

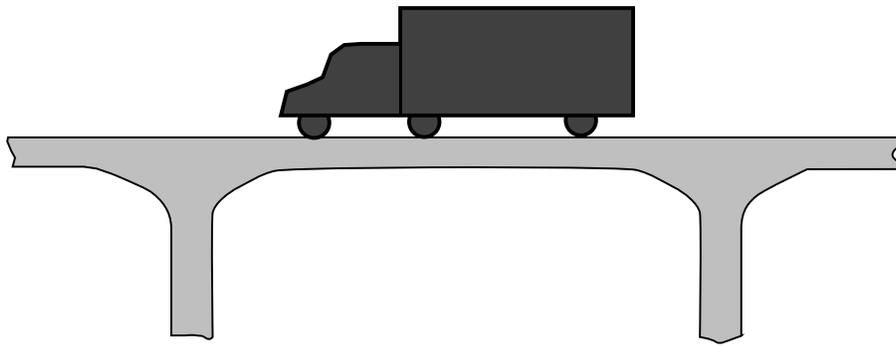
Introduction to FEM

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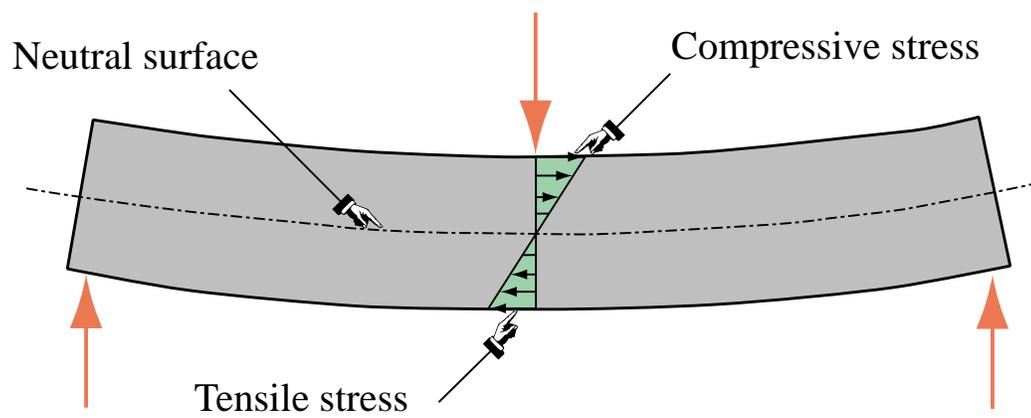
Variational Formulation of Plane Beam Element

Introduction to FEM

Beams Resist Primarily Transverse Loads



Transverse Loads are Transported to Supports by Flexural Action



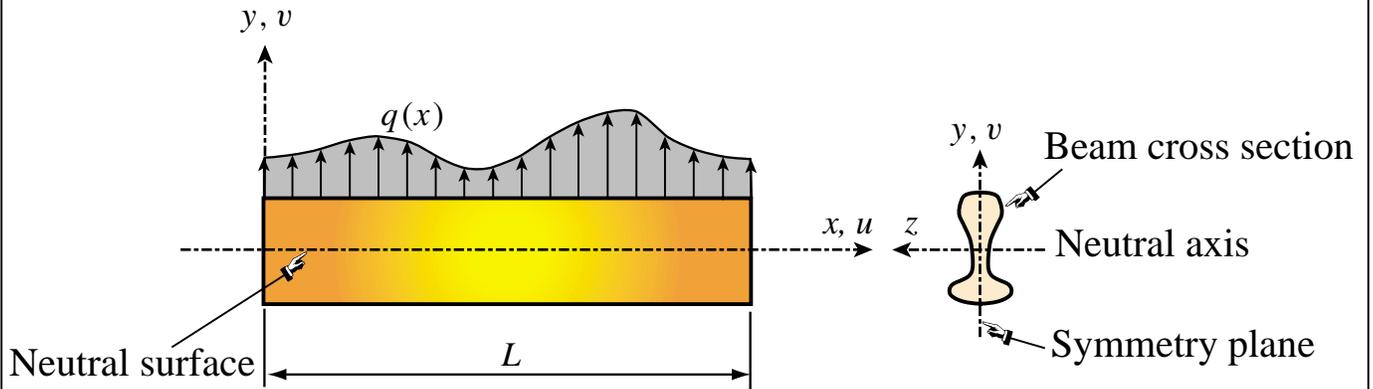
Beam Configuration

Spatial (General Beams)
Plane (This Chapter)

