

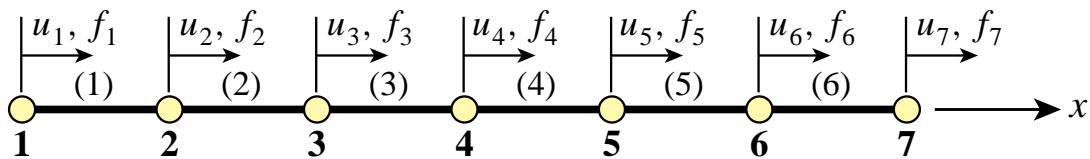
Introduction to FEM

10

MultiFreedom Constraints II

Penalty Function Method Physical Interpretation

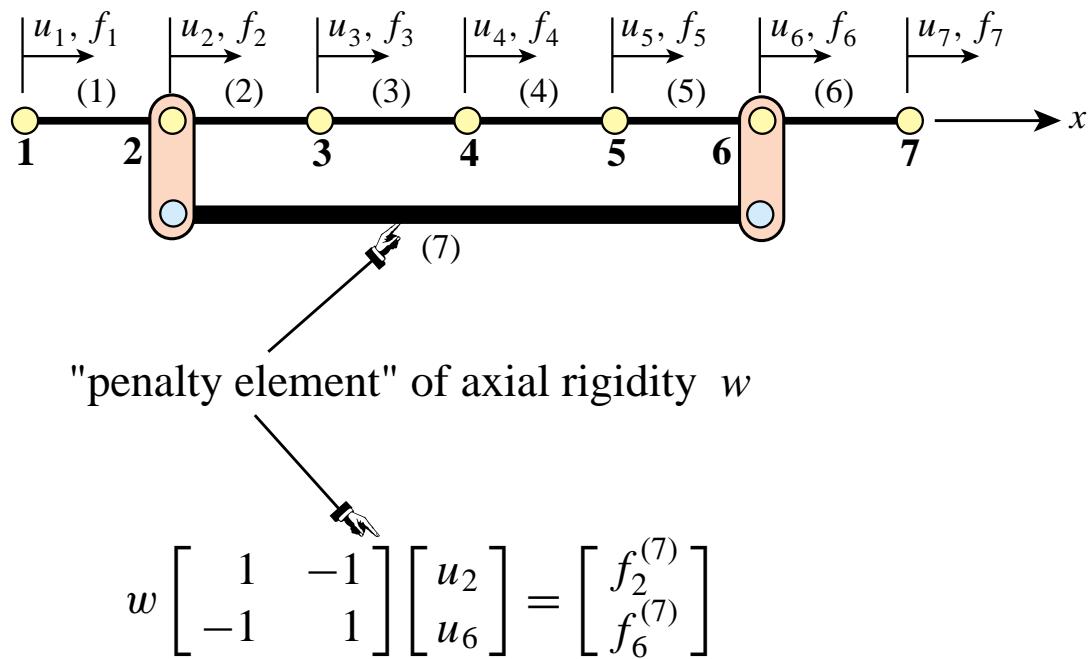
Recall the example structure



under the homogeneous MFC

$$u_2 = u_6$$

Penalty Function Method (cont'd)



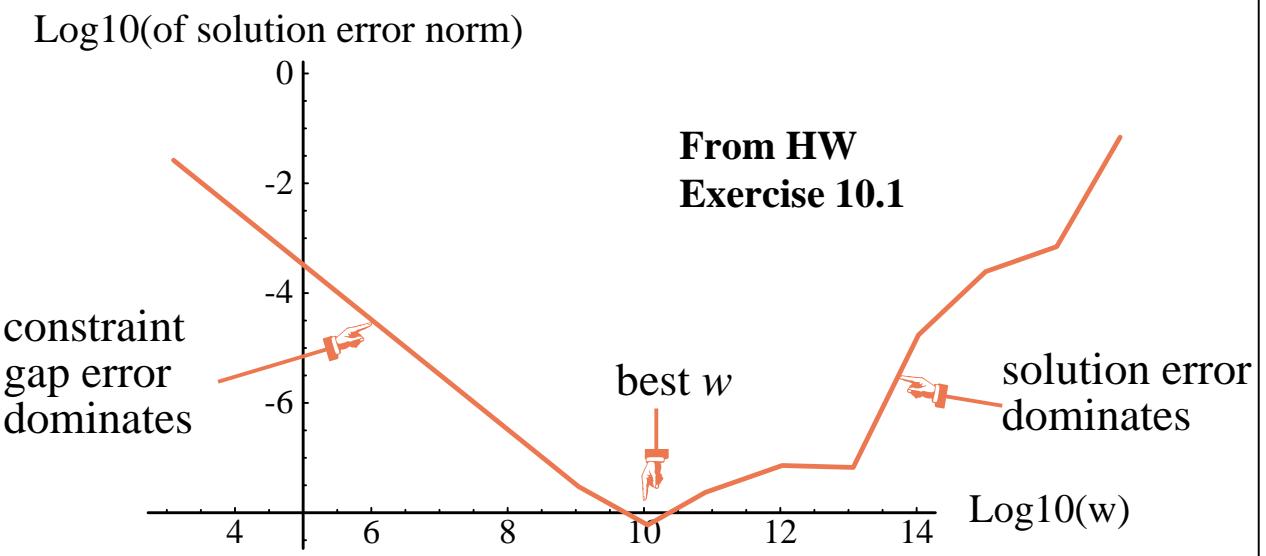
Penalty Function Method (cont'd)

