CHAPTER 4

Introduction to Rotating Machines

T he object of this chapter is to introduce and discuss some of the principles underlying the performance of electric machinery. As will be seen, these principles are common to both ac and dc machines. Various techniques and approximations involved in reducing a physical machine to simple mathematical models, sufficient to illustrate the basic principles, will be developed.

4.1 ELEMENTARY CONCEPTS

Equation 1.27, $e = d\lambda/dt$, can be used to determine the voltages induced by time-varying magnetic fields. Electromagnetic energy conversion occurs when changes in the flux linkage λ result from mechanical motion. In rotating machines, voltages are generated in windings or groups of coils by rotating these windings mechanically through a magnetic field, by mechanically rotating a magnetic field past the winding, or by designing the magnetic circuit so that the reluctance varies with rotation of the rotor. By any of these methods, the flux linking a specific coil is changed cyclically, and a time-varying voltage is generated.

A set of such coils connected together is typically referred to as an *armature winding*. In general, the term armature winding is used to refer to a winding or a set of windings on a rotating machine which carry ac currents. In *ac machines* such as synchronous or induction machines, the armature winding is typically on the stationary portion of the motor referred to as the *stator,* in which case these windings may also be referred to as *stator windings*. Figure 4.1 shows the stator winding of a large, multipole, three-phase synchronous motor under construction.

In a *dc machine,* the armature winding is found on the rotating member, referred to as the *rotor*. Figure 4.2 shows a dc-machine rotor. As we will see, the armature winding of a dc machine consists of many coils connected together to form a closed loop. A rotating mechanical contact is used to supply current to the armature winding as the rotor rotates.
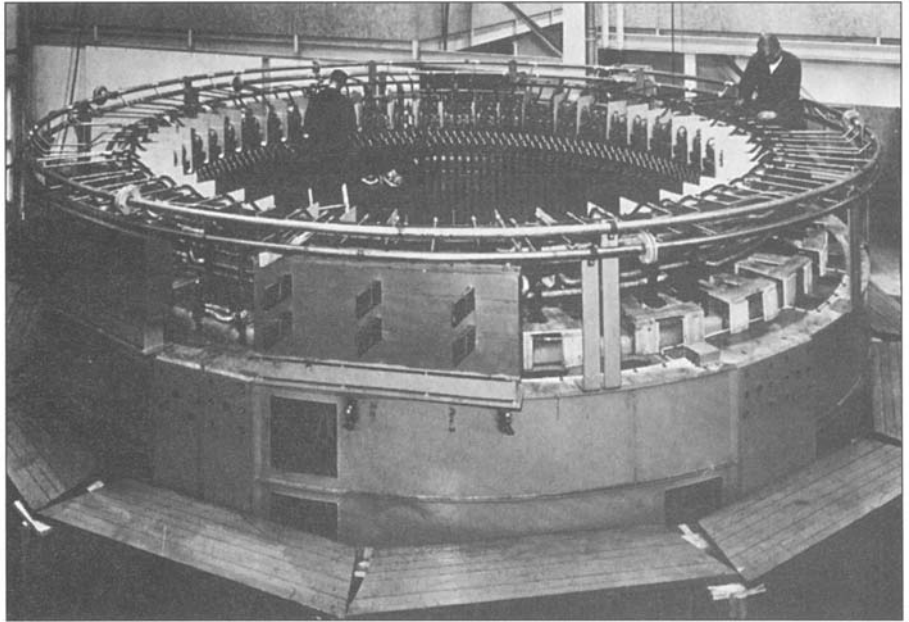
Figure 4.1   Stator of a 190-MVA three-phase 12-kV 37-r/min hydroelectric generator. The conductors have hollow passages through which cooling water is circulated. (*Brown Boveri Corporation.*)

Synchronous and dc machines typically include a second winding (or set of windings) which carry dc current and which are used to produce the main operating flux in the machine. Such a winding is typically referred to as *field winding*. The field winding on a dc machine is found on the stator, while that on a synchronous machine is found on the rotor, in which case current must be supplied to the field winding via a rotating mechanical contact. As we have seen, permanent magnets also produce dc magnetic flux and are used in the place of field windings in some machines.

In most rotating machines, the stator and rotor are made of electrical steel, and the windings are installed in slots on these structures. As is discussed in Chapter 1, the use of such high-permeability material maximizes the coupling between the coils and increases the magnetic energy density associated with the electromechanical interaction. It also enables the machine designer to shape and distribute the magnetic fields according to the requirements of each particular machine design. The time-varying flux present in the armature structures of these machines tends to induce currents, known as *eddy currents,* in the electrical steel. Eddy currents can be a large source of loss in such machines and can significantly reduce machine performance. In order to minimize the effects of eddy currents, the armature structure is typically built from thin laminations of electrical steel which are insulated from each other. This is illustrated in Fig. 4.3, which shows the stator core of an ac motor being constructed as a stack of individual laminations.

In some machines, such as *variable reluctance machines* and *stepper motors,* there are no windings on the rotor. Operation of these machines depends on the
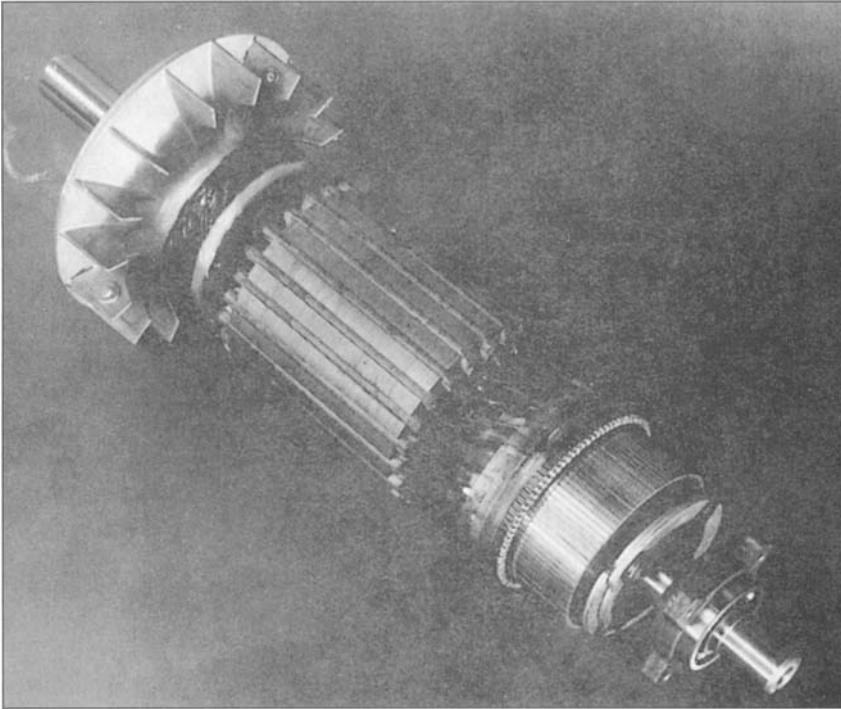
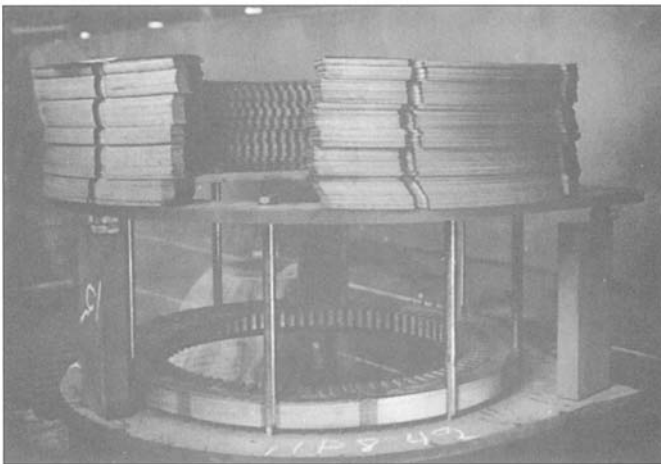**Figure 4.2** Armature of a dc motor. (*General Electric Company.*)

Figure 4.3 Partially completed stator core for an ac motor. (*Westinghouse Electric Corporation.*)

nonuniformity of air-gap reluctance associated with variations in rotor position in conjunction with time-varying currents applied to their stator windings. In such machines, both the stator and rotor structures are subjected to time-varying magnetic flux and, as a result, both may require lamination to reduce eddy-current losses.

Rotating electric machines take many forms and are known by many names: dc, synchronous, permanent-magnet, induction, variable reluctance, hysteresis, brushless, and so on. Although these machines appear to be quite dissimilar, the physical principles governing their behavior are quite similar, and it is often helpful to think of them in terms of the same physical picture. For example, analysis of a dc machine shows that associated with both the rotor and the stator are magnetic flux distributions which are fixed in space and that the torque-producing characteristic of the dc machine stems from the tendency of these flux distributions to align. An induction machine, in spite of many fundamental differences, works on exactly the same principle; one can identify flux distributions associated with the rotor and stator. Although they are not stationary but rather rotate in synchronism, just as in a dc motor they are displaced by a constant angular separation, and torque is produced by the tendency of these flux distribution to align.

Certainly, analytically based models are essential to the analysis and design of electric machines, and such models will be derived thoughout this book. However, it is also important to recognize that physical insight into the performance of these devices is equally useful. One objective of this and subsequent chapters is to guide the reader in the development of such insight.

4.2 INTRODUCTION TO AC AND DC MACHINES

4.2.1 AC Machines

Traditional ac machines fall into one of two categories: *synchronous* and *induction*. In synchronous machines, rotor-winding currents are supplied directly from the stationary frame through a rotating contact. In induction machines, rotor currents are induced in the rotor windings by a combination of the time-variation of the stator currents and the motion of the rotor relative to the stator.

Synchronous Machines   A preliminary picture of synchronous-machine performance can be gained by discussing the voltage induced in the armature of the very much simplified *salient-pole* ac synchronous generator shown schematically in Fig. 4.4. The field-winding of this machine produces a single pair of magnetic poles (similar to that of a bar magnet), and hence this machine is referred to as a *two-pole* machine.

With rare exceptions, the armature winding of a synchronous machine is on the stator, and the field winding is on the rotor, as is true for the simplified machine of Fig. 4.4. The field winding is excited by direct current conducted to it by means of stationary carbon *brushes* which contact rotatating *slip rings* or *collector rings*. Practical factors usually dictate this orientation of the two windings: It is advantageous
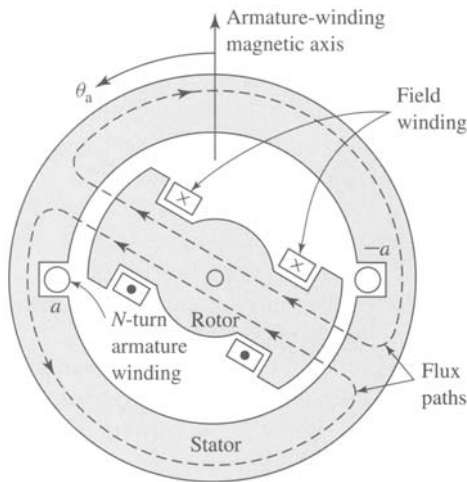
Figure 4.4 Schematic view of a simple, two-pole, single-phase synchronous generator.

to have the single, low-power field winding on the rotor while having the high-power, typically multiple-phase, armature winding on the stator.

The armature winding, consisting here of only a single coil of N turns, is indicated in cross section by the two coil sides a and $-a$ placed in diametrically opposite narrow slots on the inner periphery of the stator of Fig. 4.4. The conductors forming these coil sides are parallel to the shaft of the machine and are connected in series by end connections (not shown in the figure). The rotor is turned at a constant speed by a source of mechanical power connected to its shaft. The armature winding is assumed to be open-circuited and hence the flux in this machine is produced by the field winding alone. Flux paths are shown schematically by dashed lines in Fig. 4.4.

A highly idealized analysis of this machine would assume a sinusoidal distribution of magnetic flux in the air gap. The resultant radial distribution of air-gap flux density B is shown in Fig. 4.5a as a function of the spatial angle θ_a (measured with respect to the magnetic axis of the armature winding) around the rotor periphery. In
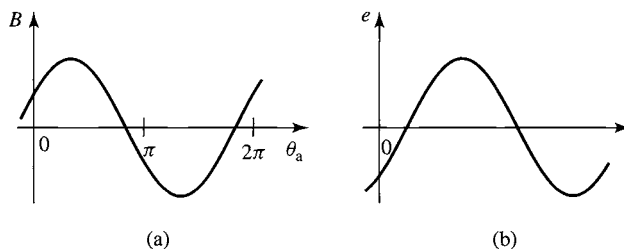

Figure 4.5 (a) Space distribution of flux density and (b) corresponding waveform of the generated voltage for the single-phase generator of Fig. 4.4.

practice, the air-gap flux-density of practical salient-pole machines can be made to approximate a sinusoidal distribution by properly shaping the pole faces.

