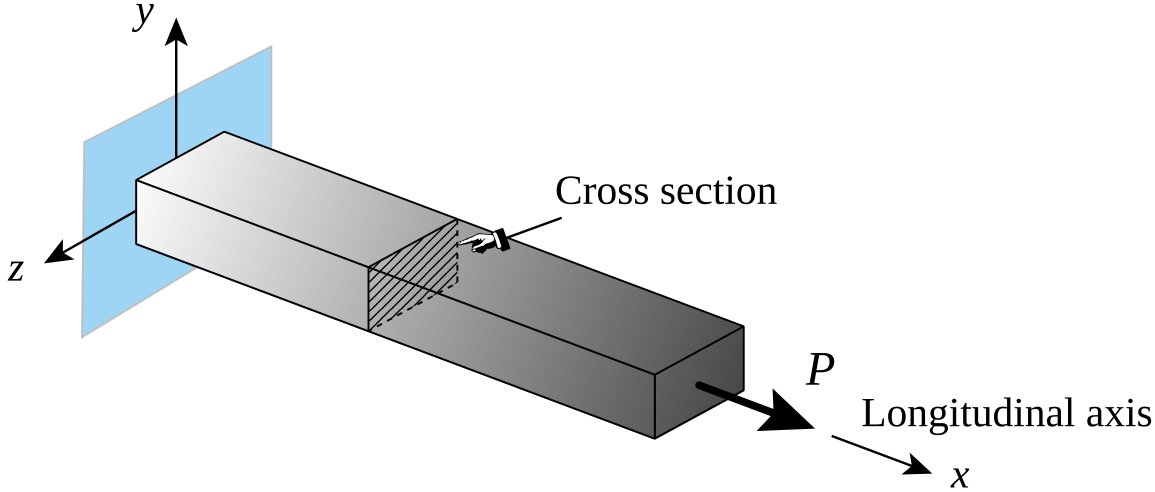
Introduction to FEM

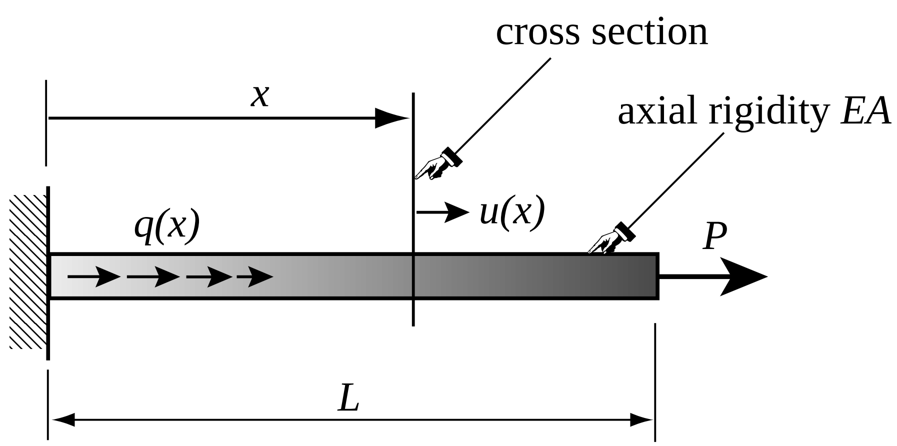
12

Variational Formulation of Bar Element

Bar Member - Variational Derivation



Bar Member (cont'd)

