

Introduction to FEM

9

MultiFreedom Constraints I

Multifreedom Constraints

Single freedom constraint examples

$$u_{x4} = 0 \quad \text{linear, homogeneous}$$

$$u_{y9} = 0.6 \quad \text{linear, non-homogeneous}$$

Multifreedom constraint examples

$$u_{x2} = \frac{1}{2}u_{y2} \quad \text{linear, homogeneous}$$

$$u_{x2} - 2u_{x4} + u_{x6} = 0.25 \quad \text{linear, non-homogeneous}$$

$$(x_5 + u_{x5} - x_3 - u_{x3})^2 + (y_5 + u_{y5} - y_3 - u_{y3})^2 = 0$$

nonlinear, homogeneous

Sources of Multifreedom Constraints

"Skew" displacement BCs

Coupling nonmatched FEM meshes

Global-local and multiscale analysis

Incompressibility

Model reduction

Introduction to FEM

MFC Application Methods

Master-Slave Elimination

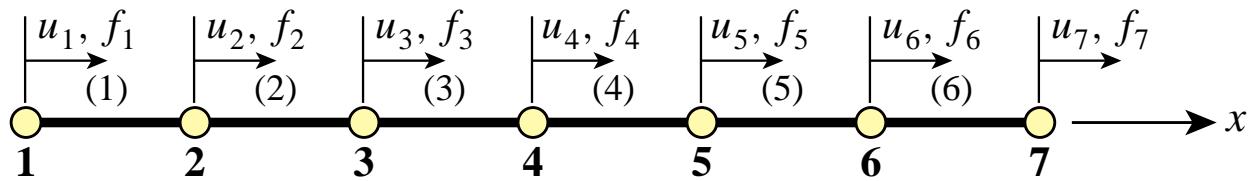
Chapter 9

Penalty Function Augmentation

Lagrange Multiplier Adjunction

Chapter 10

Example 1D Structure to Illustrate MFCs

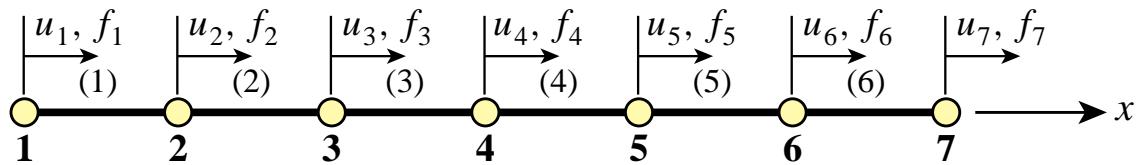


Multifreedom constraint:

$$u_2 = u_6 \quad \text{or} \quad u_2 - u_6 = 0$$

Linear homogeneous MFC

Example 1D Structure (Cont'd)



Unconstrained master stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

or

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

Master-Slave Method for Example Structure

Recall:

$$u_2 = u_6 \quad \text{or} \quad u_2 - u_6 = 0$$

Taking u_2 as master:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}$$

or

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

Forming the Modified Stiffness Equations

**Unconstrained master
stiffness equations:**

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

Master-slave transformation:

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

Congruential transformation:

$$\hat{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T}$$

$$\hat{\mathbf{f}} = \mathbf{T}^T \mathbf{f}$$

Modified stiffness equations:

$$\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

Modified Stiffness Equations for Example Structure

$$\hat{\mathbf{K}} \hat{\mathbf{u}} = \hat{\mathbf{f}}$$

In full

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} + K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 \\ f_3 \\ f_4 \\ f_5 \\ f_7 \end{bmatrix}$$

Solve for $\hat{\mathbf{u}}$, then recover $\mathbf{u} = \mathbf{T} \hat{\mathbf{u}}$

Multiple MFCs

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Suppose

$$u_2 - u_6 = 0, \quad u_1 + 4u_4 = 0, \quad 2u_3 + u_4 + u_5 = 0$$

Pick 3, 4 and 6 as slaves:

$$u_6 = u_2, \quad u_4 = -\frac{1}{4}u_1, \quad u_3 = -\frac{1}{2}(u_4 + u_5) = \frac{1}{8}u_1 - \frac{1}{2}u_5$$

Put in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \end{bmatrix}$$

This is $\mathbf{u} = \mathbf{T} \hat{\mathbf{u}}$ - then proceed as before

Non-homogeneous MFCs

$$u_2 - u_6 = 0.2$$

Pick again u_6 as slave, put into matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.2 \\ 0 \end{bmatrix}$$