As the rotor rotates, the flux-linkages of the armature winding change with time. Under the assumption of a sinusoidal flux distribution and constant rotor speed, the resulting coil voltage will be sinusoidal in time as shown in Fig. 4.5b. The coil voltage passes through a complete cycle for each revolution of the two-pole machine of Fig. 4.4. Its frequency in cycles per second (Hz) is the same as the speed of the rotor in revolutions per second: the electric frequency of the generated voltage is synchronized with the mechanical speed, and this is the reason for the designation "synchronous" machine. Thus a two-pole synchronous machine must revolve at 3600 revolutions per minute to produce a 60-Hz voltage.

A great many synchronous machines have more than two poles. As a specific example, Fig. 4.6 shows in schematic form a *four-pole* single-phase generator. The field coils are connected so that the poles are of alternate polarity. There are two complete wavelengths, or cycles, in the flux distribution around the periphery, as shown in Fig. 4.7. The armature winding now consists of two coils a_1, $-a_1$ and a_2, $-a_2$ connected in series by their end connections. The span of each coil is one wavelength of flux. The generated voltage now goes through two complete cycles per revolution of the rotor. The frequency in hertz will thus be twice the speed in revolutions per second.

When a machine has more than two poles, it is convenient to concentrate on a single pair of poles and to recognize that the electric, magnetic, and mechanical conditions associated with every other pole pair are repetitions of those for the pair under consideration. For this reason it is convenient to express angles in *electrical degrees* or *electrical radians* rather than in physical units. One pair of poles in a multipole machine or one cycle of flux distribution equals 360 electrical degrees or 2π electrical radians. Since there are poles/2 complete wavelengths, or cycles, in one
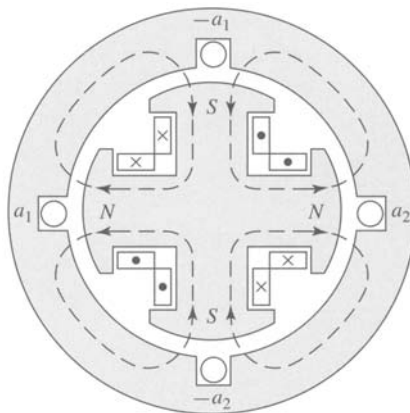

Figure 4.6 Schematic view of a simple, four-pole, single-phase synchronous generator.
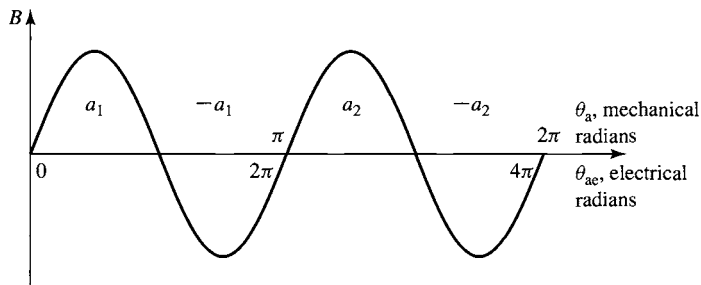
Figure 4.7 Space distribution of the air-gap flux density in a idealized, four-pole synchronous generator.

complete revolution, it follows, for example, that

$$\theta_{ae} = \left(\frac{\text{poles}}{2}\right) \theta_a \tag{4.1}$$

where θ_{ae} is the angle in electrical units and θ_a is the spatial angle. This same relationship applies to all angular measurements in a multipole machine; their values in electrical units will be equal to (poles/2) times their actual spatial values.

The coil voltage of a multipole machine passes through a complete cycle every time a pair of poles sweeps by, or (poles/2) times each revolution. The electrical frequency f_e of the voltage generated in a synchronous machine is therefore

$$f_e = \left(\frac{\text{poles}}{2}\right) \frac{n}{60} \quad \text{Hz} \tag{4.2}$$

where n is the mechanical speed in revolutions per minute, and hence $n/60$ is the speed in revolutions per second. The electrical frequency of the generated voltage in radians per second is $\omega_e = (\text{poles}/2)\,\omega_m$ where ω_m is the mechanical speed in radians per second.

The rotors shown in Figs. 4.4 and 4.6 have *salient,* or *projecting,* poles with *concentrated windings*. Figure 4.8 shows diagrammatically a *nonsalient-pole,* or *cylindrical,* rotor. The field winding is a two-pole *distributed winding;* the coil sides are distributed in multiple slots around the rotor periphery and arranged to produce an approximately sinusoidal distribution of radial air-gap flux.

The relationship between electrical frequency and rotor speed of Eq. 4.2 can serve as a basis for understanding why some synchronous generators have salient-pole rotor structures while others have cylindrical rotors. Most power systems in the world operate at frequencies of either 50 or 60 Hz. A salient-pole construction is characteristic of hydroelectric generators because hydraulic turbines operate at relatively low speeds, and hence a relatively large number of poles is required to produce the desired frequency; the salient-pole construction is better adapted mechanically to this situation. The rotor of a large hydroelectric generator is shown in Fig. 4.9. Steam turbines and gas turbines, however, operate best at relatively high speeds, and turbine-driven alternators or turbine generators are commonly two- or four-pole cylindrical-rotor
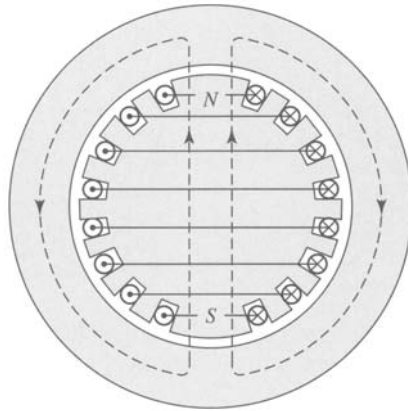
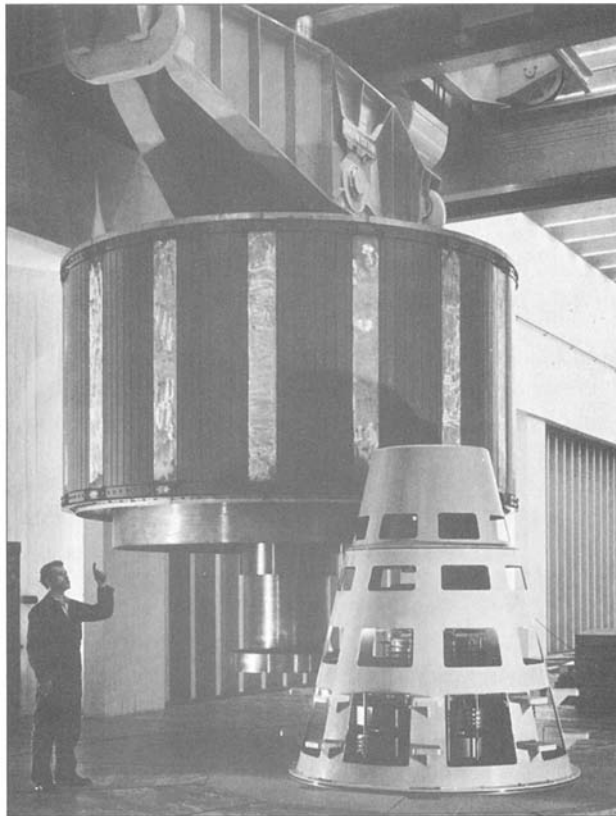**Figure 4.8** Elementary two-pole cylindrical-rotor field winding.

Figure 4.9 Water-cooled rotor of the 190-MVA hydroelectric generator whose stator is shown in Fig. 4.1. (*Brown Boveri Corporation.*)
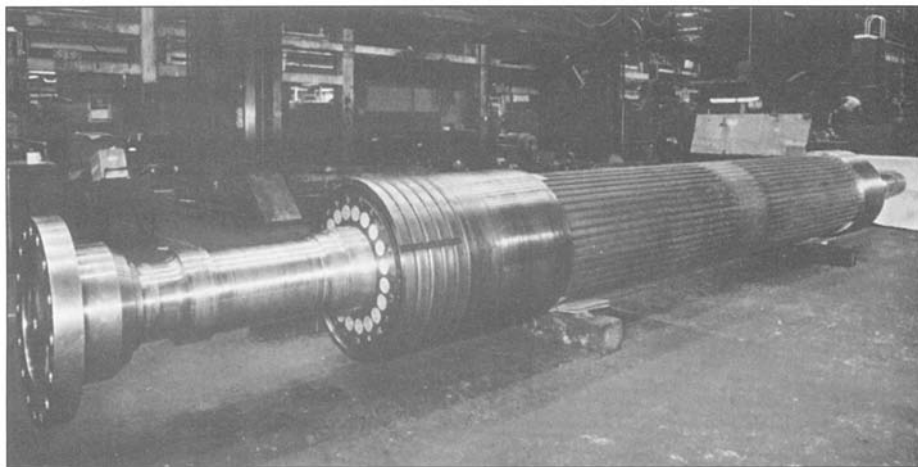
Figure 4.10 Rotor of a two-pole 3600 r/min turbine generator. (*Westinghouse Electric Corporation.*)

machines. The rotors are made from a single steel forging or from several forgings, as shown in Figs. 4.10 and 4.11.

Most of the world's power systems are three-phase systems and, as a result, with very few exceptions, synchronous generators are three-phase machines. For the production of a set of three voltages phase-displaced by 120 electrical degrees in time, a minimum of three coils phase-displaced 120 electrical degrees in space must be used. A simplified schematic view of a three-phase, two-pole machine with one coil per phase is shown in Fig. 4.12a. The three phases are designated by the letters a, b, and c. In an elementary four-pole machine, a minimum of two such sets of coils must be used, as illustrated in Fig. 4.12b; in an elementary multipole machine, the minimum number of coils sets is given by one half the number of poles.

The two coils in each phase of Fig. 4.12b are connected in series so that their voltages add, and the three phases may then be either Y- or Δ-connected. Figure 4.12c shows how the coils may be interconnected to form a Y connection. Note however, since the voltages in the coils of each phase are indentical, a parallel connection is also possible, e.g., coil $(a, -a)$ in parallel with coil $(a', -a')$, and so on.

When a synchronous generator supplies electric power to a load, the armature current creates a magnetic flux wave in the air gap which rotates at synchronous speed, as shown in Section 4.5. This flux reacts with the flux created by the field current, and electromechanical torque results from the tendency of these two magnetic fields to align. In a generator this torque opposes rotation, and mechanical torque must be applied from the prime mover to sustain rotation. This electromechanical torque is the mechanism through which the synchronous generator converts mechanical to electric energy.

The counterpart of the synchronous generator is the synchronous motor. A cut-away view of a three-phase, 60-Hz synchronous motor is shown in Fig. 4.13. Alternating current is supplied to the armature winding on the stator, and dc excitation is supplied to the field winding on the rotor. The magnetic field produced by the
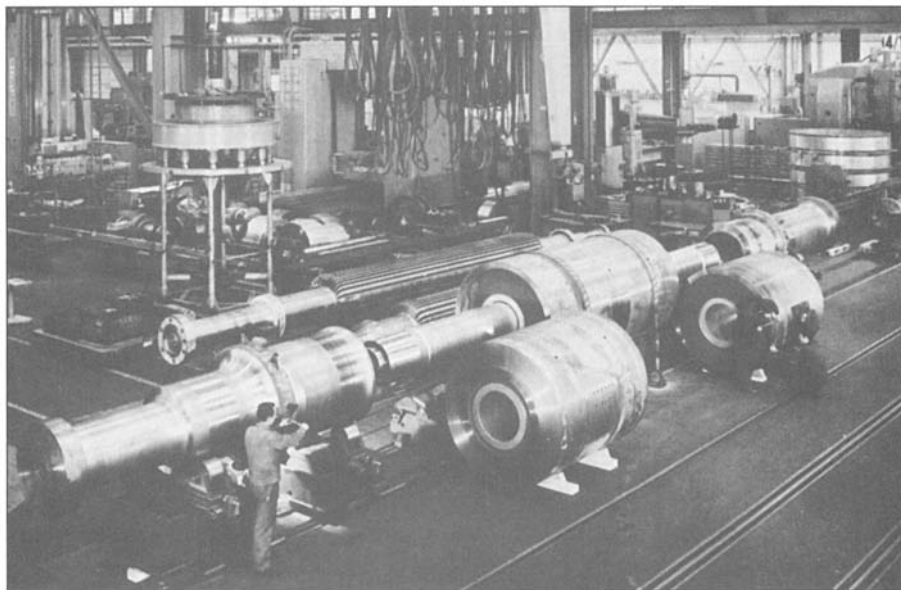
Figure 4.11 Parts of multipiece rotor for a 1333-MVA three-phase 1800 r/min turbine generator. The separate forgings will be shrunk on the shaft before final machining and milling slots for the windings. The total weight of the rotor is 435,000 lb. (*Brown Boveri Corporation.*)
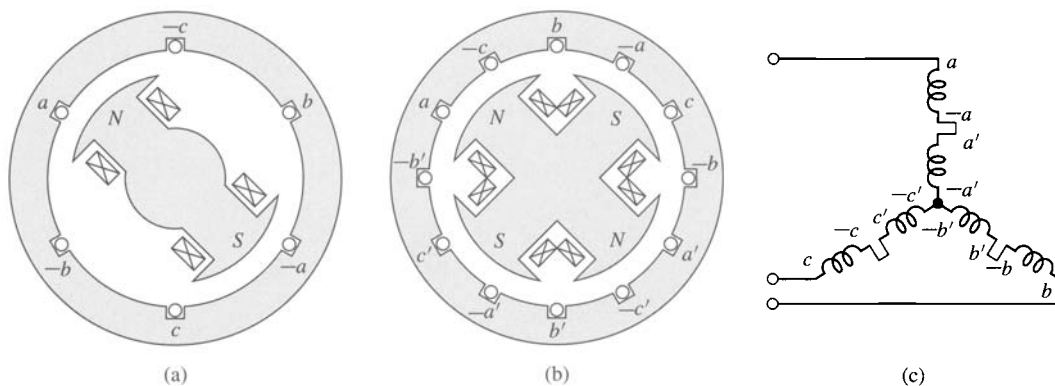
Figure 4.12 Schematic views of three-phase generators: (a) two-pole, (b) four-pole, and (c) Y connection of the windings.

armature currents rotates at synchronous speed. To produce a steady electromechanical torque, the magnetic fields of the stator and rotor must be constant in amplitude and stationary with respect to each other. In a synchronous motor, the steady-state speed is determined by the number of poles and the frequency of the armature current. Thus a synchronous motor operated from a constant-frequency ac source will operate at a constant steady-state speed.

