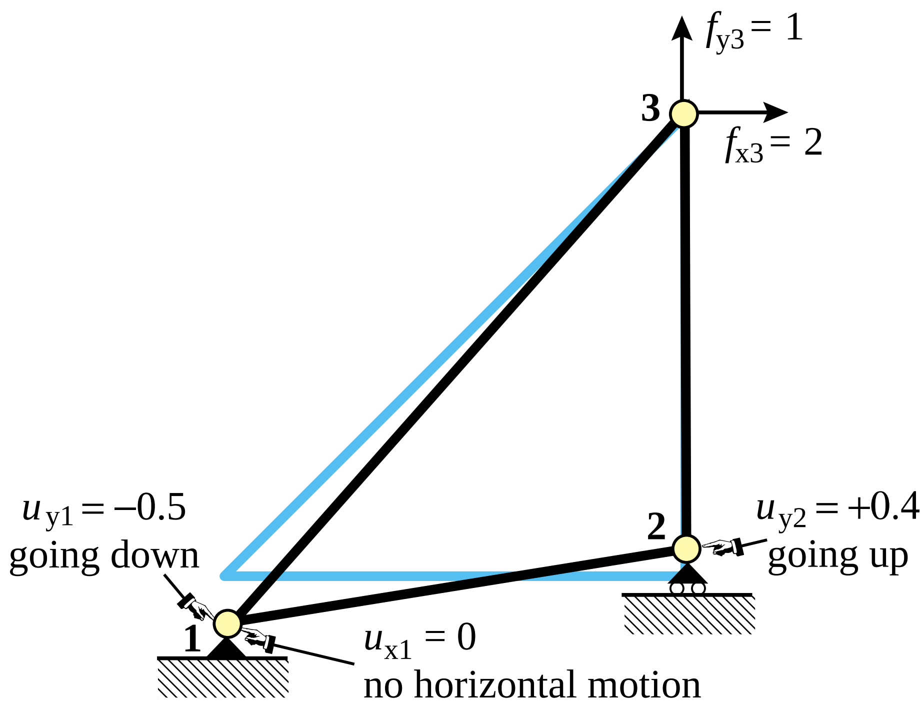
Introduction to FEM

4

The Direct Stiffness Method: Additional Topics

Prescribed Nonzero Displacements in Example Truss



Prescribed Nonzero Displacements

Recall the master stiffness equations

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}$$

The displacement B.Cs are now

$$\boxed{u_{x1} = 0, \quad u_{y1} = -0.5, \quad u_{y2} = 0.4}$$

Introduction to FEM

Prescribed NZ Displacements (cont'd)

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 0 \\ f_{y2} \\ 2 \\ 1 \end{bmatrix}$$

Remove rows 1,2,4 but (for now) keep columns

$$\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

Pass the effect of known displacements to RHS, and delete columns

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} -$$

$$\begin{bmatrix} (-10) \times 0 + 0 \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + (-5) \times 0.4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$$

Solving gives

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Complete the displacement vector with known values

$$\mathbf{u} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.4 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

Recovery of reaction forces and internal member forces proceeds as before

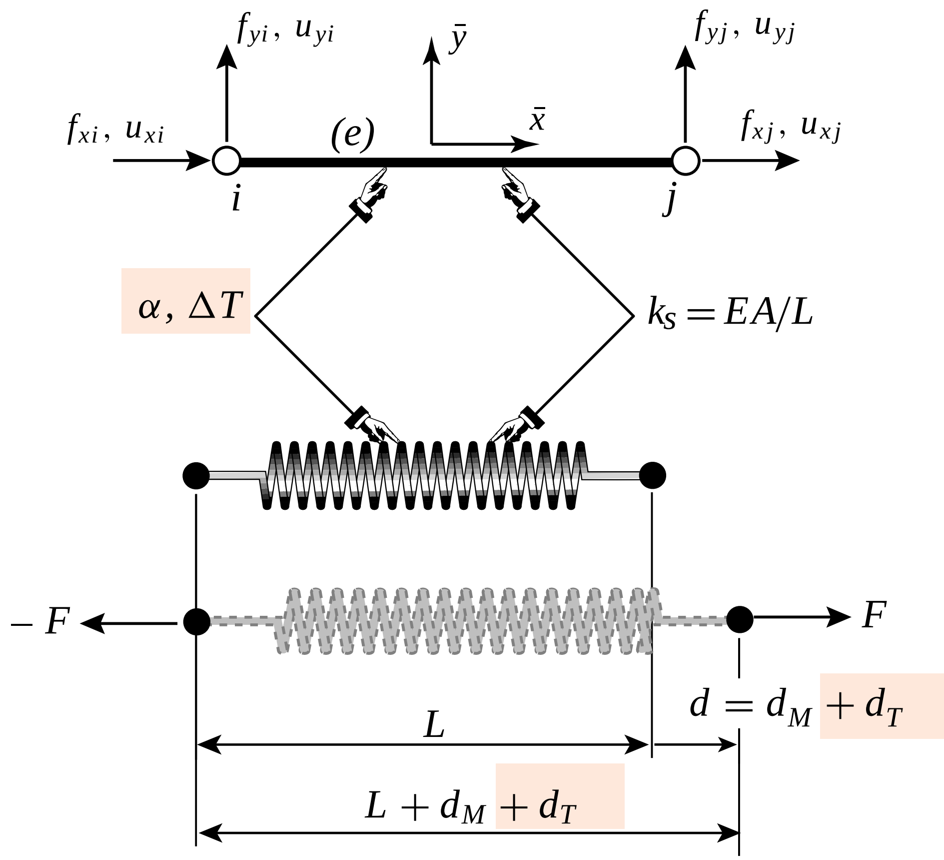
In summary, the only changes to the DSM is in the application of displacement boundary conditions before solve

