# The Direct 

 Stiffness Method: Assembly and Solution
# The Direct Stiffness Method (DSM) Steps (recalled for convenience) 

| Breakdown <br> $($ Chapter 2) | $\left\{\begin{array}{l}\text { Disconnection } \\ \text { Localization } \\ \text { Member (Element) Formation }\end{array}\right.$ |
| :---: | :--- |

Assembly \& Solution (Chapter 3)

conceptual steps

| processing | post-processing |
| :--- | :--- |
| steps | steps |

# Globalization Step: <br> Displacement Transformation 



Node displacements transform as

$$
\begin{array}{ll}
\bar{u}_{x i}=u_{x i} c+u_{y i} s, & \bar{u}_{y i}=-u_{x i} s+u_{y i} c \gamma \\
\bar{u}_{x j}=u_{x j} c+u_{y j} s, & \bar{u}_{y j}=-u_{x j} s+u_{y j} c \gamma
\end{array}
$$

in which $c=\cos \varphi \quad s=\sin \varphi$

# Globalization Step: Displacement Transformation (cont'd) 

## In matrix form

$$
\begin{aligned}
& {\left[\begin{array}{l}
\bar{u}_{x i} \\
\bar{u}_{y i} \\
\bar{u}_{x j} \\
\bar{u}_{y j}
\end{array}\right]=\left[\begin{array}{cccc}
c & s & 0 & 0 \\
-s & c & 0 & 0 \\
0 & 0 & c & s \\
0 & 0 & -s & c
\end{array}\right]\left[\begin{array}{l}
u_{x i} \\
x_{y i} \\
u_{x j} \\
u_{y j}
\end{array}\right] } \\
\text { or } & \overline{\mathbf{u}}^{(e)}=\mathbf{T}^{(e)} \mathbf{u}^{(e)}
\end{aligned}
$$

Note:
global on RHS, local on LHS

# Globalization Step: Force Transformation 



Node forces transform as

$$
\begin{gathered}
{\left[\begin{array}{c}
f_{x i} \\
f_{y i} \\
f_{x j} \\
f_{y j}
\end{array}\right]=\left[\begin{array}{cccc}
c & -s & 0 & 0 \\
s & c & 0 & 0 \\
0 & 0 & c & -s \\
0 & 0 & s & c
\end{array}\right]\left[\begin{array}{c}
\bar{f}_{x i} \\
\bar{f}_{y i} \\
\bar{f}_{x j} \\
\bar{f}_{y j}
\end{array}\right]} \\
\mathbf{f}^{(e)}=\left(\mathbf{T}^{(e)}\right)^{T} \mathbf{f}^{(e)}
\end{gathered}
$$

## Globalization: Congruential Transformation of Element Stiffness Matrices

$$
\begin{gathered}
\overline{\mathbf{K}}^{(e)} \mathbf{\mathbf { u }}^{(e)}=\overline{\mathbf{f}}^{(e)} \\
\overline{\mathbf{u}}^{(e)}=\mathbf{T}^{(e)} \mathbf{u}^{(e)} \quad \mathbf{f}^{(e)}=\left(\mathbf{T}^{(e)}\right)^{T} \overline{\mathbf{f}}^{(e)} \\
\mathbf{K}^{(e)}=\left(\mathbf{T}^{(e)}\right)^{T} \overline{\mathbf{K}}^{(e)} \mathbf{T}^{(e)} \\
\mathbf{K}^{(e)}=\frac{E^{(e)} A^{(e)}\left[\begin{array}{cccc}
c^{2} & s c & -c^{2} & -s c \\
L^{(e)} & s^{2} & -s c & -s^{2} \\
-c^{2} & -s c & c^{2} & s c \\
-s c & -s^{2} & s c & s^{2}
\end{array}\right]}{}
\end{gathered}
$$

## The Example Truss - FEM Model



## Globalized Element Stiffness Equations for Example Truss

$$
\begin{aligned}
& {\left[\begin{array}{l}
f_{x 1}^{(1)} \\
f_{y 1}^{(1)} \\
f_{x 1}^{(1)} \\
f_{y 2}^{(1)}
\end{array}\right]=10\left[\begin{array}{cccc}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
u_{x 1}^{(1)} \\
u_{y 1}^{(1)} \\
u_{x(12}^{(1)} \\
u_{y 2}^{(1)}
\end{array}\right]} \\
& {\left[\begin{array}{l}
f_{x 2}^{(2)} \\
f_{y 2}^{(2)} \\
f_{x 3}^{(2)} \\
f_{y 3}^{(2)}
\end{array}\right]=5\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{x 2}^{(2)} \\
u_{y 2}^{(2)} \\
u_{x 3}^{(2)} \\
u_{y 3}^{(2)}
\end{array}\right]}
\end{aligned}
$$