Figure 4.13 Cutaway view of a high-speed synchronous motor. The excitor shown on the left end of the rotor is a small ac generator with a rotating semiconductor rectifier assembly. (*General Electric Company.*)

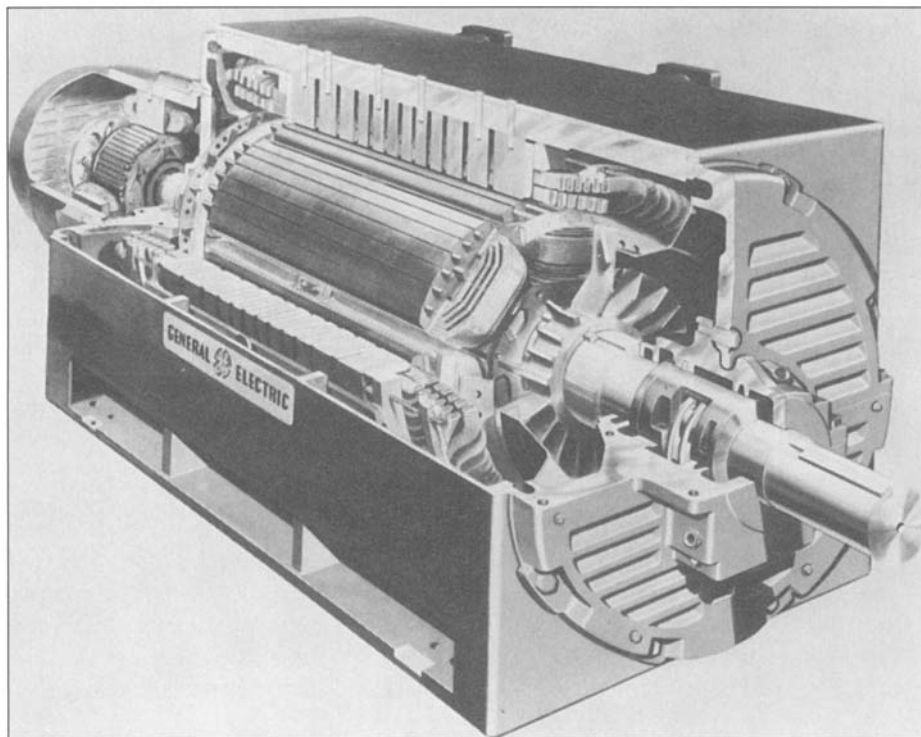In a motor the electromechanical torque is in the direction of rotation and balances the opposing torque required to drive the mechanical load. The flux produced by currents in the armature of a synchronous motor rotates ahead of that produced by the field, thus pulling on the field (and hence on the rotor) and doing work. This is the opposite of the situation in a synchronous generator, where the field does work as its flux pulls on that of the armature, which is lagging behind. In both generators and motors, an electromechanical torque and a rotational voltage are produced. These are the essential phenomena for electromechanical energy conversion.

Induction Machines   A second type of ac machine is the *induction machine*. Like the synchronous machine, the stator winding of an induction machine is excited with alternating currents. In contrast to a synchronous machine in which a field winding on the rotor is excited with dc current, alternating currents flow in the rotor windings of an induction machine. In induction machines, alternating currents are applied directly to the stator windings. Rotor currents are then produced by induction, i.e., transformer action. The induction machine may be regarded as a generalized transformer in which electric power is transformed between rotor and stator together with a change of frequency and a flow of mechanical power. Although the induction motor is the most

common of all motors, it is seldom used as a generator; its performance characteristics as a generator are unsatisfactory for most applications, although in recent years it has been found to be well suited for wind-power applications. The induction machine may also be used as a frequency changer.

In the induction motor, the stator windings are essentially the same as those of a synchronous machine. However, the rotor windings are electrically short-circuited and frequently have no external connections; currents are induced by transformer action from the stator winding. A cutaway view of a squirrel-cage induction motor is shown in Fig. 4.14. Here the rotor "windings" are actually solid aluminum bars which are cast into the slots in the rotor and which are shorted together by cast aluminum rings at each end of the rotor. This type of rotor construction results in induction motors which are relatively inexpensive and highly reliable, factors contributing to their immense popularity and widespread application.

As in a synchronous motor, the armature flux in the induction motor leads that of the rotor and produces an electromechanical torque. In fact, we will see that, just as in a synchronous machine, the rotor and stator fluxes rotate in synchronism with each other and that torque is related to the relative displacement between them. However, unlike a synchronous machine, the rotor of an induction machine does not itself rotate synchronously; it is the "slipping" of the rotor with respect to the synchronous armature flux that gives rise to the induced rotor currents and hence the torque. Induction motors operate at speeds less than the synchronous mechanical speed. A typical speed-torque characteristic for an induction motor is shown in Fig. 4.15.
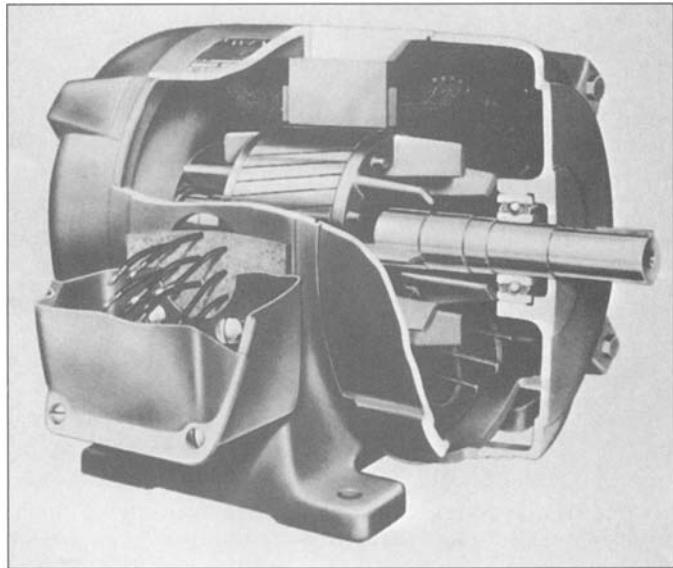
Figure 4.14 Cutaway view of a squirrel-cage induction motor. (*Westinghouse Electric Corporation.*)
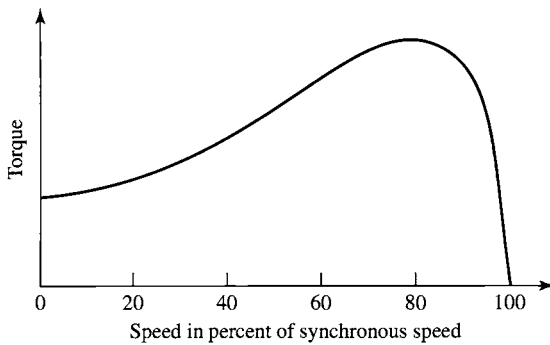
Figure 4.15 Typical induction-motor speed-torque characteristic.
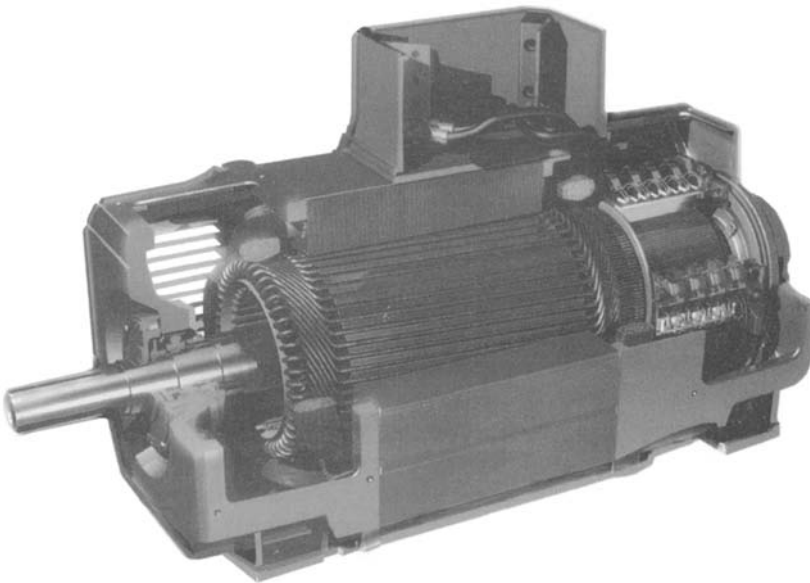
Figure 4.16 Cutaway view of a typical integral-horsepower dc motor. (*ASEA Brown Boveri.*)

4.2.2 DC Machines

As has been discussed, the armature winding of a dc generator is on the rotor with current conducted from it by means of carbon brushes. The field winding is on the stator and is excited by direct current. A cutaway view of a dc motor is shown in Fig. 4.16.

A very elementary two-pole dc generator is shown in Fig. 4.17. The armature winding, consisting of a single coil of N turns, is indicated by the two coil sides
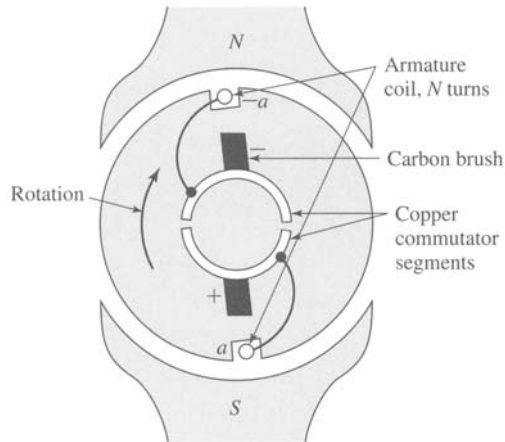
Figure 4.17 Elementary dc machine with commutator.

a and $-a$ placed at diametrically opposite points on the rotor with the conductors parallel to the shaft. The rotor is normally turned at a constant speed by a source of mechanical power connected to the shaft. The air-gap flux distribution usually approximates a flat-topped wave, rather than the sine wave found in ac machines, and is shown in Fig. 4.18a. Rotation of the coil generates a coil voltage which is a time function having the same waveform as the spatial flux-density distribution.

Although the ultimate purpose is the generation of a direct voltage, the voltage induced in an individual armature coil is an alternating voltage, which must therefore be rectified. The output voltage of an ac machine can be rectified using external semiconductor rectifiers. This is in contrast to the conventional dc machine in which rectification is produced mechanically by means of a *commutator*, which is a cylinder formed of copper segments insulated from each other by mica or some other highly insulating material and mounted on, but insulated from, the rotor shaft. Stationary carbon brushes held against the commutator surface connect the winding to the external armature terminals. The commutator and brushes can readily be seen in Fig. 4.16. The need for commutation is the reason why the armature windings of dc machines are placed on the rotor.

For the elementary dc generator, the commutator takes the form shown in Fig. 4.17. For the direction of rotation shown, the commutator at all times connects the coil side, which is under the south pole, to the positive brush and that under the north pole to the negative brush. The commutator provides full-wave rectification, transforming the voltage waveform between brushes to that of Fig. 4.18b and making available a unidirectional voltage to the external circuit. The dc machine of Fig. 4.17 is, of course, simplified to the point of being unrealistic in the practical sense, and later it will be essential to examine the action of more realistic commutators.