Initial Force Effects (also called Initial Strain & Initial Stress Effects by FEM authors)

- **Thermomechanical effects**
- **Moisture effects**
- **Prestress effects**
- **Lack of fit**
- **Residual stresses**

For specificity we will study in detail only the first one: ***thermomechanical*** effects

Thermomechanical Effects on Bar Element



Thermomechanical Effects on Bar Element (Cont'd)

Axial strain is sum of mechanical and thermal:

$$e_T = d_T / L = \alpha \, \Delta T$$

$$e = e_M + e_T = \frac{\sigma}{E} + \alpha \, \Delta T$$

Thermomechanical Effects on Bar Element (cont'd)

$$e = d/L \qquad d = \bar{u}_{xj} - \bar{u}_{xi}$$

$$\frac{\bar{u}_{xj} - \bar{u}_{xi}}{L} = \frac{\sigma}{E} + \alpha\,\Delta T$$

$$\frac{EA}{L}(\bar{u}_{xj} - \bar{u}_{xi}) = A\sigma + EA\alpha\,\Delta T = p_M + p_T = F$$

$$F = \frac{EA}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix}$$

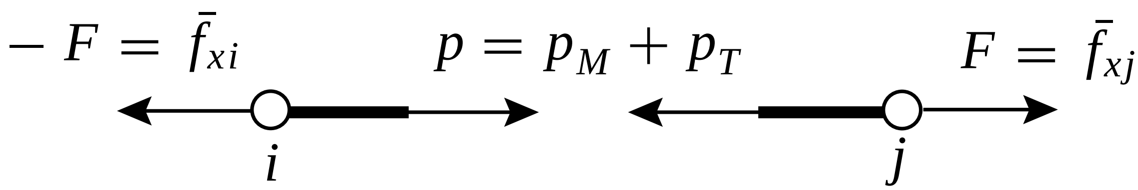
Incorporating Thermomechanical Effects into the Element Stiffness Equations

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad \bar{f}_{yi} = \bar{f}_{yj} = 0$$

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{f}_{Mxi} \\ \bar{f}_{Myi} \\ \bar{f}_{Mxj} \\ \bar{f}_{Myj} \end{bmatrix} + \begin{bmatrix} \bar{f}_{Txi} \\ \bar{f}_{Tyi} \\ \bar{f}_{Txj} \\ \bar{f}_{Tyj} \end{bmatrix} = \begin{bmatrix} \bar{f}_{Mxi} \\ \bar{f}_{Myi} \\ \bar{f}_{Mxj} \\ \bar{f}_{Myj} \end{bmatrix} + EA\ \alpha \Delta T \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

Physical Interpretation of Thermal Force Vector



Matrix Form of Element Stiffness Equations

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}_M + \bar{\mathbf{f}}_T = \bar{\mathbf{K}}\,\bar{\mathbf{u}}$$

or

$$\bar{\mathbf{K}}\,\bar{\mathbf{u}} = \bar{\mathbf{f}}_M + \bar{\mathbf{f}}_T$$

Here $\bar{\mathbf{f}}_T$ is the element thermal force vector

$$\bar{\mathbf{f}}_T = E A \alpha \Delta T \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Assembly Rules with Thermomechanical Effects

1. *Compatibility*: The joint displacements of all members meeting at a joint must be the same

2. *Equilibrium*: The sum of *effective forces* exerted by all members that meet at a joint must balance the external force applied to that joint.