Upon merging the penalty element the modified stiffness equations are

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} + w & K_{23} & 0 & 0 & -w & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & -w & 0 & 0 & K_{56} & K_{66} + w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This modified system is submitted to the equation solver.
Note that **u** retains the same arrangement of DOFs.

But which penalty weight to use?



Rough guideline: "square root rule"; see Notes.

Introduction to FEM

Penalty Function Method - General MFCs

$$3u_3 + u_5 - 4u_6 = 1$$

$$[3 \quad 1 \quad -4] \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = 1$$

Premultiply both sides by

$$[3 \quad 1 \quad -4]^T \begin{bmatrix} 9 & 3 & -12 \\ 3 & 1 & -4 \\ -12 & -4 & 16 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

"Penalty element" stiffness equations

Scale by w and merge:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} + 9w & K_{34} & 3w & -12w & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 3w & K_{45} & K_{55} + w & K_{56} - 4w & 0 \\ 0 & 0 & -12w & 0 & K_{56} - 4w & K_{66} + 16w & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 + 3w \\ f_4 \\ f_5 + w \\ f_6 - 4w \\ f_7 \end{bmatrix}$$

Assessment of Penalty Function Method

😊 ADVANTAGES

- general application (inc' nonlinear MFCs)**
- easy to implement using FE library and standard assembler**
- no change in vector of unknowns**
- retains positive definiteness**
- insensitive to constraint dependence**

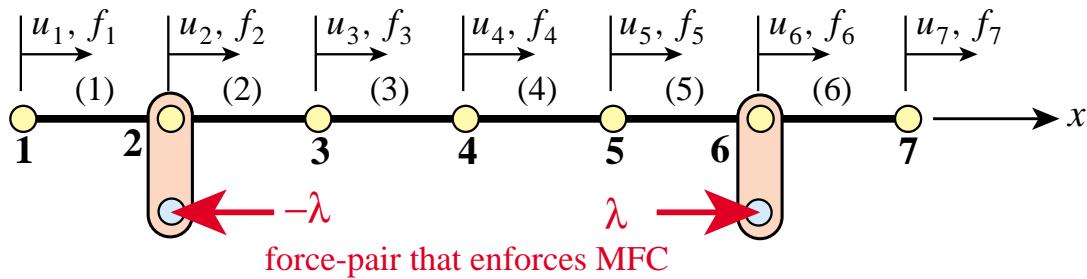
😢 DISADVANTAGES

- selection of weight left to user**
- accuracy limited by ill-conditioning**

Introduction to FEM

Lagrange Multiplier Method

Physical Interpretation



$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \lambda \\ f_3 \\ f_4 \\ f_5 \\ f_6 + \lambda \\ f_7 \end{bmatrix}$$

Lagrange Multiplier Method (cont'd)

Because λ is unknown, it is passed to the LHS and appended to the node-displacement vector:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

This is now a system of 7 equations and 8 unknowns.
Needs an extra equation: the MFC.

Lagrange Multiplier Method (cont'd)

Append MFC as additional equation:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \end{bmatrix}$$

This is the ***multiplier-augmented system***. The new coefficient matrix is called the ***bordered stiffness***.

Lagrange Multiplier Method - Multiple MFCs

Three MFCs:

$$u_2 - u_6 = 0, \quad 5u_2 - 8u_7 = 3, \quad 3u_3 + u_5 - 4u_6 = 1$$

Recipe step #1:
append the
3 constraints

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Recipe step #2:
append multipliers,
symmetrize
and fill

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 & -8 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

Assessment of Lagrange Multiplier Method



ADVANTAGES

- general application**
- exact**
- no user decisions ("black box")**



DISADVANTAGES

- difficult implementation**
- additional unknowns**
- loses positive definiteness**
- sensitive to constraint dependence**

MFC Application Methods: Assessment Summary

Introduction to FEM

	Master-Slave Elimination	Penalty Function	Lagrange Multiplier
Generality	fair	excellent	excellent
Ease of implementation	poor to fair	good	fair
Sensitivity to user decisions	high	high	small to none
Accuracy	variable	mediocre	excellent
Sensitivity as regards constraint dependence	high	none	high
Retains positive definiteness	yes	yes	no
Modifies unknown vector	yes	no	yes