The effect of direct current in the field winding of a dc machine is to create a magnetic flux distribution which is stationary with respect to the stator. Similarly, the
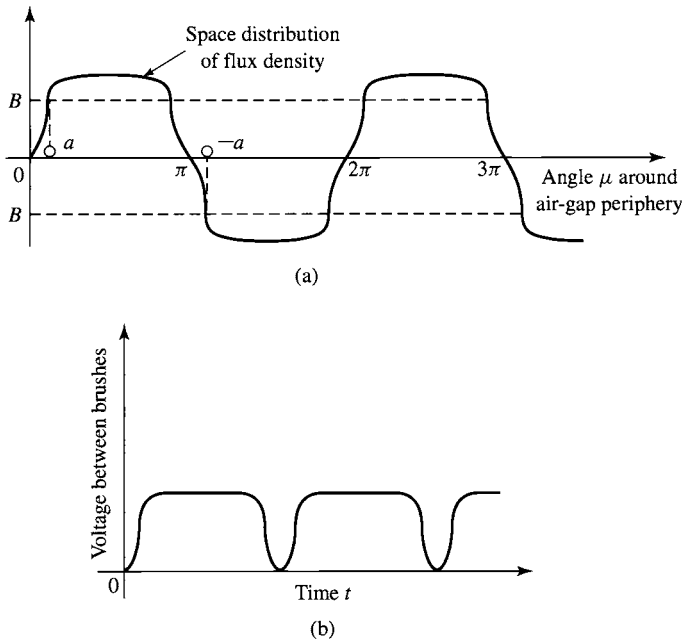
Figure 4.18 (a) Space distribution of air-gap flux density in an elementary dc machine; (b) waveform of voltage between brushes.

effect of the commutator is such that when direct current flows through the brushes, the armature creates a magnetic flux distribution which is also fixed in space and whose axis, determined by the design of the machine and the position of the brushes, is typically perpendicular to the axis of the field flux.

Thus, just as in the ac machines discussed previously, it is the interaction of these two flux distributions that creates the torque of the dc machine. If the machine is acting as a generator, this torque opposes rotation. If it is acting as a motor, the electromechanical torque acts in the direction of the rotation. Remarks similar to those already made concerning the roles played by the generated voltage and electromechanical torque in the energy conversion process in synchronous machines apply equally well to dc machines.

4.3 MMF OF DISTRIBUTED WINDINGS

Most armatures have distributed windings, i.e., windings which are spread over a number of slots around the air-gap periphery, as in Figs. 4.2 and 4.1. The individual coils are interconnected so that the result is a magnetic field having the same number of poles as the field winding.

The study of the magnetic fields of distributed windings can be approached by examining the magnetic field produced by a winding consisting of a single N-turn coil which spans 180 electrical degrees, as shown in Fig. 4.19a. A coil which spans
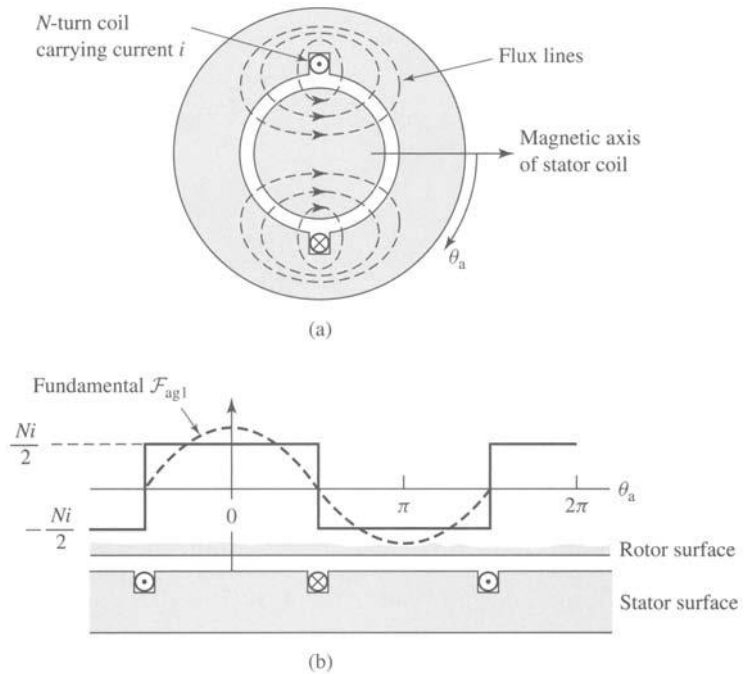
Figure 4.19   (a) Schematic view of flux produced by a concentrated, full-pitch winding in a machine with a uniform air gap. (b) The air-gap mmf produced by current in this winding.

180 electrical degrees is known as a *full-pitch coil*. The dots and crosses indicate current flow towards and away from the reader, respectively. For simplicity, a concentric cylindrical rotor is shown. The general nature of the magnetic field produced by the current in the coil is shown by the dashed lines in Fig. 4.19a. Since the permeability of the armature and field iron is much greater than that of air, it is sufficiently accurate for our present purposes to assume that all the reluctance of the magnetic circuit is in the air gap. From symmetry of the structure it is evident that the magnetic field intensity H_{ag} in the air gap at angle θ_a under one pole is the same in magnitude as that at angle $\theta_a + \pi$ under the opposite pole, but the fields are in the opposite direction.

Around any of the closed paths shown by the flux lines in Fig. 4.19a the mmf is Ni. The assumption that all the reluctance of this magnetic circuit is in the air gap leads to the result that the line integral of **H** inside the iron is negligibly small, and thus it is reasonable to neglect the mmf drops associated with portions of the magnetic circuit inside the iron. By symmetry we argued that the air-gap fields H_{ag} on opposite sides of the rotor are equal in magnitude but opposite in direction. It follows that the air-gap mmf should be similarly distributed; since each flux line crosses the air gap twice, the mmf drop across the air gap must be equal to half of the total or $Ni/2$.

Figure 4.19b shows the air gap and winding in developed form, i.e., laid out flat. The air-gap mmf distribution is shown by the steplike distribution of amplitude

$Ni/2$. On the assumption of narrow slot openings, the mmf jumps abruptly by Ni in crossing from one side to the other of a coil. This mmf distribution is discussed again in Section 4.4, where the resultant magnetic fields are evaluated.

4.3.1 AC Machines

Fourier analysis can show that the air-gap mmf produced by a single coil such as the full-pitch coil of Fig. 4.19 consists of a fundamental space-harmonic component as well as a series of higher-order harmonic components. In the design of ac machines, serious efforts are made to distribute the coils making up the windings so as to minimize the higher-order harmonic components and to produce an air-gap mmf wave which consists predominantly of the space-fundamental sinusoidal component. It is thus appropriate here to assume that this has been done and to focus our attention on the fundamental component.

The rectangular air-gap mmf wave of the concentrated two-pole, full-pitch coil of Fig. 4.19b can be resolved into a Fourier series comprising a fundamental component and a series of odd harmonics. The fundamental component $\mathcal{F}_{ag1}$ is

$$\mathcal{F}_{ag1} = \frac{4}{\pi} \left(\frac{Ni}{2} \right) \cos \theta_a \tag{4.3}$$

where θ_a is measured from the magnetic axis of the stator coil, as shown by the dashed sinusoid in Fig. 4.19b. It is a sinusoidal space wave of amplitude

$$(F_{ag1})_{peak} = \frac{4}{\pi} \left(\frac{Ni}{2} \right) \tag{4.4}$$

with its peak aligned with the magnetic axis of the coil.

Now consider a *distributed winding,* consisting of coils distributed in several slots. For example, Fig. 4.20a shows phase *a* of the armature winding of a somewhat simplified two-pole, three-phase ac machine. Phases *b* and *c* occupy the empty slots. The windings of the three phases are identical and are located with their magnetic axes 120 degrees apart. We direct our attention to the air-gap mmf of phase *a* alone, postponing the discussion of the effects of all three phases until Section 4.5. The winding is arranged in two layers, each full-pitch coil of N_c turns having one side in the top of a slot and the other coil side in the bottom of a slot a pole pitch away. In a practical machine, this two-layer arrangement simplifies the geometric problem of getting the end turns of the individual coils past each other.

Figure 4.20b shows one pole of this winding laid out flat. With the coils connected in series and hence carrying the same current, the mmf wave is a series of steps each of height $2N_c i_a$ (equal to the ampere-turns in the slot), where i_a is the winding current. Its space-fundamental component is shown by the sinusoid. It can be seen that the distributed winding produces a closer approximation to a sinusoidal mmf wave than the concentrated coil of Fig. 4.19.

The amplitude of the fundamental-space-harmonic-component of the mmf wave of a distributed winding is less than the sum of the fundamental components of the
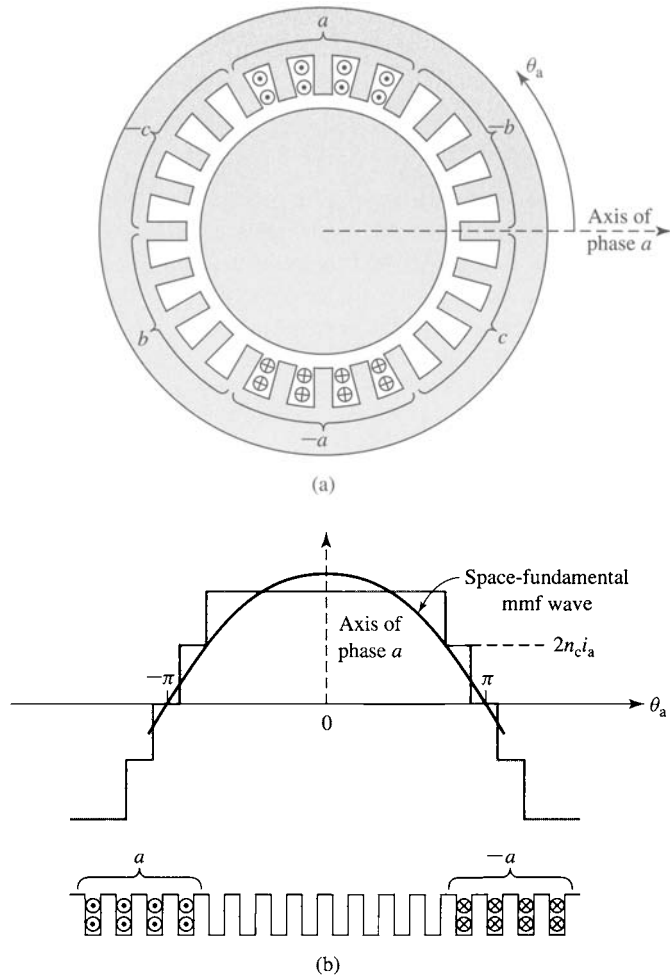
Figure 4.20 The mmf of one phase of a distributed two-pole, three-phase winding with full-pitch coils.

individual coils because the magnetic axes of the individual coils are not aligned with the resultant. The modified form of Eq. 4.3 for a distributed multipole winding having N_{ph} series turns per phase is

$$\mathcal{F}_{ag1} = \frac{4}{\pi} \left(\frac{k_w N_{ph}}{poles} \right) i_a \cos \left(\frac{poles}{2} \theta_a \right) \tag{4.5}$$

in which the factor $4/\pi$ arises from the Fourier-series analysis of the rectangular mmf wave of a concentrated full-pitch coil, as in Eq. 4.3, and the *winding factor* k_w takes into account the distribution of the winding. This factor is required because the mmf's produced by the individual coils of any one phase group have different magnetic axes.

When they are connected in series to form the phase winding, their phasor sum is then less than their numerical sum. (See Appendix B for details.) For most three-phase windings, k_w typically falls in the range of 0.85 to 0.95.

The factor $k_w N_{ph}$ is the effective series turns per phase for the fundamental mmf. The peak amplitude of this mmf wave is

$$(F_{ag1})_{peak} = \frac{4}{\pi} \left(\frac{k_w N_{ph}}{poles} \right) i_a \tag{4.6}$$

EXAMPLE 4.1

The phase-*a* two-pole armature winding of Fig. 4.20a can be considered to consist of 8 N_c-turn, full-pitch coils connected in series, with each slot containing two coils. There are a total of 24 armature slots, and thus each slot is separated by $360°/24 = 15°$. Assume angle θ_a is measured from the magnetic axis of phase a such that the four slots containing the coil sides labeled a are at $\theta_a = 67.5°, 82.5°, 97.5°$, and $112.5°$. The opposite sides of each coil are thus found in the slots found at $-112.5°, -97.5°, -82.5°$ and $-67.5°$, respectively. Assume this winding to be carrying current i_a.

(*a*) Write an expression for the space-fundamental mmf produced by the two coils whose sides are in the slots at $\theta_a = 112.5°$ and $-67.5°$. (*b*) Write an expression for the space-fundamental mmf produced by the two coils whose sides are in the slots at $\theta_a = 67.5°$ and $-112.5°$. (*c*) Write an expression for the space-fundamental mmf of the complete armature winding. (*d*) Determine the winding factor k_w for this distributed winding.