No change in application of 1. To account for 2, the thermal forces are globalized and added to the mechanical forces during the merge process

Master Stiffness Equations with Thermomechanical Effects

$$\mathbf{K}\mathbf{u} = \mathbf{f}_M + \mathbf{f}_T = \mathbf{f}$$

effective force vector

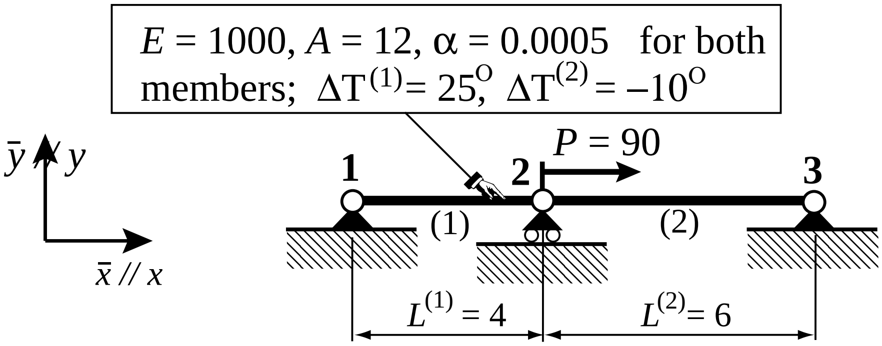
Solve for node displacements **u.** If it is desired to retrieve the mechanical node forces including reactions:

$$\mathbf{f}_M = \mathbf{K}\mathbf{u} - \mathbf{f}_T$$

For computation of internal forces and stresses see worked out Examples

Introduction to FEM

Worked Out Example #1 in Notes



Thermal forces

$$\bar{\mathbf{f}}_T^{(1)} = \begin{bmatrix} \bar{f}_{T1}^{(1)} \\ \bar{f}_{T2}^{(1)} \end{bmatrix} = E^{(1)}A^{(1)}\alpha^{(1)}\Delta T^{(1)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -150 \\ 150 \end{bmatrix}$$

$$\bar{\mathbf{f}}_T^{(2)} = \begin{bmatrix} \bar{f}_{T2}^{(2)} \\ \bar{f}_{T3}^{(2)} \end{bmatrix} = E^{(2)}A^{(2)}\alpha^{(2)}\Delta T^{(2)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 60 \\ -60 \end{bmatrix}$$

Element stiffness equations

$$3000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(1)} \\ \bar{u}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \bar{f}_{M1}^{(1)} \\ \bar{f}_{M2}^{(1)} \end{bmatrix} + \begin{bmatrix} -150 \\ 150 \end{bmatrix}$$

$$2000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_2^{(2)} \\ \bar{u}_3^{(2)} \end{bmatrix} = \begin{bmatrix} \bar{f}_{M2}^{(2)} \\ \bar{f}_{M3}^{(2)} \end{bmatrix} + \begin{bmatrix} 60 \\ -60 \end{bmatrix}$$

Worked Out Example #1 in Notes (cont'd)

Assembled master stiffness equations

$$1000\begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ f_{M3} \end{bmatrix} + \begin{bmatrix} -150 \\ 150+60 \\ -60 \end{bmatrix} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ f_{M3} \end{bmatrix} + \begin{bmatrix} -150 \\ 210 \\ -60 \end{bmatrix}$$

Reduced stiffness equations upon applying BCs:

$$5000\,u_2 = f_{M2} + 210 = 90 + 210 = 300$$

Solution for unknown displacement

$$u_2 = 300/5000 = +0.06$$

Recovery of internal member forces

$$p^{(1)} = \frac{E^{(1)}A^{(1)}}{L^{(1)}}(u_2 - u_1) - E^{(1)}A^{(1)}\alpha^{(1)}\Delta T^{(1)} = 3000 \times 0.06 - 12000 \times 0.0005 \times 25 = 60$$

$$p^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}}(u_3 - u_2) - E^{(2)}A^{(2)}\alpha^{(2)}\Delta T^{(2)} = 2000 \times (-0.06) - 12000 \times 0.0005 \times (-10) = -72$$

Member stresses

$$\sigma^{(1)} = 60/12 = 5 \qquad \text{(T)}$$

$$\sigma^{(2)} = -72/12 = -6 \qquad \text{(C)}$$

Generalization: Initial Force Effects in the DSM

$$\mathbf{Ku} = \mathbf{f}_M + \mathbf{f}_I = \mathbf{f}$$

initial force vector effective force vector

Where does $\mathbf{f}_I$ come from? Thermal effects, moisture, prestress, lack of fit, residual stresses, some nonlinearities.

Common property: if displacements $\mathbf{u}$ vanish

$$\mathbf{f}_M + \mathbf{f}_I = \mathbf{0} \qquad \mathbf{f}_M = -\mathbf{f}_I$$

There are (self-equilibrating) *mechanical forces in the absence of displacements*

Summary: Treating Initial Force Effects in the DSM

Disconnection
Localization
Member (Element) Relations ☜—— Include $\bar{f}_I^{(e)}$

Globalization ☜—— Transform to $f_I^{(e)}$
Merge ☜—— Assemble into f_I
Application of BCs
Solution
Recovery of Derived Quantities ☜—— Subtract $\bar{f}_I^{(e)}$