Nonhomogeneous MFCs (cont'd)

Introduction to FEM

$$\mathbf{u} = \mathbf{T} \hat{\mathbf{u}} + \mathbf{g} \quad \mathbf{g} = \text{"gap" vector}$$

Premultiply both sides by $\mathbf{T}^T \mathbf{K}$, replace $\mathbf{K} \mathbf{u} = \mathbf{f}$ and pass data to RHS. This gives

$$\hat{\mathbf{K}} \hat{\mathbf{u}} = \hat{\mathbf{f}}$$

with

$$\hat{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T} \quad \text{and} \quad \hat{\mathbf{f}} = \mathbf{T}^T (\mathbf{f} - \mathbf{K} \mathbf{g})$$

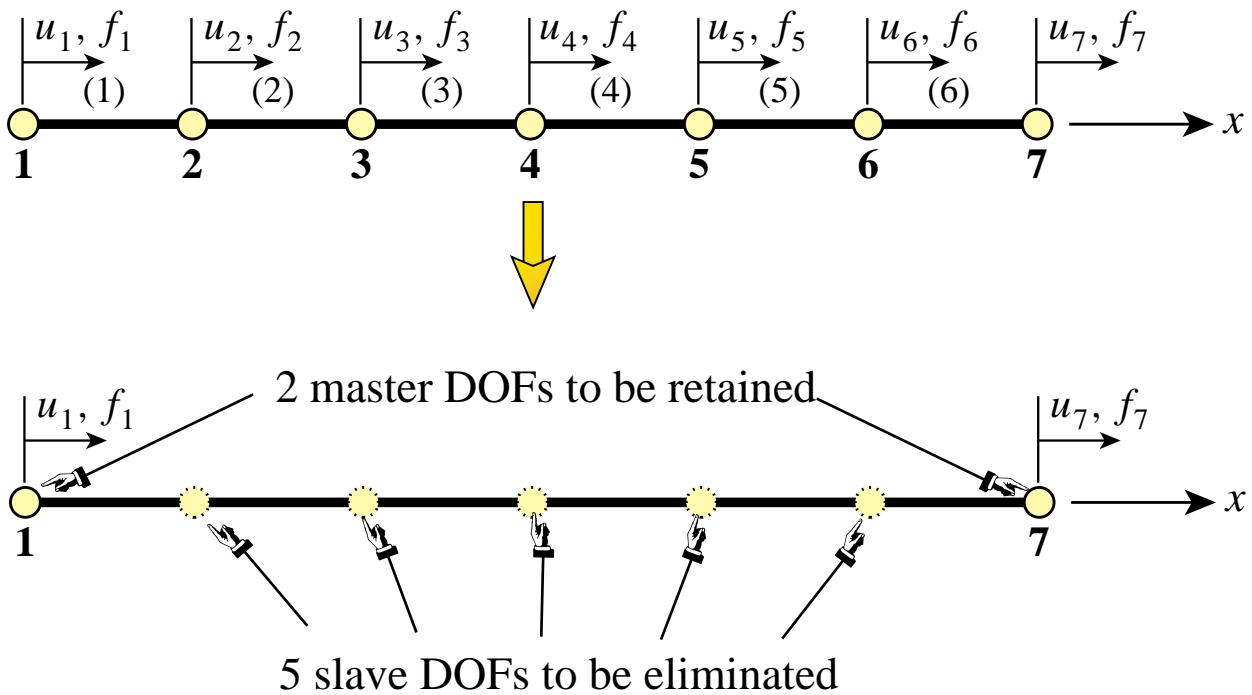
a modified force vector

For the example structure

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} + K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 - 0.2K_{66} \\ f_3 \\ f_4 \\ f_5 - 0.2K_{56} \\ f_7 - 0.2K_{67} \end{bmatrix}$$

Solve for $\hat{\mathbf{u}}$, then recover $\mathbf{u} = \mathbf{T} \hat{\mathbf{u}} + \mathbf{g}$

Model Reduction Example



Model Reduction Example (cont'd)

Lots of slaves, few masters. Only masters are left. Example of previous slide:

$$\text{5 slaves} \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5/6 & 1/6 \\ 4/6 & 2/6 \\ 3/6 & 3/6 \\ 2/6 & 4/6 \\ 1/6 & 5/6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \end{bmatrix} \rightarrow \text{2 masters}$$