Beam Models

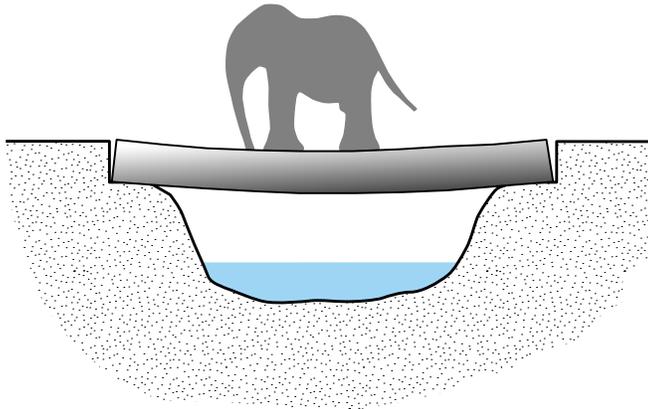
Bernoulli-Euler
*Timoshenko (advanced topic
not covered in class)*

Plane Beam Terminology

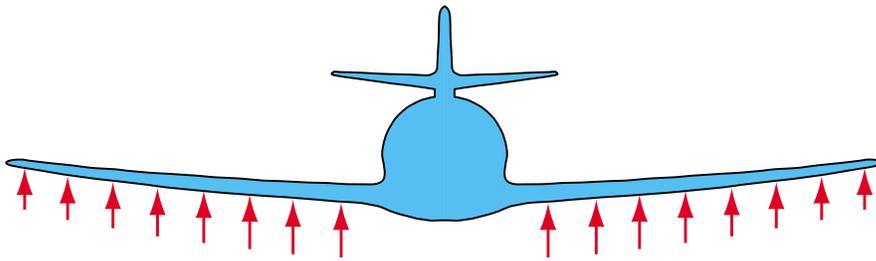


Introduction to FEM

Common Support Conditions



Simply Supported



Cantilever

Basic Relations for Bernoulli-Euler Model of Plane Beam

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y \frac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

$$e = \frac{\partial u}{\partial x} = -y \frac{\partial^2 v}{\partial x^2} = -y \frac{d^2 v}{dx^2} = -y\kappa$$

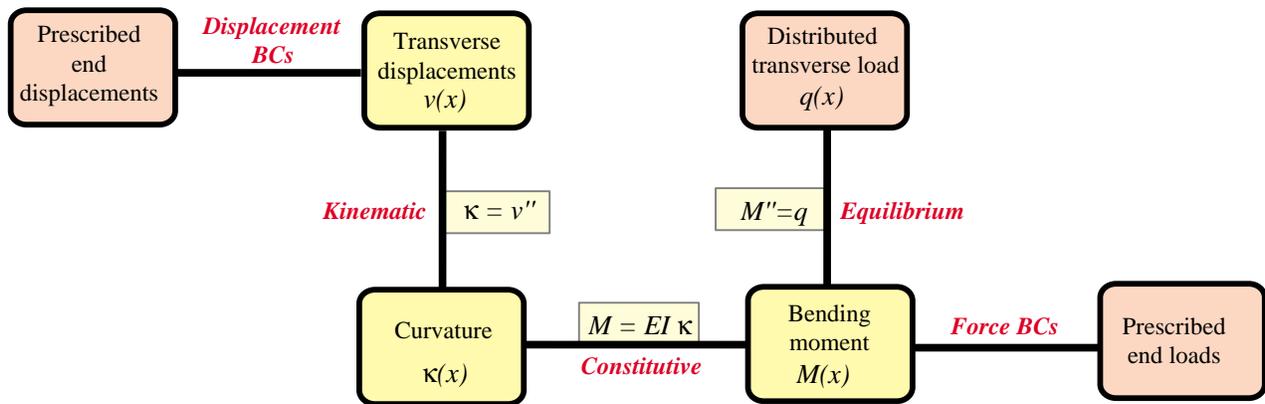
$$\sigma = Ee = -Ey \frac{d^2 v}{dx^2} = -Ey\kappa$$

