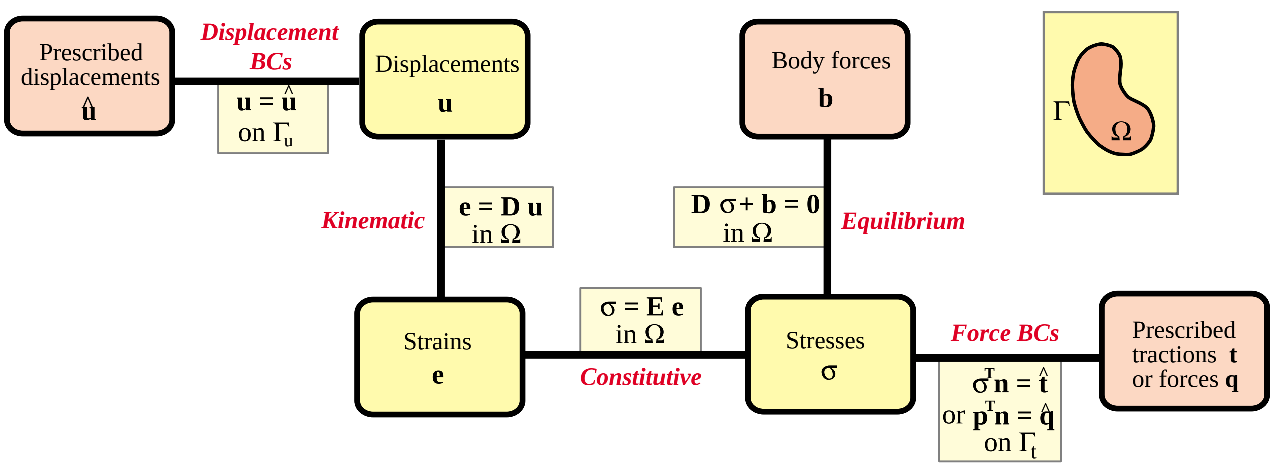
Strong Form Tonti Diagram of Plane Stress Governing Equations



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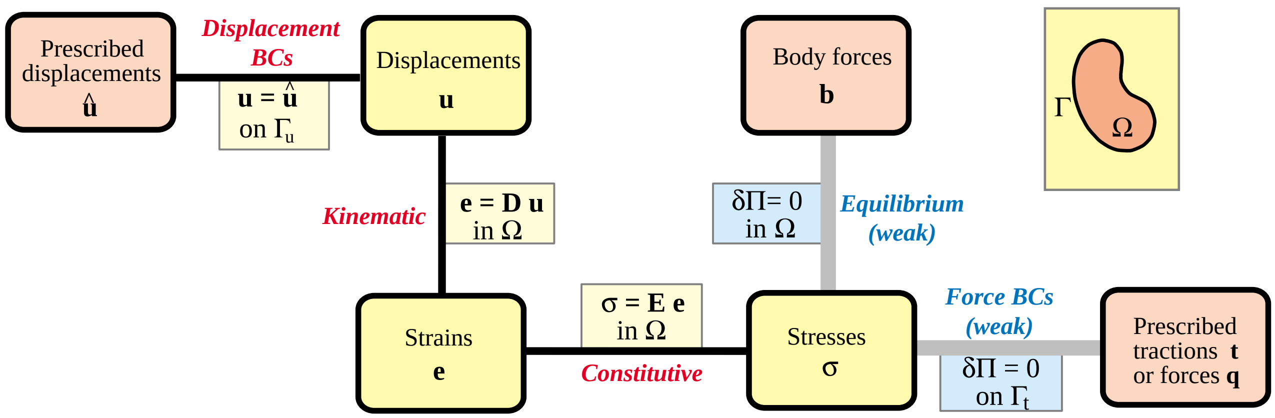
TPE-Based Weak Form Diagram of Plane Stress Governing Equations



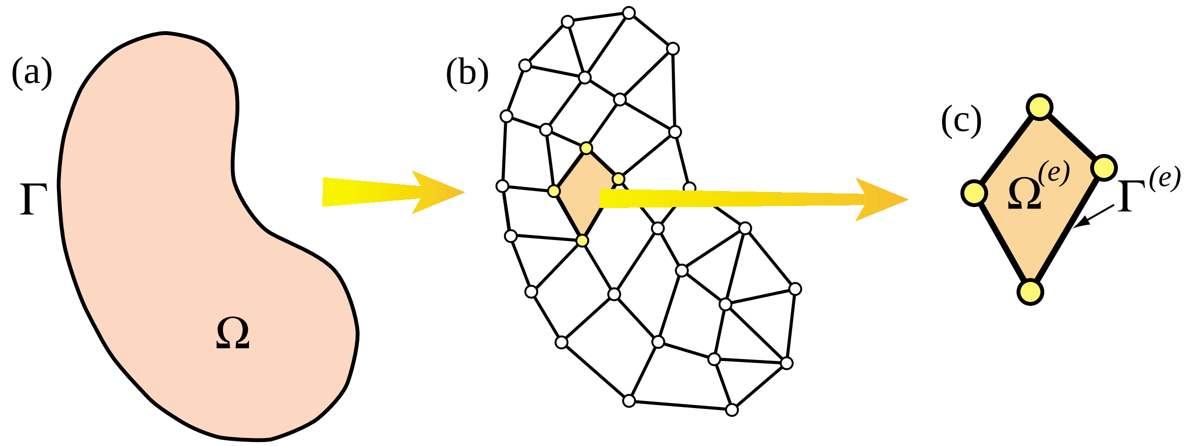
Total Potential Energy of Plate in Plane Stress