Applying the congruent transformation we get the reduced stiffness equations

$$\begin{bmatrix} \hat{K}_{11} & \hat{K}_{17} \\ \hat{K}_{17} & \hat{K}_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_7 \end{bmatrix}$$

where

$$\hat{K}_{11} = \frac{1}{36}(36K_{11}+60K_{12}+25K_{22}+40K_{23}+16K_{33}+24K_{34}+9K_{44}+12K_{45}+4K_{55}+4K_{56}+K_{66})$$

$$\hat{K}_{17} = \frac{1}{36}(6K_{12}+5K_{22}+14K_{23}+8K_{33}+18K_{34}+9K_{44}+18K_{45}+8K_{55}+14K_{56}+5K_{66}+6K_{67})$$

$$\hat{K}_{77} = \frac{1}{36}(K_{22}+4K_{23}+4K_{33}+12K_{34}+9K_{44}+24K_{45}+16K_{55}+40K_{56}+25K_{66}+60K_{67}+36K_{77})$$

$$\hat{f}_1 = \frac{1}{6}(6f_1+5f_2+4f_3+3f_4+2f_5+f_6), \quad \hat{f}_7 = \frac{1}{6}(f_2+2f_3+3f_4+4f_5+5f_6+6f_7).$$

Model Reduction Example: Mathematica Script

```
(* Model Reduction Example *)
ClearAll[K11,K12,K22,K23,K33,K34,K44,K45,K55,K56,K66,
f1,f2,f3,f4,f5,f6];
K={{K11,K12,0,0,0,0},{K12,K22,K23,0,0,0,0},
{0,K23,K33,K34,0,0,0},{0,0,K34,K44,K45,0,0},
{0,0,0,K45,K55,K56,0},{0,0,0,0,K56,K66,K67},
{0,0,0,0,0,K67,K77}}; Print["K=",K//MatrixForm];
f={f1,f2,f3,f4,f5,f6,f7}; Print["f=",f];
T={{6,0},{5,1},{4,2},{3,3},{2,4},{1,5},{0,6}}/6;
Print["Transformation matrix T=",T//MatrixForm];
Khat=Simplify[Transpose[T].K.T];
fhat=Simplify[Transpose[T].f];
Print["Modified Stiffness:"];
Print["Khat(1,1)=",Khat[[1,1]],"\nKhat(1,2)=",Khat[[1,2]],
"\nKhat(2,2)=",Khat[[2,2]]];
Print["Modified Force:"];
Print["fhat(1)=",fhat[[1]]," fhat(2)=",fhat[[2]]];
```

Modified Stiffness:

(Some print output removed so slide fits)

$$\begin{aligned} Khat(1,1) &= \frac{1}{36} (36 K_{11} + 60 K_{12} + 25 K_{22} + 40 K_{23} + 16 K_{33} + 24 K_{34} + 9 K_{44} + 12 K_{45} + 4 K_{55} + 4 K_{56} + K_{66}) \\ Khat(1,2) &= \frac{1}{36} (6 K_{12} + 5 K_{22} + 14 K_{23} + 8 K_{33} + 18 K_{34} + 9 K_{44} + 18 K_{45} + 8 K_{55} + 14 K_{56} + 5 K_{66} + 6 K_{67}) \\ Khat(2,2) &= \frac{1}{36} (K_{22} + 4 K_{23} + 4 K_{33} + 12 K_{34} + 9 K_{44} + 24 K_{45} + 16 K_{55} + 40 K_{56} + 25 K_{66} + 60 K_{67} + 36 K_{77}) \end{aligned}$$

Modified Force:

$$fhat(1) = \frac{1}{6} (6 f_1 + 5 f_2 + 4 f_3 + 3 f_4 + 2 f_5 + f_6) \quad fhat(2) = \frac{1}{6} (f_2 + 2 f_3 + 3 f_4 + 4 f_5 + 5 f_6 + 6 f_7)$$

Introduction to FEM

Assessment of Master-Slave Method



ADVANTAGES

- exact if precautions taken**
- easy to understand**
- retains positive definiteness**
- important applications to model reduction**



DISADVANTAGES

- requires user decisions**
- messy implementation for general MFCs**
- hinders sparsity of master stiffness equations**
- sensitive to constraint dependence**
- restricted to linear constraints**