$$M = EI\kappa$$

Plus equilibrium equation $M'' = q$ (not used specifically in FEM)

Introduction to FEM

Tonti Diagram for Bernoulli-Euler Model of Plane Beam (Strong Form)



Total Potential Energy of Beam Member

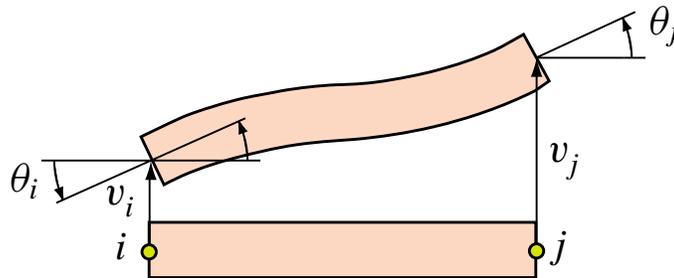
$$\Pi = U - W$$

$$U = \frac{1}{2} \int_V \sigma_{xx} e_{xx} dV = \frac{1}{2} \int_0^L M \kappa dx = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx$$

$$= \frac{1}{2} \int_0^L EI \kappa^2 dx$$

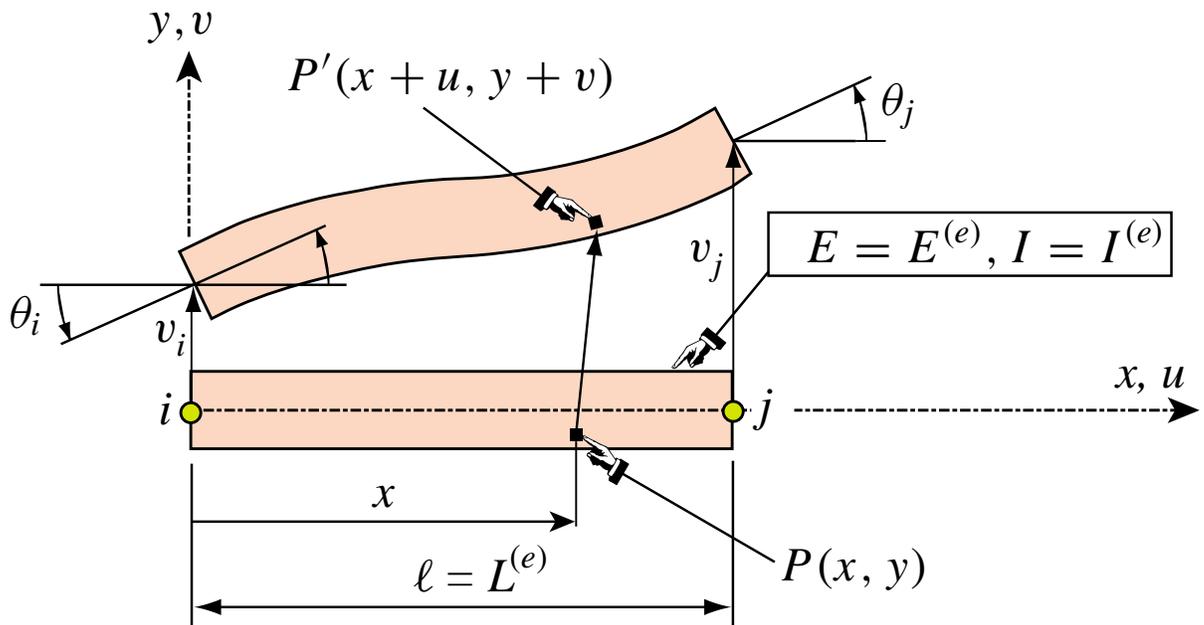
$$W = \int_0^L qv dx.$$

Degrees of Freedom of Beam Element

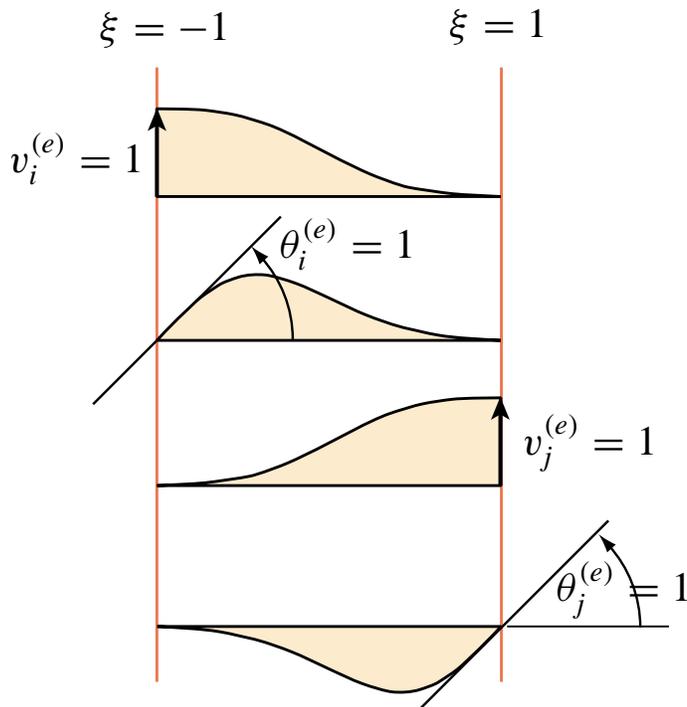


$$\mathbf{u}^{(e)} = \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix}$$

Bernoulli-Euler Kinematics of Plane Beam Element



Plane Beam Element Shape Functions