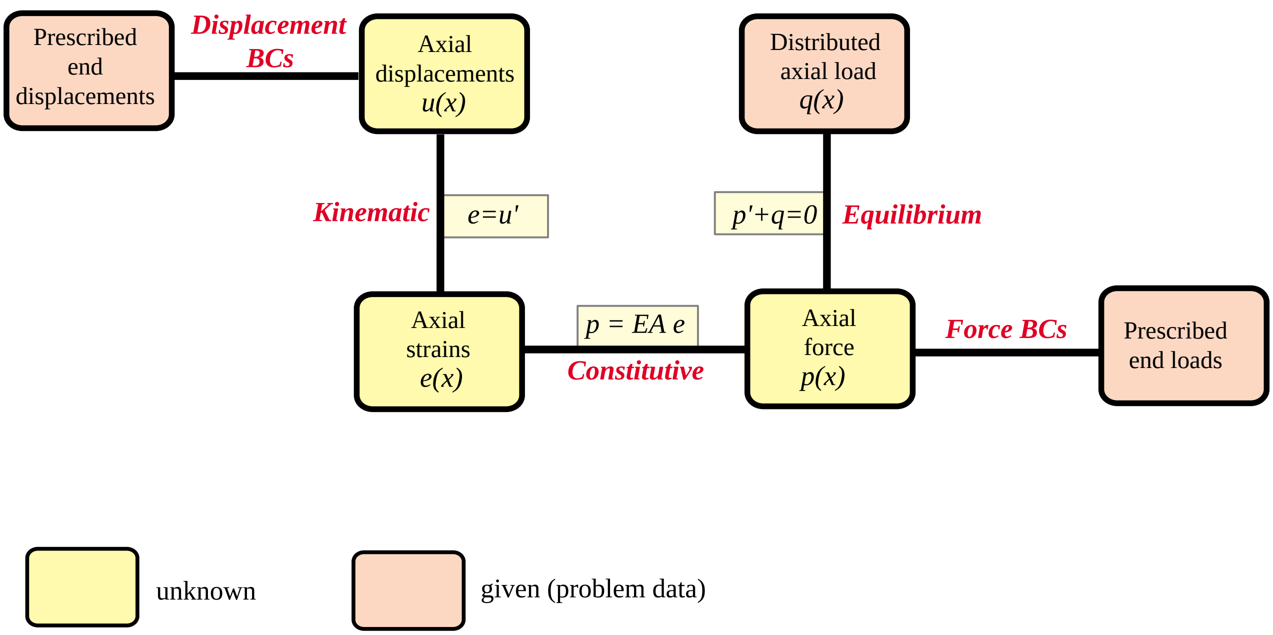


The Bar Revisited - Notation

Quantity	*Meaning*
x	Longitudinal bar axis*
$(.)'$	$d(.)/dx$
$u(x)$	Axial displacement
$q(x)$	Distributed axial force, given per unit of bar length
L	Total bar length
E	Elastic modulus
A	Cross section area; may vary with x
EA	Axial rigidity
$e = du/dx = u'$	Infinitesimal axial strain
$\sigma = Ee = Eu'$	Axial stress
$p = A\sigma = EAe = EAu'$	Internal axial force
P	Prescribed end load

* x is used in this Chapter instead of $\bar{x}$ (as in Chapters 2–3) to simplify the notation.

Tonti Diagram of Governing Equations



Potential Energy of the Bar Member
(before discretization)

Internal energy (= strain energy)

$$U = \tfrac{1}{2}\int_0^L pe\,dx = \tfrac{1}{2}\int_0^L (EAu')u'\,dx = \tfrac{1}{2}\int_0^L u'EAu'\,dx$$

External work

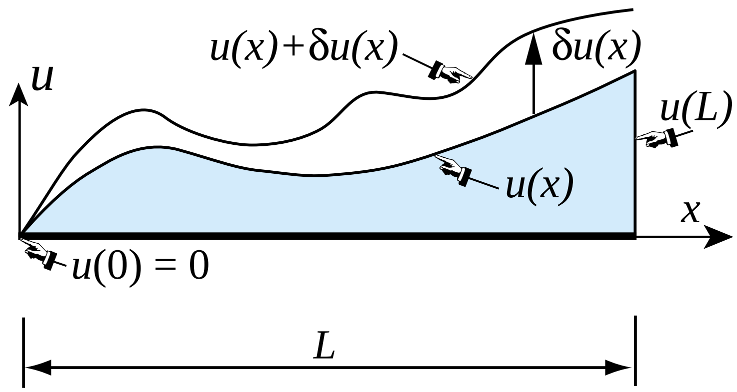
$$W = \int_0^L qu\,dx$$

Total potential energy

$$\boxed{\Pi = U - W}$$

Introduction to FEM

Concept of Kinematically Admissible Variation



$\delta u(x)$ is kinematically admissible if $u(x)$ and $u(x) + \delta u(x)$

(i) are continuous over bar length, i.e. $u(x) \in \mathcal{C}_0$ in $x \in [0, L]$.