■ Solution

a. Noting that the magnetic axis of this pair of coils is at $\theta_a = (112.5° - 67.5°)/2 = 22.5°$ and that the total ampere-turns in the slot is equal to $2N_c i_a$, the mmf produced by this pair of coils can be found from analogy with Eq. 4.3 to be

$$(\mathcal{F}_{ag1})_{22.5°} = \frac{4}{\pi} \left(\frac{2N_c i_a}{2} \right) \cos(\theta_a - 22.5°)$$

b. This pair of coils produces the same space-fundamental mmf as the pair of part (a) with the exception that this mmf is centered at $\theta_a = -22.5°$. Thus

$$(\mathcal{F}_{ag1})_{-22.5°} = \frac{4}{\pi} \left(\frac{2N_c i_a}{2} \right) \cos(\theta_a + 22.5°)$$

c. By analogy with parts (a) and (b), the total space-fundamental mmf can be written as

$$(\mathcal{F}_{ag1})_{total} = (\mathcal{F}_{ag1})_{-22.5°} + (\mathcal{F}_{ag1})_{-7.5°} + (\mathcal{F}_{ag1})_{7.5°} + (\mathcal{F}_{ag1})_{22.5°}$$

$$= \frac{4}{\pi} \left(\frac{2N_c}{2} \right) i_a [\cos(\theta_a + 22.5°) + \cos(\theta_a + 7.5°)$$

$$+ \cos(\theta_a - 7.5°) + \cos(\theta_a - 22.5°)]$$

$$= \frac{4}{\pi} \left(\frac{7.66 N_c}{2} \right) i_a \cos \theta_a$$

$$= 4.88 N_c i_a \cos \theta_a$$

d. Recognizing that, for this winding $N_{ph} = 8N_c$, the total mmf of part (c) can be rewritten as

$$(\mathcal{F}_{ag1})_{total} = \frac{4}{\pi} \left(\frac{0.958N_{ph}}{2} \right) i_a \cos\theta_a$$

Comparison with Eq. 4.5 shows that for this winding, the winding factor is $k_w = 0.958$.

Calculate the winding factor of the phase-a winding of Fig. 4.20 if the number of turns in the four coils in the two outer pairs of slots is reduced to six while the number of turns in the four coils in the inner slots remains at eight.

Solution

$$k_w = 0.962$$

Equation 4.5 describes the space-fundamental component of the mmf wave produced by current in phase a of a distributed winding. If the phase-a current is sinusoidal in time, e.g., $i_a = I_m \cos\omega t$, the result will be an mmf wave which is stationary in space and varies sinusoidally both with respect to θ_a and in time. In Section 4.5 we will study the effect of currents in all three phases and will see that the application of three-phase currents will produce a rotating mmf wave.

In a directly analogous fashion, rotor windings are often distributed in slots to reduce the effects of space harmonics. Figure 4.21a shows the rotor of a typical two-pole round-rotor generator. Although the winding is symmetric with respect to the rotor axis, the number of turns per slot can be varied to control the various harmonics. In Fig. 4.21b it can be seen that there are fewer turns in the slots nearest the pole face. In addition, the designer can vary the spacing of the slots. As for distributed armature windings, the fundamental air-gap mmf wave of a multipole rotor winding can be found from Eq. 4.5 in terms of the total number of series turns N_r, the winding current I_r and a winding factor k_r as

$$\mathcal{F}_{ag1} = \frac{4}{\pi} \left(\frac{k_r N_r}{poles} \right) I_r \cos\left(\frac{poles}{2}\theta_r \right) \tag{4.7}$$

where θ_r is the spatial angle measured with respect to the rotor magnetic axis, as shown in Fig. 4.21b. Its peak amplitude is

$$(F_{ag1})_{peak} = \frac{4}{\pi} \left(\frac{k_r N_r}{poles} \right) I_r \tag{4.8}$$

4.3.2 DC Machines

Because of the restrictions imposed on the winding arrangement by the commutator, the mmf wave of a dc machine armature approximates a sawtooth waveform
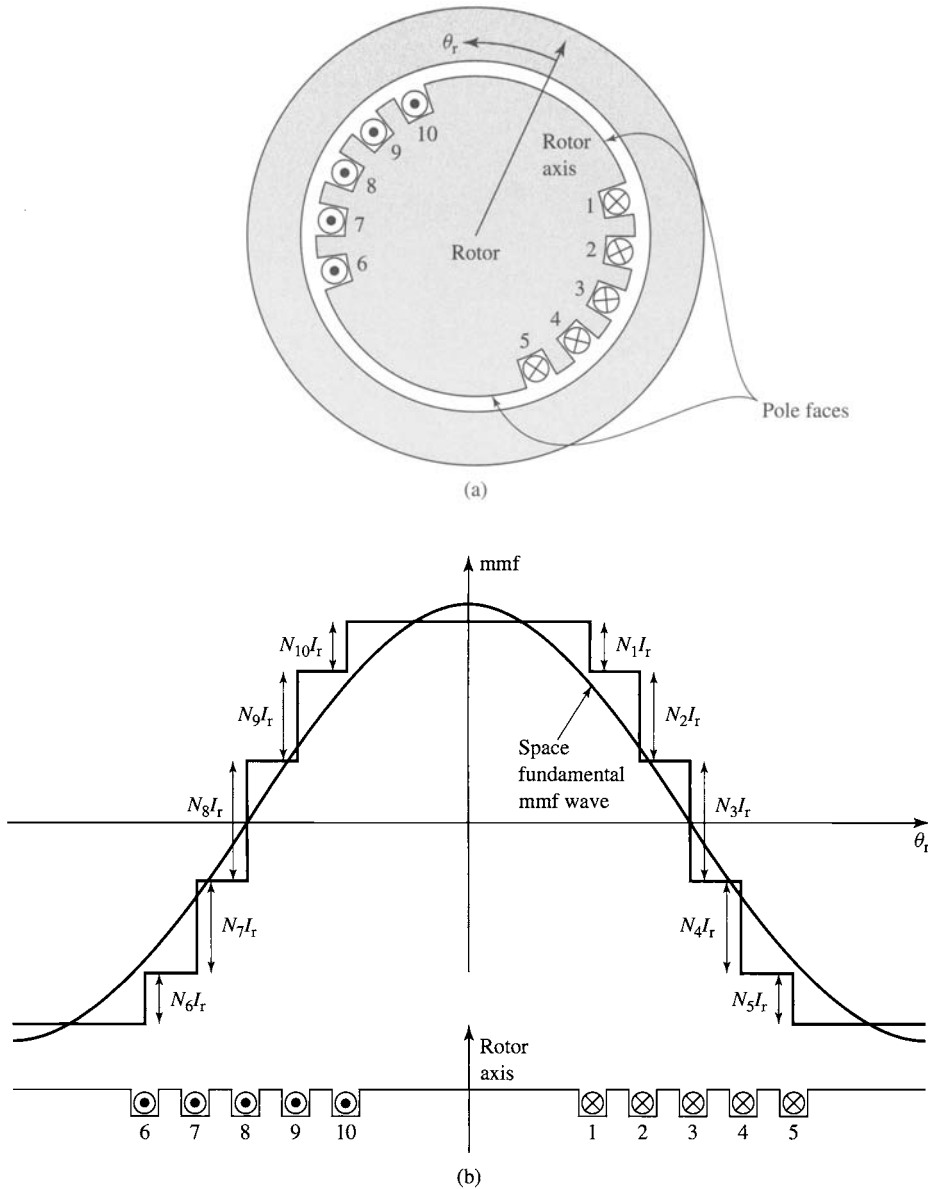
Figure 4.21 The air-gap mmf of a distributed winding on the rotor of a round-rotor generator.

more nearly than the sine wave of ac machines. For example, Fig. 4.22 shows diagrammatically in cross section the armature of a two-pole dc machine. (In practice, in all but the smallest of dc machines, a larger number of coils and slots would probably be used.) The current directions are shown by dots and crosses. The armature winding coil connections are such that the armature winding produces a magnetic
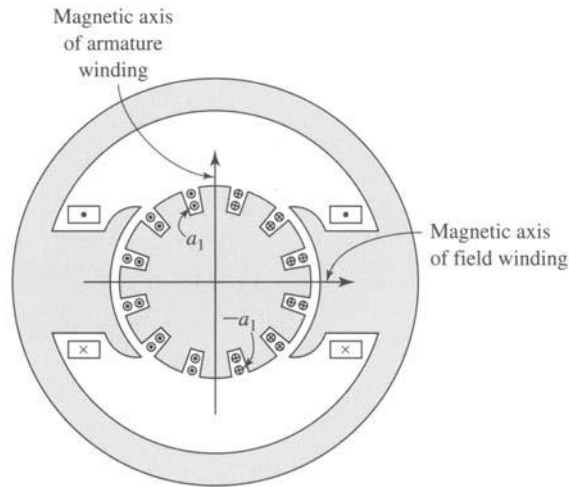
Figure 4.22 Cross section of a two-pole dc machine.

field whose axis is vertical and thus is perpendicular to the axis of the field winding. As the armature rotates, the coil connections to the external circuit are changed by the commutator such that the magnetic field of the armature remains vertical. Thus, the armature flux is always perpendicular to that produced by the field winding and a continuous unidirectional torque results. Commutator action is discussed in some detail in Section 7.2.

Figure 4.23a shows this winding laid out flat. The mmf wave is shown in Fig. 4.23b. On the assumption of narrow slots, it consists of a series of steps. The height of each step equals the number of ampere-turns $2N_c i_c$ in a slot, where N_c is the number of turns in each coil and i_c is the coil current, with a two-layer winding and full-pitch coils being assumed. The peak value of the mmf wave is along the magnetic axis of the armature, midway between the field poles. This winding is equivalent to a coil of $12N_c i_c$ A·turns distributed around the armature. On the assumption of symmetry at each pole, the peak value of the mmf wave at each armature pole is $6N_c i_c$ A·turns.

This mmf wave can be represented approximately by the sawtooth wave drawn in Fig. 4.23b and repeated in Fig. 4.23c. For a more realistic winding with a larger number of armature slots per pole, the triangular distribution becomes a close approximation. This mmf wave would be produced by a rectangular distribution of current density at the armature surface, as shown in Fig. 4.23c.

For our preliminary study, it is convenient to resolve the mmf waves of distributed windings into their Fourier series components. The fundamental component of the sawtooth mmf wave of Fig. 4.23c is shown by the sine wave. Its peak value is $8/\pi^2 = 0.81$ times the height of the sawtooth wave. This fundamental mmf wave is that which would be produced by the fundamental space-harmonic component of the rectangular current-density distribution of Fig. 4.23c. This sinusoidally-distributed current sheet is shown dashed in Fig. 4.23c.
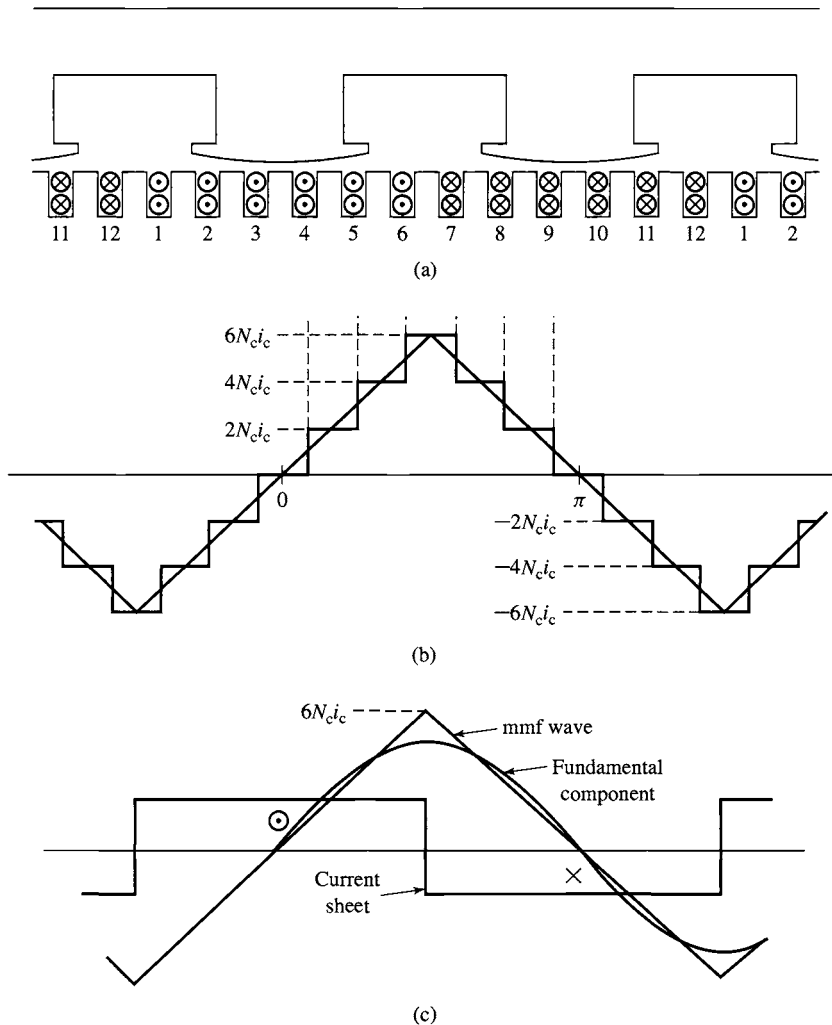
Figure 4.23 (a) Developed sketch of the dc machine of Fig. 4.22; (b) mmf wave; (c) equivalent sawtooth mmf wave, its fundamental component, and equivalent rectangular current sheet.

Note that the air-gap mmf distribution depends on only the winding arrangement and symmetry of the magnetic structure at each pole. The air-gap flux density, however, depends not only on the mmf but also on the magnetic boundary conditions, primarily the length of the air gap, the effect of the slot openings, and the shape of the pole face. The designer takes these effects into account by means of detailed analyses, but these details need not concern us here.