$$N_{v_i}^{(e)}(\xi) = \frac{1}{4}(1 - \xi)^2(2 + \xi)$$

$$N_{\theta_i}^{(e)}(\xi) = \frac{1}{8}\ell(1 - \xi)^2(1 + \xi)$$

$$N_{v_j}^{(e)}(\xi) = \frac{1}{4}(1 + \xi)^2(2 - \xi)$$

$$N_{\theta_j}^{(e)}(\xi) = -\frac{1}{8}\ell(1 + \xi)^2(1 - \xi)$$

Shape Functions in Terms of Natural Coordinate ξ

$$\mathbf{v}^{(e)} = [N_{v_i}^{(e)} \quad N_{\theta_i}^{(e)} \quad N_{v_j}^{(e)} \quad N_{\theta_j}^{(e)}] \begin{bmatrix} v_i^{(e)} \\ \theta_i^{(e)} \\ v_j^{(e)} \\ \theta_j^{(e)} \end{bmatrix} = \mathbf{N}\mathbf{u}^{(e)}$$

$$\xi = \frac{2x}{\ell} - 1$$

$$\begin{aligned} N_{v_i}^{(e)} &= \frac{1}{4}(1 - \xi)^2(2 + \xi), & N_{\theta_i}^{(e)} &= \frac{1}{8}\ell(1 - \xi)^2(1 + \xi), \\ N_{v_j}^{(e)} &= \frac{1}{4}(1 + \xi)^2(2 - \xi), & N_{\theta_j}^{(e)} &= -\frac{1}{8}\ell(1 + \xi)^2(1 - \xi). \end{aligned}$$