(ii) satisfy exactly displacement BC ; in the figure, $u(0) = 0$

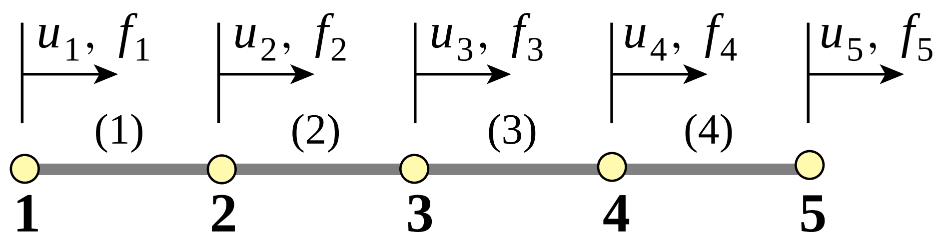
The Minimum Potential Energy (MPE) Principle

The MPE principle states that the actual displacement solution $u^*(x)$ that satisfies the governing equations is that which renders the TPE functional $\Pi[u]$ stationary:

$$\boxed{\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^*}$$

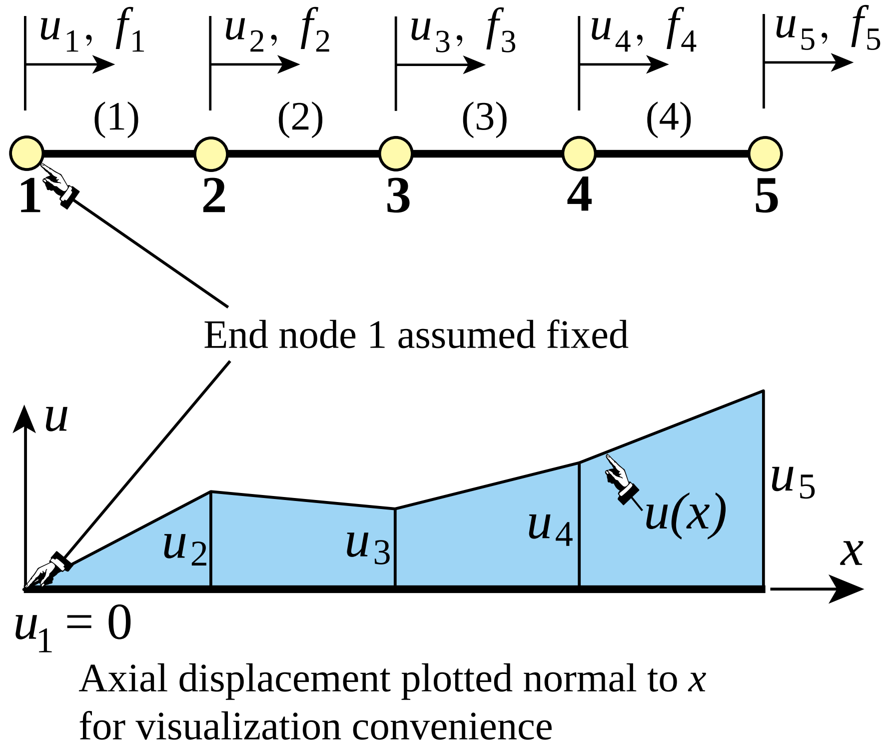
with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement solution $u^*(x)$

FEM Discretization of Bar Member

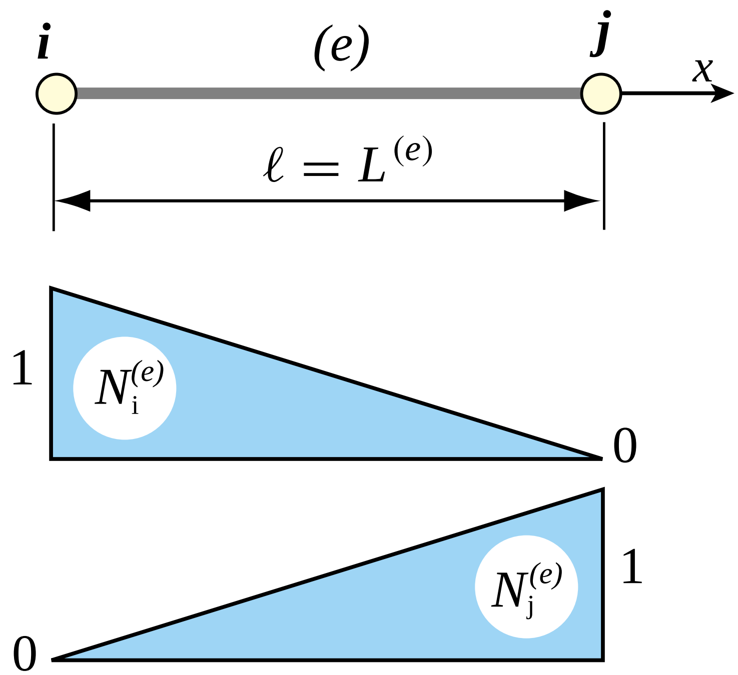


Introduction to FEM

FEM Displacement Trial Function



Element Shape Functions



Introduction to FEM

Total Potential Energy Principle and Decomposition over Elements

$$\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^* \quad \text{(exact solution)}$$