DC machines often have a magnetic structure with more than two poles. For example, Fig. 4.24a shows schematically a four-pole dc machine. The field winding
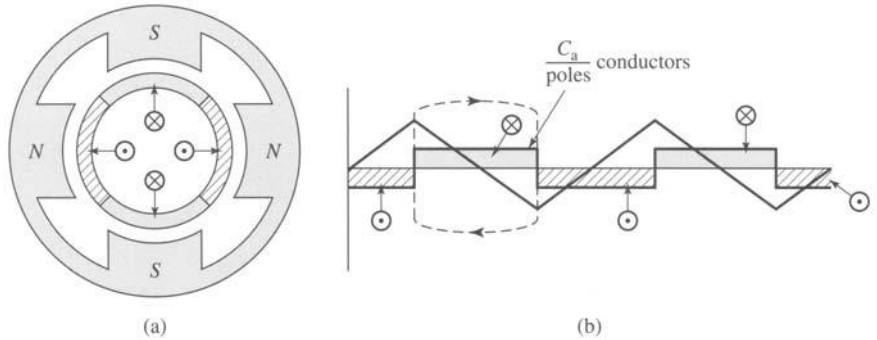
Figure 4.24 (a) Cross section of a four-pole dc machine; (b) development of current sheet and mmf wave.

produces alternate north-south-north-south polarity, and the armature conductors are distributed in four belts of slots carrying currents alternately toward and away from the reader, as symbolized by the cross-hatched areas. This machine is shown in laid-out form in Fig. 4.24b. The corresponding sawtooth armature-mmf wave is also shown. On the assumption of symmetry of the winding and field poles, each successive pair of poles is like every other pair of poles. Magnetic conditions in the air gap can then be determined by examining any pair of adjacent poles, that is, 360 electrical degrees.

The peak value of the sawtooth armature mmf wave can be written in terms of the total number of conductors in the armature slots as

$$(F_{ag})_{peak} = \left(\frac{C_a}{2m \cdot poles} \right) i_a \quad A \cdot turns/pole \tag{4.9}$$

where

C_a = total number of conductors in armature winding

m = number of parallel paths through armature winding

i_a = armature current, A

This equation takes into account the fact that in some cases the armature may be wound with multiple current paths in parallel. It is for this reason that it is often more convenient to think of the armature in terms of the number of conductors (each conductor corresponding to a single current-carrying path within a slot). Thus i_a/m is the current in each conductor. This equation comes directly from the line integral around the dotted closed path in Fig. 4.24b which crosses the air gap twice and encloses $C_a/poles$ conductors, each carrying current i_a/m in the same direction. In more compact form,

$$(F_{ag})_{peak} = \left(\frac{N_a}{poles} \right) i_a \tag{4.10}$$

where $N_a = C_a/(2m)$ is the number of series armature turns. From the Fourier series for the sawtooth mmf wave of Fig. 4.24b, the peak value of the space fundamental is given by

$$(F_{ag1})_{peak} = \frac{8}{\pi^2} \left(\frac{N_a}{poles} \right) i_a \qquad (4.11)$$

4.4 MAGNETIC FIELDS IN ROTATING MACHINERY

We base our preliminary investigations of both ac and dc machines on the assumption of sinusoidal spatial distributions of mmf. This assumption will be found to give very satisfactory results for most problems involving ac machines because their windings are commonly distributed so as to minimize the effects of space harmonics. Because of the restrictions placed on the winding arrangement by the commutator, the mmf waves of dc machines inherently approach more nearly a sawtooth waveform. Nevertheless, the theory based on a sinusoidal model brings out the essential features of dc machine theory. The results can readily be modified whenever necessary to account for any significant discrepancies.

It is often easiest to begin by examination of a two-pole machine, in which the electrical and mechanical angles and velocities are equal. The results can immediately be extrapolated to a multipole machine when it is recalled that electrical angles and angular velocities are related to mechanical angles and angular velocities by a factor of poles/2 (see, for example, Eq. 4.1).

The behavior of electric machinery is determined by the magnetic fields created by currents in the various windings of the machine. This section discusses how these magnetic fields and currents are related.

4.4.1 Machines with Uniform Air Gaps

Figure 4.25a shows a single full-pitch, N-turn coil in a high-permeability magnetic structure ($\mu \to \infty$), with a concentric, cylindrical rotor. The air-gap mmf $\mathcal{F}_{ag}$ of this configuration is shown plotted versus angle θ_a in Fig. 4.25b. For such a structure, with a uniform air gap of length g at radius r_r (very much larger than g), it is quite accurate to assume that the magnetic field $\mathbf{H}$ in the air gap is directed only radially and has constant magnitude across the air gap.

The air-gap mmf distribution of Fig. 4.25b is equal to the line integral of H_{ag} across the air gap. For this case of constant radial H_{ag}, this integral is simply equal to the product of the air-gap radial magnetic field H_{ag} times the air-gap length g, and thus H_{ag} can be found simply by dividing the air-gap mmf by the air-gap length:

$$H_{ag} = \frac{\mathcal{F}_{ag}}{g} \qquad (4.12)$$
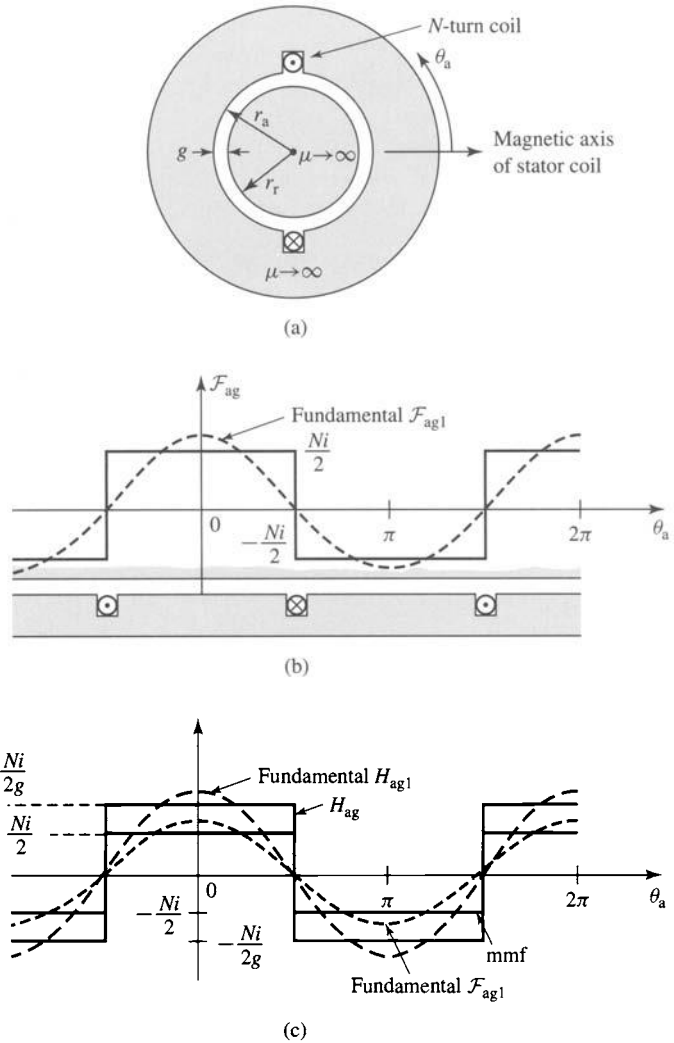
Figure 4.25 The air-gap mmf and radial component of H_{ag} for a concentrated full-pitch winding.

Thus, in Fig. 4.25c, the radial H_{ag} field and mmf can be seen to be identical in form, simply related by a factor of $1/g$.

The fundamental space-harmonic component of H_{ag} can be found directly from the fundamental component $\mathcal{F}_{ag1}$, given by Eq. 4.3.

$$H_{ag1} = \frac{\mathcal{F}_{ag1}}{g} = \frac{4}{\pi}\left(\frac{Ni}{2g}\right)\cos\theta_a \tag{4.13}$$

It is a sinusoidal space wave of amplitude

$$(H_{ag1})_{peak} = \frac{4}{\pi}\left(\frac{Ni}{2g}\right) \tag{4.14}$$

For a distributed winding such as that of Fig. 4.20, the air-gap magnetic field intensity is easily found once the air-gap mmf is known. Thus the fundamental component of H_{ag} can be found by dividing the fundamental component of the air-gap mmf (Eq. 4.5) by the air-gap length g

$$H_{ag1} = \frac{4}{\pi}\left(\frac{k_w N_{ph}}{g \cdot poles}\right) i_a \cos\left(\frac{poles}{2}\theta_a\right) \tag{4.15}$$

This equation has been written for the general case of a multipole machine, and N_{ph} is the total number of series turns per phase.

Note that the space-fundamental air-gap mmf $\mathcal{F}_{ag1}$ and air-gap magnetic field H_{ag1} produced by a distributed winding of winding factor k_w and $N_{ph}/poles$ series turns per pole is equal to that produced by a concentrated, full pitch winding of $(k_w N_{ph})/poles$ turns per pole. In the analysis of machines with distributed windings, this result is useful since in considering space-fundamental quantities it permits the distributed solution to be obtained from the single N-turn, full-pitch coil solution simply by replacing N by the effective number of turns, $k_w N_{ph}$, of the distributed winding.

EXAMPLE 4.2

A four-pole synchronous ac generator with a smooth air gap has a distributed rotor winding with 263 series turns, a winding factor of 0.935, and an air gap of length 0.7 mm. Assuming the mmf drop in the electrical steel to be negligible, find the rotor-winding current required to produce a peak, space-fundamental magnetic flux density of 1.6 T in the machine air gap.

■ **Solution**

The space-fundamental air-gap magnetic flux density can be found by multiplying the air-gap magnetic field by the permeability of free space μ_0, which in turn can be found from the space-fundamental component of the air-gap mmf by dividing by the air-gap length g. Thus, from Eq. 4.8

$$(B_{ag1})_{peak} = \frac{\mu_0(\mathcal{F}_{ag1})_{peak}}{g} = \frac{4\mu_0}{\pi g}\left(\frac{k_r N_r}{poles}\right)I_r$$

and I_r can be found from

$$I_r = \left(\frac{\pi g \cdot poles}{4\mu_0 k_r N_r}\right)(B_{ag1})_{peak}$$

$$= \left(\frac{\pi \times 0.0007 \times 4}{4 \times 4\pi \times 10^{-7} \times 0.935 \times 263}\right)1.6$$

$$= 11.4 \text{ A}$$

A 2-pole synchronous machine has an air-gap length of 2.2 cm and a field winding with a total of 830 series turns. When excited by a field current of 47 A, the peak, space-fundamental magnetic flux density in the machine air-gap is measured to be 1.35 T.

Based upon the measured flux density, calculate the field-winding winding factor k_r.

Solution

$$k_r = 0.952$$

4.4.2 Machines with Nonuniform Air Gaps

Figure 4.26a shows the structure of a typical dc machine, and Fig. 4.26b shows the structure of a typical salient-pole synchronous machine. Both machines consist of magnetic structures with extremely nonuniform air gaps. In such cases the air-gap magnetic-field distribution is more complex than that of uniform-air-gap machines.