$$
\left[\begin{array}{l}
f_{x}^{(3)} \\
f_{y}^{(3)} \\
f_{x 3}^{(3)} \\
f_{y 3}^{(3)}
\end{array}\right]=20\left[\begin{array}{cccc}
0.5 & 0.5 & -0.5 & -0.5 \\
0.5 & 0.5 & -0.5 & -0.5 \\
-0.5 & -0.5 & 0.5 & 0.5 \\
-0.5 & -0.5 & 0.5 & 0.5
\end{array}\right]\left[\begin{array}{l}
u_{x 1}^{(3)} \\
u_{y 1}^{(3)} \\
u_{x 3}^{(3)} \\
u_{y 3}^{(3)}
\end{array}\right]
$$

## Assembly Rules

1. Compatibility: The joint displacements of all members meeting at a joint must be the same
2. Equilibrium: The sum of forces exerted by all members that meet at a joint must balance the external force applied to that joint.

To apply these rules in assembly by hand, it is convenient to augment the element stiffness equations as shown for the example truss in the next slide.

Expanded Element Stiffness Equations of Example Truss

$$
\begin{aligned}
& {\left[\begin{array}{l}
f_{x 1}^{(1)} \\
f_{y 1}^{(1)} \\
f_{x 2}^{(1)} \\
f_{y 2}^{(1)} \\
f_{x 3}^{(1)} \\
f_{y 3}^{(1)}
\end{array}\right]=\left[\begin{array}{cccccc}
10 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
u_{x 1}^{(1)} \\
u_{x 1}^{(1)} \\
u_{x 2}^{(1)} \\
u_{y 2}^{(1)} \\
u_{x 3}^{(1)} \\
u_{y 3}^{(1)}
\end{array}\right]} \\
& {\left[\begin{array}{l}
f_{x}^{(2)} \\
f_{y 1}^{(2)} \\
f_{x 2}^{(2)} \\
f_{y 2}^{(2)} \\
f_{x 3}^{(2)} \\
f_{y 3}^{(2)}
\end{array}\right]=\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 0 & 5
\end{array}\right]\left[\begin{array}{l}
u_{x 1}^{(2)} \\
u_{y 1}^{(2)} \\
u_{x 2}^{(2)} \\
u_{y 2}^{(2)} \\
u_{x 3}^{(2)} \\
u_{y 3}^{(2)}
\end{array}\right]} \\
& {\left[\begin{array}{l}
f_{x 1}^{(3)} \\
f_{y 1}^{(3)} \\
f_{x 2}^{(3)} \\
f_{y 2}^{(3)} \\
f_{x 3}^{(3)} \\
f_{y 3}^{(3)}
\end{array}\right]=\left[\begin{array}{cccccc}
10 & 10 & 0 & 0 & -10 & -10 \\
10 & 10 & 0 & 0 & -10 & -10 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-10 & -10 & 0 & 0 & 10 & 10 \\
-10 & -10 & 0 & 0 & 10 & 10
\end{array}\right]\left[\begin{array}{l}
u_{x 1}^{(3)} \\
u_{y 1}^{(3)} \\
u_{x 2}^{(3)} \\
u_{y 2}^{(3)} \\
u_{y 2}^{(3)} \\
u_{x 3}^{(3)} \\
u_{y 3}^{(3)}
\end{array}\right]}
\end{aligned}
$$

## Reconnecting Members by Enforcing Compatibility Rule



## Next, Apply Equilibrium Rule



Be careful with + directions of internal forces!


Applying this to all joints (see Notes):

$$
\mathbf{f}=\mathbf{f}^{(1)}+\mathbf{f}^{(2)}+\mathbf{f}^{(3)}
$$

## Forming the Master Stiffness Equations through Equilibrium Rule

$$
\mathbf{f}=\mathbf{f}^{(1)}+\mathbf{f}^{(2)}+\mathbf{f}^{(3)}=\left(\mathbf{K}^{(1)}+\mathbf{K}^{(2)}+\mathbf{K}^{(3)}\right) \mathbf{u}=\mathbf{K} \mathbf{u}
$$

$$
\left[\begin{array}{l}
f_{x 1} \\
f_{y 1} \\
f_{x 2} \\
f_{y 2} \\
f_{x 3} \\
f_{y 3}
\end{array}\right]=\left[\begin{array}{cccccc}
20 & 10 & -10 & 0 & -10 & -10 \\
10 & 10 & 0 & 0 & -10 & -10 \\
-10 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & -5 \\
-10 & -10 & 0 & 0 & 10 & 10 \\
-10 & -10 & 0 & -5 & 10 & 15
\end{array}\right]\left[\begin{array}{l}
u_{x 1} \\
u_{y 1} \\
u_{x 2} \\
u_{y 2} \\
u_{x 3} \\
u_{y 3}
\end{array}\right]
$$