But $$\Pi = \Pi^{(1)} + \Pi^{(2)} + \ldots + \Pi^{(N_e)}$$

and $$\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots + \delta\Pi^{(N_e)} = 0$$

From fundamental lemma of variational calculus, each component variation must vanish, giving

$$\boxed{\delta\Pi^{(e)} = \delta U^{(e)} - \delta W^{(e)} = 0}$$

Element Shape Functions (cont'd)

Linear displacement interpolation:

$$u^{(e)}(x) = N_i^{(e)} u_i^{(e)} + N_j^{(e)} u_j^{(e)} = [\, N_i^{(e)} \quad N_j^{(e)} \,] \begin{bmatrix} u_i^{(e)} \\ u_j^{(e)} \end{bmatrix} = \mathbf{N}\mathbf{u}^{(e)}$$

in which

$$N_i^{(e)} = 1 - \frac{x - x_i}{\ell} = 1 - \zeta, \qquad N_j^{(e)} = \frac{x - x_i}{\ell} = \zeta$$

$\zeta = \dfrac{x - x_i}{\ell}$ dimensionless (natural) coordinate

Introduction to FEM

Displacement Variation Process Yields the Element Stiffness Equations

$$\Pi^{(e)} = U^{(e)} - W^{(e)} \quad \begin{cases} U^{(e)} = \frac{1}{2}(\mathbf{u}^{(e)})^T \mathbf{K}^{(e)} \mathbf{u}^{(e)} \\ W^{(e)} = (\mathbf{u}^{(e)})^T \mathbf{f}^{(e)} \end{cases}$$

$$\delta\Pi^{(e)} = 0 \quad ☞ \quad \left(\delta\mathbf{u}^{(e)}\right)^T \left[\mathbf{K}^{(e)}\mathbf{u}^{(e)} - \mathbf{f}^{(e)}\right] = 0$$

(Appendix D)

since $\delta\mathbf{u}^{(e)}$ is arbitrary [...] = 0

$$\boxed{\mathbf{K}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)}}$$

the element stiffness equations

The Bar Element Stiffness

$$U^{(e)} = \tfrac{1}{2}\int_0^{\ell} e\,E A\,e\;dx \qquad e = u'$$

$$U^{(e)} = \tfrac{1}{2}\int_0^{\ell} [\,u_i^{(e)} \quad u_j^{(e)}\,]\,\frac{1}{\ell}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{1}{\ell}[\,-1 \quad 1\,]\begin{bmatrix} u_i^{(e)} \\ u_j^{(e)} \end{bmatrix} dx$$

$$U^{(e)} = \tfrac{1}{2}[\,u_i^{(e)} \quad u_j^{(e)}\,]\int_0^{\ell} \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \begin{bmatrix} u_i^{(e)} \\ u_j^{(e)} \end{bmatrix} = \tfrac{1}{2}\left(\mathbf{u}^{(e)}\right)^T \mathbf{K}^{(e)}\mathbf{u}^{(e)}$$

$$\boxed{\mathbf{K}^{(e)} = \int_0^{\ell} E A\,\mathbf{B}^T\mathbf{B}\,dx = \int_0^{\ell} \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx}$$

If EA is constant over element

$$\boxed{\mathbf{K}^{(e)} = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

The Consistent Nodal Force Vector

$$W^{(e)} = \int_0^\ell q u \, dx = \int_0^\ell (\mathbf{u}^{(e)})^T \mathbf{N}^T q \, dx = \left(\mathbf{u}^{(e)}\right)^T \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \left(\mathbf{u}^{(e)}\right)^T \mathbf{f}^{(e)}$$

$$\boxed{\mathbf{f}^{(e)} = \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx}$$

in which $\zeta = \dfrac{x - x_i}{\ell}$

Bar Consistent Force Vector (cont'd)

If q is constant along element

$$\mathbf{f}^{(e)} = q \int_0^{\ell} \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = q\ell \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

the same result as with EbE load lumping (i.e., assigning one half of the total load to each node)