Detailed analysis of the magnetic field distributions in such machines requires complete solutions of the field problem. For example, Fig. 4.27 shows the magnetic field distribution in a salient-pole dc generator (obtained by a finite-element solution). However, experience has shown that through various simplifying assumptions, analytical techniques which yield reasonably accurate results can be developed. These techniques are illustrated in later chapters, where the effects of saliency on both dc and ac machines are discussed.
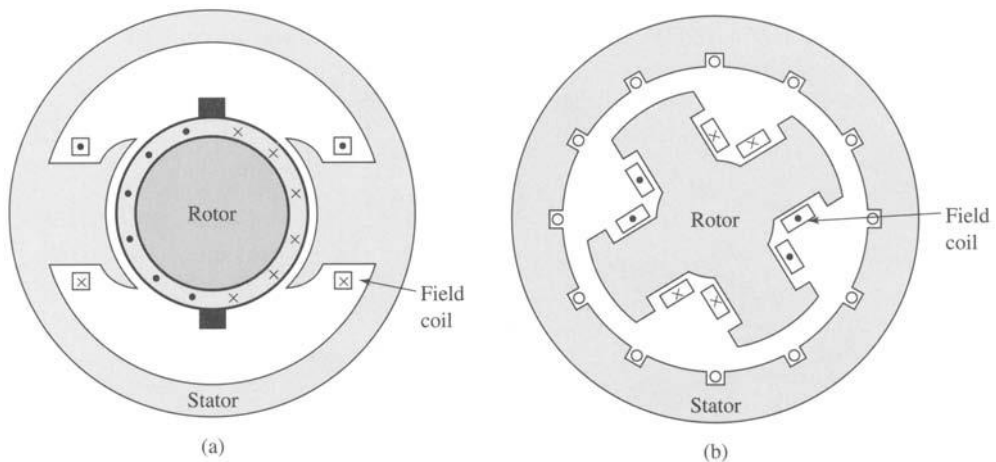
Figure 4.26 Structure of typical salient-pole machines: (a) dc machine and (b) salient-pole synchronous machine.
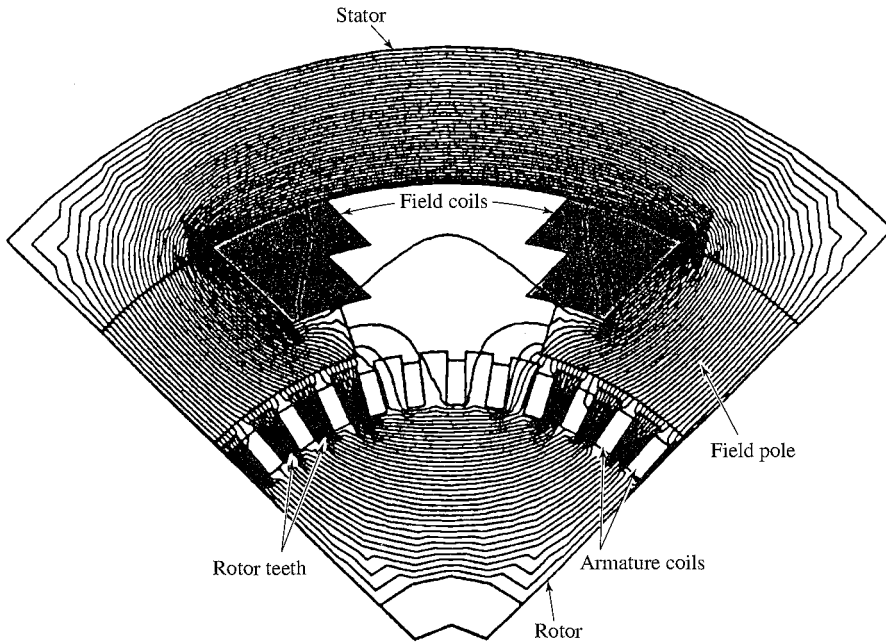
Figure 4.27 Finite-element solution of the magnetic field distribution in a salient-pole dc generator. Field coils excited; no current in armature coils. (*General Electric Company.*)

4.5 ROTATING MMF WAVES IN AC MACHINES

To understand the theory and operation of polyphase ac machines, it is necessary to study the nature of the mmf wave produced by a polyphase winding. Attention will be focused on a two-pole machine or one pair of a multipole winding. To develop insight into the polyphase situation, it is helpful to begin with an analysis of a single-phase winding.

4.5.1 MMF Wave of a Single-Phase Winding

Figure 4.28a shows the space-fundamental mmf distribution of a single-phase winding, where, from Eq. 4.5,

$$\mathcal{F}_{ag1} = \frac{4}{\pi}\left(\frac{k_w N_{ph}}{poles}\right) i_a \cos\left(\frac{poles}{2}\theta_a\right) \tag{4.16}$$

When this winding is excited by a sinusoidally varying current in time at electrical frequency ω_e

$$i_a = I_a \cos\omega_e t \tag{4.17}$$
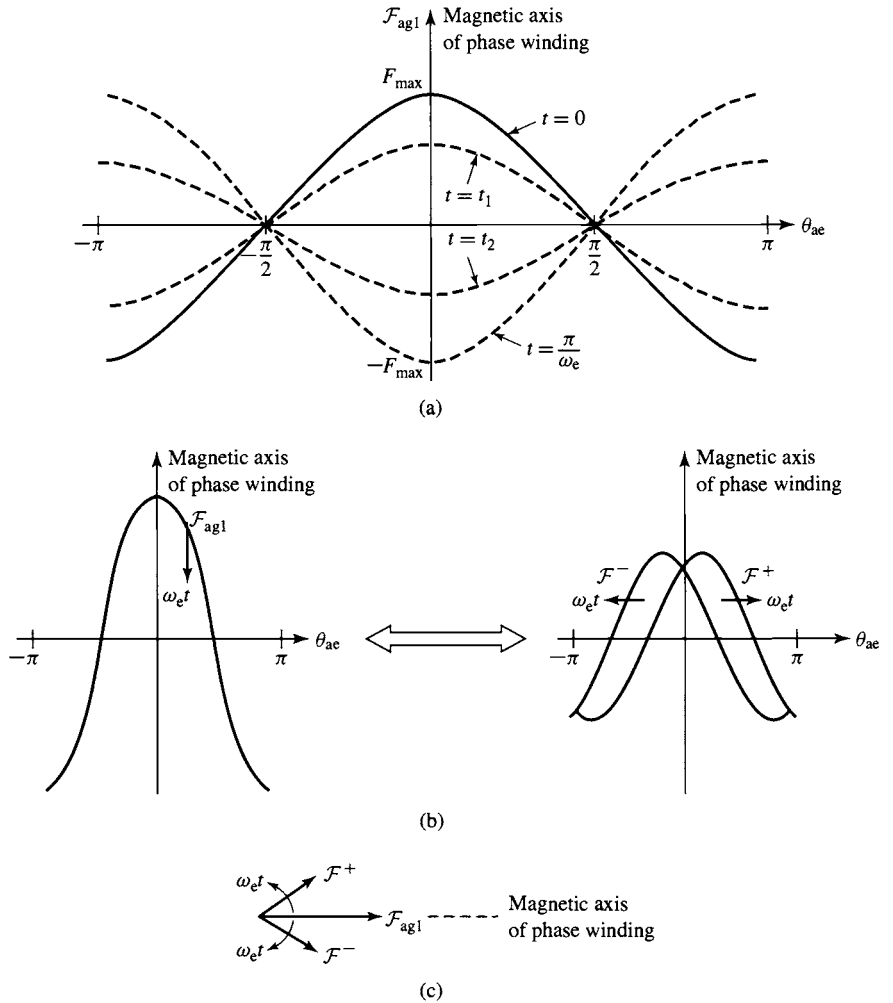
Figure 4.28 Single-phase-winding space-fundamental air-gap mmf: (a) mmf distribution of a single-phase winding at various times; (b) total mmf $\mathcal{F}_{ag1}$ decomposed into two traveling waves $\mathcal{F}^-$ and $\mathcal{F}^+$; (c) phasor decomposition of $\mathcal{F}_{ag1}$.

the mmf distribution is given by

$$\mathcal{F}_{ag1} = F_{max} \cos\left(\frac{\text{poles}}{2}\theta_a\right) \cos \omega_e t$$

$$= F_{max} \cos(\theta_{ae}) \cos \omega_e t \tag{4.18}$$

Equation 4.18 has been written in a form to emphasize the fact that the result is an mmf distribution of maximum amplitude.

$$F_{max} = \frac{4}{\pi}\left(\frac{k_w N_{ph}}{\text{poles}}\right) I_a \tag{4.19}$$

This mmf distribution remains fixed in space with an amplitude that varies sinusoidally in time at frequency ω_e, as shown in Fig. 4.28a. Note that, to simplify the notation, Eq. 4.1 has been used to express the mmf distribution of Eq. 4.18 in terms of the electrical angle θ_{ae}.

Use of a common trigonometric identity[1] permits Eq. 4.18 to be rewritten in the form

$$\mathcal{F}_{ag1} = F_{max} \left[\frac{1}{2} \cos (\theta_{ae} - \omega_e t) + \frac{1}{2} \cos (\theta_{ae} + \omega_e t) \right] \qquad (4.20)$$

which shows that the mmf of a single-phase winding can be resolved into two rotating mmf waves each of amplitude one-half the maximum amplitude of $\mathcal{F}_{ag1}$ with one, $\mathcal{F}_{ag1}^+$, traveling in the $+\theta_a$ direction and the other, $\mathcal{F}_{ag1}^-$, traveling in the $-\theta_a$ direction, both with electrical angular velocity ω_e (equal to a mechanical angular velocity of $2\omega_e/poles$):

$$\mathcal{F}_{ag1}^+ = \frac{1}{2} F_{max} \cos (\theta_{ae} - \omega_e t) \qquad (4.21)$$

$$\mathcal{F}_{ag1}^- = \frac{1}{2} F_{max} \cos (\theta_{ae} + \omega_e t) \qquad (4.22)$$

This decomposition is shown graphically in Fig. 4.28b and in a phasor representation in Fig. 4.28c.

The fact that the air-gap mmf of a single-phase winding excited by a source of alternating current can be resolved into rotating traveling waves is an important conceptual step in understanding ac machinery. As shown in Section 4.5.2, in polyphase ac machinery the windings are equally displaced in space phase, and the winding currents are similarly displaced in time phase, with the result that the negative-traveling flux waves of the various windings sum to zero while the positive-traveling flux waves reinforce, giving a single positive-traveling flux wave.

In single-phase ac machinery, the positive-traveling flux wave produces useful torque while the negative-traveling flux wave produces both negative and pulsating torque as well as losses. These machines are designed so as to minimize the effects of the negative-traveling flux wave, although, unlike in the case of polyphase machinery, these effects cannot be totally eliminated.

4.5.2 MMF Wave of a Polyphase Winding

In this section we study the mmf distributions of three-phase windings such as those found on the stator of three-phase induction and synchronous machines. The analyses presented can be readily extended to a polyphase winding with any number of phases. Once again attention is focused on a two-pole machine or one pair of poles of a multipole winding.

In a three-phase machine, the windings of the individual phases are displaced from each other by 120 electrical degrees in space around the airgap circumference, as shown by coils a, $-a$, b, $-b$, and c, $-c$ in Fig. 4.29. The concentrated full-pitch

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[1] $\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha - \beta) + \frac{1}{2} \cos (\alpha + \beta)$
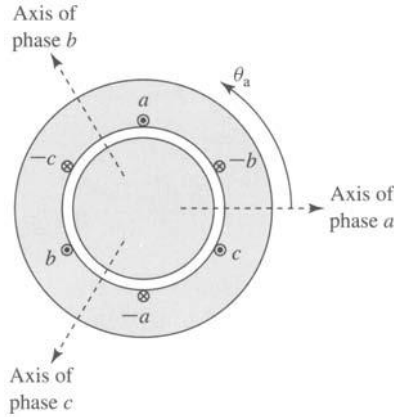
Figure 4.29 Simplified two-pole three-phase stator winding.

coils shown here may be considered to represent distributed windings producing sinusoidal mmf waves centered on the magnetic axes of the respective phases. The space-fundamental sinusoidal mmf waves of the three phases are accordingly displaced 120 electrical degrees in space. Each phase is excited by an alternating current which varies in magnitude sinusoidally with time. Under balanced three-phase conditions, the instantaneous currents are

$$i_a = I_m \cos \omega_e t \tag{4.23}$$

$$i_b = I_m \cos (\omega_e t - 120°) \tag{4.24}$$

$$i_c = I_m \cos (\omega_e t + 120°) \tag{4.25}$$

where I_m is the maximum value of the current and the time origin is arbitrarily taken as the instant when the phase-a current is a positive maximum. The phase sequence is assumed to be abc. The instantaneous currents are shown in Fig. 4.30. The dots and crosses in the coil sides (Fig. 4.29) indicate the reference directions for positive phase currents.

The mmf of phase a has been shown to be

$$\mathcal{F}_{a1} = \mathcal{F}_{a1}^+ + \mathcal{F}_{a1}^- \tag{4.26}$$

where

$$\mathcal{F}_{a1}^+ = \frac{1}{2} F_{max} \cos (\theta_{ae} - \omega_e t) \tag{4.27}$$

$$\mathcal{F}_{a1}^- = \frac{1}{2} F_{max} \cos (\theta_{ae} + \omega_e t) \tag{4.28}$$

and

$$F_{max} = \frac{4}{\pi} \left(\frac{k_w N_{ph}}{poles} \right) I_m \tag{4.29}$$
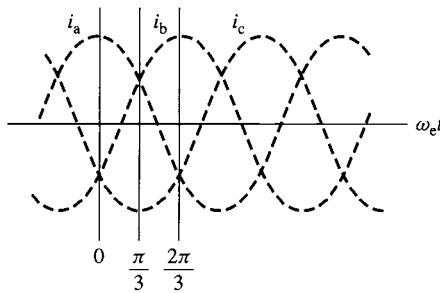
Figure 4.30 Instantaneous phase currents under balanced three-phase conditions.

Note that to avoid excessive notational complexity, the subscript ag has been dropped; here the subscript a1 indicates the space-fundamental component of the phase-a air-gap mmf.