$$\boxed{\Pi = U - W}$$

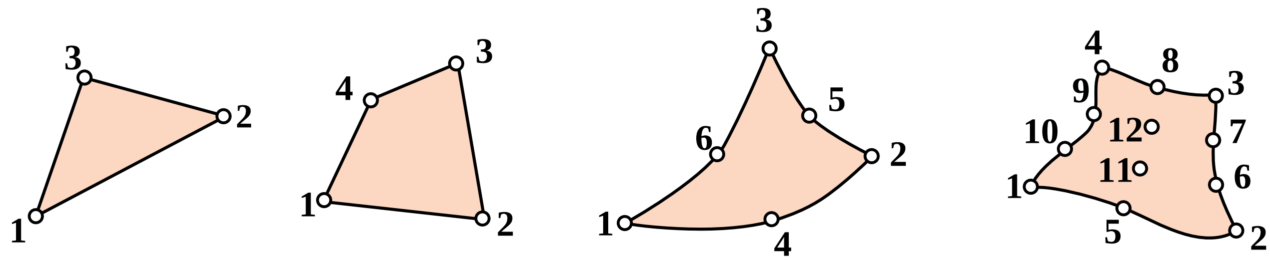
$$U = \tfrac{1}{2} \int_\Omega h\, \boldsymbol{\sigma}^T \mathbf{e}\, d\Omega = \tfrac{1}{2} \int_\Omega h\, \mathbf{e}^T \mathbf{E} \mathbf{e}\, d\Omega$$

$$W = \int_\Omega h\, \mathbf{u}^T \mathbf{b}\, d\Omega + \int_{\Gamma_t} h\, \mathbf{u}^T \hat{\mathbf{t}}\, d\Gamma$$

Discretization into Plane Stress Finite Elements

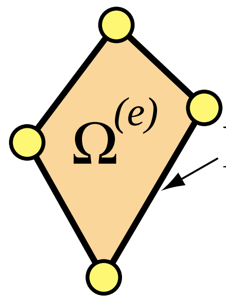


Plane Stress Element Geometries and Node Configurations



$n = 3$ $n = 4$ $n = 6$ $n = 12$

Total Potential Energy of Plane Stress Element

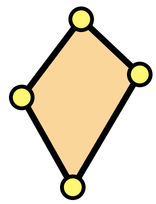


$$\Pi^{(e)} = U^{(e)} - W^{(e)}$$

$$U^{(e)} = \tfrac{1}{2}\int_{\Omega^{(e)}} h\,\boldsymbol{\sigma}^T\mathbf{e} = \tfrac{1}{2}\int_{\Omega^{(e)}} h\,\mathbf{e}^T\mathbf{E}\mathbf{e}\,d\Omega^{(e)}$$

$$W^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{u}^T\mathbf{b}\,d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{u}^T\mathbf{t}\,d\Gamma^{(e)}$$

Constructing a Displacement Assumed Element



n nodes, n=4 in figure

Node displacement vector:

$$\mathbf{u}^{(e)} = [\,u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn}\,]^T$$

Displacement interpolation

$$\boxed{\mathbf{u}(x,y) =} \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & \ldots & N_n^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \ldots & 0 & N_n^{(e)} \end{bmatrix} \mathbf{u}^{(e)}$$

$$\boxed{= \mathbf{N}\ \mathbf{u}^{(e)}}$$

N **is called the shape function matrix**

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Element Construction (cont'd)

Differentiate the displacement interpolation wrt *x,y* **to get the strain-displacement relation**

$$\boxed{\mathbf{e}(x,y) =} \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} \boxed{= \mathbf{B}\mathbf{u}^{(e)}}$$

B is called the strain-displacement matrix

Element Construction (cont'd)

Element total potential energy

$$\Pi^{(e)} = \tfrac{1}{2}\mathbf{u}^{(e)\,T}\mathbf{K}^{(e)}\mathbf{u}^{(e)} - \mathbf{u}^{(e)\,T}\mathbf{f}^{(e)}$$

Element stiffness matrix

$$\boxed{\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega^{(e)}}$$

Consistent node force vector

$$\boxed{\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{N}^T\hat{\mathbf{t}}\,d\Gamma^{(e)}}$$

body force **surface force**

Requirements on Finite Element Shape Functions

Interpolation Conditions:

N_i **takes on value 1 at node** i**, 0 at all other nodes**

Continuity (intra- and inter-element) and Completeness Conditions

are covered later in the course (Chs. 18-19)