Element Stiffness and Consistent Node Forces

$$\mathbf{B} = \frac{1}{\ell} \left[6\frac{\xi}{\ell} \quad 3\xi - 1 \quad -6\frac{\xi}{\ell} \quad 3\xi + 1 \right]$$

$$\Pi^{(e)} = \frac{1}{2} \mathbf{u}^{(e)T} \mathbf{K}^{(e)} \mathbf{u}^{(e)} - \mathbf{u}^{(e)T} \mathbf{f}^{(e)}$$

$$\mathbf{K}^{(e)} = \int_0^\ell EI \mathbf{B}^T \mathbf{B} dx = \int_{-1}^1 EI \mathbf{B}^T \mathbf{B} \frac{1}{2} \ell d\xi$$

$$\mathbf{f}^{(e)} = \int_0^\ell \mathbf{N}^T q dx = \int_{-1}^1 \mathbf{N}^T q \frac{1}{2} \ell d\xi$$

Analytical Computation of Prismatic Beam Element Stiffness

$$\mathbf{K}^{(e)} = \frac{EI}{2l^3} \int_{-1}^1 \begin{bmatrix} 36\xi^2 & 6\xi(3\xi - 1)l & -36\xi^2 & 6\xi(3\xi + 1)l \\ & (3\xi - 1)^2 l^2 & -6\xi(3\xi - 1)l & (9\xi^2 - 1)l^2 \\ \text{symm} & & 36\xi^2 & -6\xi(3\xi + 1)l \\ & & & (3\xi + 1)^2 l^2 \end{bmatrix} d\xi$$

$$= \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ \text{symm} & & 12 & -6l \\ & & & 4l^2 \end{bmatrix}$$

***Mathematica* Script for Symbolic Computation of Prismatic Plane Beam Element Stiffness**

```

ClearAll[EI,l,ξ];
B={{6*ξ,(3*ξ-1)*l,-6*ξ,(3*ξ+1)*l}}/l^2;
Ke=(EI*l/2)*Integrate[Transpose[B].B,{ξ,-1,1}];
Ke=Simplify[Ke];
Print["Ke for prismatic beam:"];
Print[Ke//MatrixForm];

```

Ke for prismatic beam:

$$\begin{pmatrix} \frac{12 EI}{l^3} & \frac{6 EI}{l^2} & -\frac{12 EI}{l^3} & \frac{6 EI}{l^2} \\ \frac{6 EI}{l^2} & \frac{4 EI}{l} & -\frac{6 EI}{l^2} & \frac{2 EI}{l} \\ -\frac{12 EI}{l^3} & -\frac{6 EI}{l^2} & \frac{12 EI}{l^3} & -\frac{6 EI}{l^2} \\ \frac{6 EI}{l^2} & \frac{2 EI}{l} & -\frac{6 EI}{l^2} & \frac{4 EI}{l} \end{pmatrix}$$

Corroborates the result from hand integration.

Introduction to FEM

Analytical Computation of Consistent Node Force Vector for Uniform Load q

$$\mathbf{f}^{(e)} = \frac{1}{2}q\ell \int_{-1}^1 \mathbf{N} d\xi = \frac{1}{2}q\ell \int_{-1}^1 \begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{1}{8}\ell(1-\xi)^2(1+\xi) \\ \frac{1}{4}(1+\xi)^2(2-\xi) \\ -\frac{1}{8}\ell(1+\xi)^2(1-\xi) \end{bmatrix} d\xi$$

$$= q\ell \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12}\ell \\ \frac{1}{2} \\ -\frac{1}{12}\ell \end{bmatrix}$$

"fixed end moments"

***Mathematica* Script for Computation of Consistent Node Force Vector for Uniform q**

```
ClearAll[q,l,ξ]
Ne={{2*(1-ξ)^2*(2+ξ), (1-ξ)^2*(1+ξ)*l,
      2*(1+ξ)^2*(2-ξ), -(1+ξ)^2*(1-ξ)*l}}/8;
fe=(q*l/2)*Integrate[Ne,{ξ,-1,1}]; fe=Simplify[fe];
Print["fe^T for uniform load q:"];
Print[fe//MatrixForm];
```

fe^T for uniform load q:

$$\left(\frac{lq}{2} \quad \frac{l^2q}{12} \quad \frac{lq}{2} \quad -\frac{l^2q}{12} \right)$$

Force vector printed as row vector to save space.