# Applying Support and Loading Boundary Conditions to Example Truss 



Displacement BCs:
$u_{x 1}=u_{y 1}=u_{y 2}=0$
Force BCs:

$$
f_{x 2}=0, \quad f_{x 3}=2, \quad f_{y 3}=1
$$

## Where Do Boundary Conditions Go?

$$
\left[\begin{array}{cccccc}
20 & 10 & -10 & 0 & -10 & -10 \\
10 & 10 & 0 & 0 & -10 & -10 \\
-10 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & -5 \\
-10 & -10 & 0 & 0 & 10 & 10 \\
-10 & -10 & 0 & -5 & 10 & 15
\end{array}\right]\left[\begin{array}{l}
u_{x 1} \\
u_{y 1} \\
u_{x 2} \\
u_{y 2} \\
u_{x 3} \\
u_{y 3}
\end{array}\right]=\left[\begin{array}{c}
f_{x 1} \\
f_{y 1} \\
f_{x 2} \\
f_{y 2} \\
f_{x 3} \\
f_{y 3}
\end{array}\right.
$$

## Reduced Master Stiffness Equations for Hand Computation

Strike out rows and columns pertaining to known displacements:

$$
\left[\begin{array}{ccc}
10 & 0 & 0 \\
0 & 10 & 10 \\
0 & 10 & 15
\end{array}\right]\left[\begin{array}{l}
u_{x 2} \\
u_{x 3} \\
u_{y 3}
\end{array}\right]=\left[\begin{array}{l}
f_{x 2} \\
f_{x 3} \\
f_{y 3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
2 \\
1
\end{array}\right]
$$

or


> Reduced stiffness equations

Solve by Gauss elimination for unknown node displacements

## Solve for Unknown Node Displacements and Complete the Displacement Vector

$$
\left[\begin{array}{l}
u_{x 2} \\
u_{x 3} \\
u_{y 3}
\end{array}\right]=\left[\begin{array}{c}
0 \\
0.4 \\
-0.2
\end{array}\right]
$$

- Expand with known displacement BCs
$\mathbf{u}=\left[\begin{array}{c}0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2\end{array}\right]$


## Recovery of Node Forces Including Reactions

$$
\mathbf{f}=\mathbf{K u}=\left[\begin{array}{cccccc}
20 & 10 & -10 & 0 & -10 & -10 \\
10 & 10 & 0 & 0 & -10 & -10 \\
-10 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & -5 \\
-10 & -10 & 0 & 0 & 10 & 10 \\
-10 & -10 & 0 & -5 & 10 & 15
\end{array}\right]\left[\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0.4 \\
-0.2
\end{array}\right]=\begin{gathered}
{\left[\begin{array}{c}
-2 \\
-2 \\
0 \\
1 \\
2 \\
1
\end{array}\right]} \\
\hline
\end{gathered}
$$



## Recovery of Internal Forces (Axial Forces in Truss Members)

For each member (element) $(e)=(1),(2),(3)$


1. extract $\mathbf{u}^{(e)}$ from $\mathbf{u}$
2. transform to local (element) displacements

$$
\overline{\mathbf{u}}^{(e)}=\mathbf{T}^{(e)} \mathbf{u}^{(e)}
$$

3. compute elongation $d^{(e)}=\bar{u}_{x j}^{(e)}-\bar{u}_{x i}^{(e)}$
4. compute axial force $p^{(e)}=\frac{E^{(e)} A^{(e)}}{L^{(e)}} d^{(e)}$

# Computer Oriented Assembly and Solution in Actual FEM Codes 

K stored in special sparse format (for example "skyline format" studied in Part III)

Assembly done by "freedom pointers" (Sec 3.5.1)

Equations for supports are not physically deleted (Sec 3.5.2) Next slide explains this for the example truss

## Computer Oriented Modification of Master Stiffness Equations

$$
\left.\begin{array}{l}
\text { Recall } f_{x 3}=2, \\
f_{y 3}=1
\end{array}\right] \text { (freedoms 1, 2, 4) }
$$

zero out rows and columns 1,2 and 4 store 1's on diagonal

## Computer Oriented Modification of Master Stiffness Equations (cont'd)

$$
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 10 \\
0 & 0 & 0 & 0 & 10 & 15
\end{array}\right]\left[\begin{array}{l}
u_{x 1} \\
u_{y 1} \\
u_{x 2} \\
u_{y 2} \\
u_{x 3} \\
u_{y 3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
2 \\
1
\end{array}\right]
$$

Modified master stiffness equations

same u as in original equations