Similarly, for phases b and c, whose axes are at $\theta_a = 120°$ and $\theta_a = -120°$, respectively,

$$\mathcal{F}_{b1} = \mathcal{F}_{b1}^+ + \mathcal{F}_{b1}^- \tag{4.30}$$

$$\mathcal{F}_{b1}^+ = \frac{1}{2}F_{max}\cos(\theta_{ae} - \omega_e t) \tag{4.31}$$

$$\mathcal{F}_{b1}^- = \frac{1}{2}F_{max}\cos(\theta_{ae} + \omega_e t + 120°) \tag{4.32}$$

and

$$\mathcal{F}_{c1} = \mathcal{F}_{c1}^+ + \mathcal{F}_{c1}^- \tag{4.33}$$

$$\mathcal{F}_{c1}^+ = \frac{1}{2}F_{max}\cos(\theta_{ae} - \omega_e t) \tag{4.34}$$

$$\mathcal{F}_{c1}^- = \frac{1}{2}F_{max}\cos(\theta_{ae} + \omega_e t - 120°) \tag{4.35}$$

The total mmf is the sum of the contributions from each of the three phases

$$\mathcal{F}(\theta_{ae}, t) = \mathcal{F}_{a1} + \mathcal{F}_{b1} + \mathcal{F}_{c1} \tag{4.36}$$

This summation can be performed quite easily in terms of the positive- and negative-traveling waves. The negative-traveling waves sum to zero

$$\mathcal{F}^-(\theta_{ae}, t) = \mathcal{F}_{a1}^- + \mathcal{F}_{b1}^- + \mathcal{F}_{c1}^-$$

$$= \frac{1}{2}F_{max}[\cos(\theta_{ae} + \omega_e t) + \cos(\theta_{ae} + \omega_e t - 120°)$$

$$+ \cos(\theta_{ae} + \omega_e t + 120°)]$$

$$= 0 \tag{4.37}$$

while the positive-traveling waves reinforce

$$\mathcal{F}^+(\theta_{ae}, t) = \mathcal{F}_{a1}^+ + \mathcal{F}_{b1}^+ + \mathcal{F}_{c1}^+$$

$$= \frac{3}{2} F_{max} \cos{(\theta_{ae} - \omega_e t)} \tag{4.38}$$

Thus, the result of displacing the three windings by $120°$ in space phase and displacing the winding currents by $120°$ in time phase is a single positive-traveling mmf wave

$$\mathcal{F}(\theta_{ae}, t) = \frac{3}{2} F_{max} \cos{(\theta_{ae} - \omega_e t)}$$

$$= \frac{3}{2} F_{max} \cos{\left(\left(\frac{poles}{2}\right)\theta_a - \omega_e t\right)} \tag{4.39}$$

The air-gap mmf wave described by Eq. 4.39 is a space-fundamental sinusoidal function of the electrical space angle θ_{ae} (and hence of the space angle $\theta_a = (2/poles)\theta_{ae}$). It has a constant amplitude of $(3/2)F_{max}$, i.e., 1.5 times the amplitude of the air-gap mmf wave produced by the individual phases alone. It has a positive peak at angle $\theta_a = (2/poles)\omega_e t$. Thus, under balanced three-phase conditions, the three-phase winding produces an air-gap mmf wave which rotates at *synchronous angular velocity* ω_s

$$\omega_s = \left(\frac{2}{poles}\right)\omega_e \tag{4.40}$$

where

ω_e = angular frequency of the applied electrical excitation [rad/sec]

ω_s = synchronous spatial angular velocity of the air-gap mmf wave [rad/sec]

The corresponding *synchronous speed* n_s in r/min can be expressed in terms of the applied electrical frequency $f_e = \omega_e/(2\pi)$ in Hz as

$$n_s = \left(\frac{120}{poles}\right) f_e \quad \text{r/min} \tag{4.41}$$

In general, a rotating field of constant amplitude will be produced by a q-phase winding excited by balanced q-phase currents of frequency f_e when the respective phase axes are located $2\pi/q$ electrical radians apart in space. The amplitude of this flux wave will be $q/2$ times the maximum contribution of any one phase, and the synchronous angular velocity will remain $\omega_s = (\frac{2}{poles})\omega_e$ radians per second.

In this section, we have seen that a polyphase winding excited by balanced polyphase currents produces a rotating mmf wave. Production of a rotating mmf wave and the corresponding rotating magnetic flux is key to the operation of polyphase rotating electrical machinery. It is the interaction of this magnetic flux wave with that of the rotor which produces torque. Constant torque is produced when rotor-produced magnetic flux rotates in sychronism with that of the stator.

4.5.3 Graphical Analysis of Polyphase MMF

For balanced three-phase currents as given by Eqs. 4.23 to 4.25, the production of a rotating mmf can also be shown graphically. Consider the state of affairs at $t = 0$ (Fig. 4.30), the moment when the phase-a current is at its maximum value I_m. The mmf of phase a then has its maximum value F_{max}, as shown by the vector $F_a = F_{max}$ drawn along the magnetic axis of phase a in the two-pole machine shown schematically in Fig. 4.31a. At this moment, currents i_b and i_c are both $I_m/2$ in the negative direction, as shown by the dots and crosses in Fig. 4.31a indicating the actual instantaneous directions. The corresponding mmf's of phases b and c are shown by the vectors F_b and F_c, both of magnitude $F_{max}/2$ drawn in the negative direction along the magnetic axes of phases b and c, respectively. The resultant, obtained by adding the individual contributions of the three phases, is a vector of magnitude $F = \frac{3}{2}F_{max}$ centered on the axis of phase a. It represents a sinusoidal space wave with its positive peak centered on the axis of phase a and having an amplitude $\frac{3}{2}$ times that of the phase-a contribution alone.

At a later time $\omega_e t = \pi/3$ (Fig. 4.30), the currents in phases a and b are a positive half maximum, and that in phase c is a negative maximum. The individual mmf components and their resultant are now shown in Fig. 4.31b. The resultant has the same amplitude as at $t = 0$, but it has now rotated counterclockwise 60 electrical degrees in space. Similarly, at $\omega_e t = 2\pi/3$ (when the phase-b current is a positive maximum and the phase-a and phase-c currents are a negative half maximum) the same resultant mmf distribution is again obtained, but it has rotated counterclockwise 60 electrical degrees still farther and is now aligned with the magnetic axis of phase b (see Fig. 4.31c). As time passes, then, the resultant mmf wave retains its sinusoidal form and amplitude but rotates progressively around the air gap; the net result can be seen to be an mmf wave of constant amplitude rotating at a uniform angular velocity.

In one cycle the resultant mmf must be back in the position of Fig. 4.31a. The mmf wave therefore makes one revolution per electrical cycle in a two-pole machine. In a multipole machine the mmf wave travels one pole-pair per electrical cycle and hence one revolution in poles/2 electrical cycles.
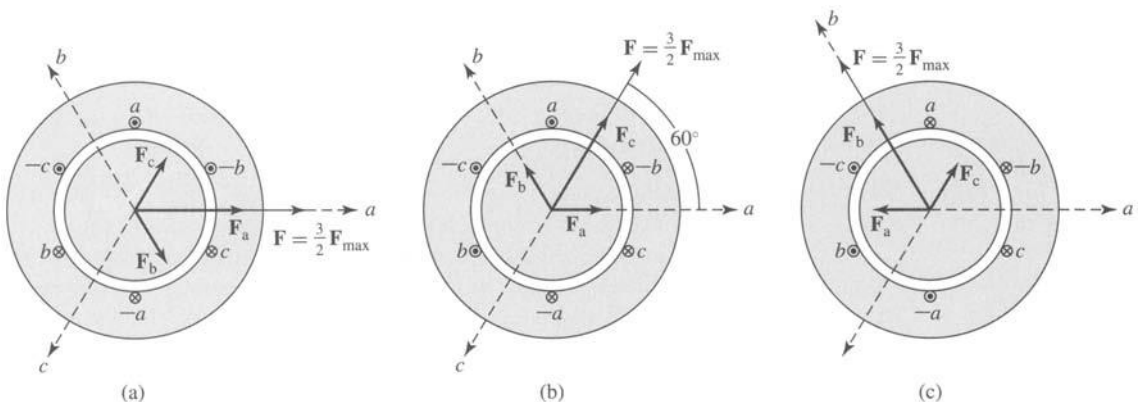
Figure 4.31 The production of a rotating magnetic field by means of three-phase currents.

EXAMPLE 4.3

Consider a three-phase stator excited with balanced, 60-Hz currents. Find the synchronous angular velocity in rad/sec and speed in r/min for stators with two, four, and six poles.

■ **Solution**

For a frequency of $f_e = 60$ Hz, the electrical angular frequency is equal to

$$\omega_e = 2\pi f_e = 120\pi \approx 377 \text{ rad/sec}$$

Using Eqs. 4.40 and 4.41, the following table can be constructed:

Poles	n_s (r/min)	ω_s (rad/sec)
2	3600	$120\pi \approx 377$
4	1800	60π
6	1200	40π

Practice Problem 4.3

Repeat Example 4.3 for a three-phase stator excited by balanced 50-Hz currents.

Solution

Poles	n_s (r/min)	ω_s (rad/sec)
2	3000	100π
4	1500	50π
6	1000	$100\pi/3$

4.6   GENERATED VOLTAGE

The general nature of the induced voltage has already been discussed in Section 4.2. Quantitative expressions for the induced voltage will now be determined.

4.6.1   AC Machines

An elementary ac machine is shown in cross section in Fig. 4.32. The coils on both the rotor and the stator have been shown as concentrated, multiple-turn, full-pitch coils. As we have seen, a machine with distributed windings can be represented in this form simply by multiplying the number of series turns in the winding by a winding factor. Under the assumption of a small air gap, the field winding can be assumed to produce radial space-fundamental air-gap flux of peak flux density B_{peak}. Although Fig. 4.32 shows a two-pole machine, the analysis presented here is for the general case of a multipole machine. As is derived in Example 4.2, if the air gap is uniform, B_{peak} can
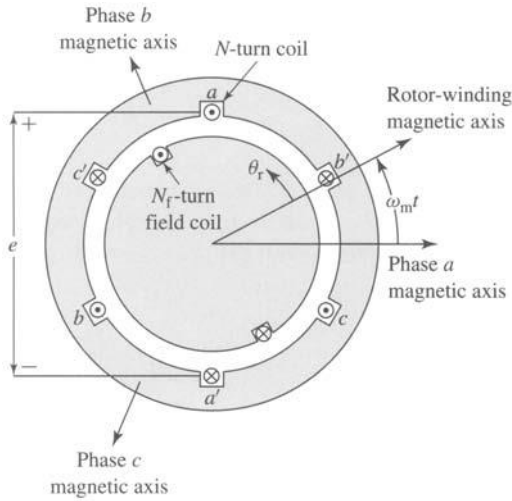
Figure 4.32 Cross-sectional view of an elementary three-phase ac machine.

be found from

$$B_{peak} = \frac{4\mu_0}{\pi g} \left(\frac{k_f N_f}{poles} \right) I_f \tag{4.42}$$

where

g = air-gap length
N_f = total series turns in the field winding
k_f = field-winding winding factor
I_f = field current

When the rotor poles are in line with the magnetic axis of a stator phase, the flux linkage with a stator phase winding is $k_w N_{ph} \Phi_p$, where Φ_p is the air-gap flux per pole [Wb]. For the assumed sinusoidal air-gap flux-density

$$B = B_{peak} \cos \left(\frac{poles}{2} \theta_r \right) \tag{4.43}$$

Φ_p can be found as the integral of the flux density over the pole area

$$\Phi_p = l \int_{-\pi/poles}^{+\pi/poles} B_{peak} \cos \left(\frac{poles}{2} \theta_r \right) r \, d\theta_r$$

$$= \left(\frac{2}{poles} \right) 2 B_{peak} l r \tag{4.44}$$

Here,

θ_r = angle measured from the rotor magnetic axis

r = radius to air gap

l = axial length of the stator/rotor iron

As the rotor turns, the flux linkage varies cosinusoidally with the angle between the magnetic axes of the stator coil and rotor. With the rotor spinning at constant angular velocity ω_m, the flux linkage with the phase-a stator coil is

$$\lambda_a = k_w N_{ph} \Phi_p \cos \left(\left(\frac{\text{poles}}{2} \right) \omega_m t \right)$$

$$= k_w N_{ph} \Phi_p \cos \omega_{me} t \tag{4.45}$$

where time t is arbitrarily chosen as zero when the peak of the flux-density wave coincides with the magnetic axis of phase a. Here,

$$\omega_{me} = \left(\frac{\text{poles}}{2} \right) \omega_m \tag{4.46}$$

is the mechanical rotor velocity expressed in electrical rad/sec.

By Faraday's law, the voltage induced in phase a is

$$e_a = \frac{d\lambda_a}{dt} = k_w N_{ph} \frac{d\Phi_p}{dt} \cos \omega_{me} t$$

$$- \omega_{me} k_w N_{ph} \Phi_p \sin \omega_{me} t \tag{4.47}$$

The polarity of this induced voltage is such that if the stator coil were short-circuited, the induced voltage would cause a current to flow in the direction that would oppose any change in the flux linkage of the stator coil. Although Eq. 4.47 is derived on the assumption that only the field winding is producing air-gap flux, the equation applies equally well to the general situation where Φ_p is the net air-gap flux per pole produced by currents on both the rotor and the stator.

The first term on the right-hand side of Eq. 4.47 is a transformer voltage and is present only when the amplitude of the air-gap flux wave changes with time. The second term is the *speed voltage* generated by the relative motion of the air-gap flux wave and the stator coil. In the normal steady-state operation of most rotating machines, the amplitude of the air-gap flux wave is constant; under these conditions the first term is zero and the generated voltage is simply the speed voltage. The term *electromotive force* (abbreviated *emf*) is often used for the speed voltage. Thus, for constant air-gap flux,

$$e_a = -\omega_{me} k_w N_{ph} \Phi_p \sin \omega_{me} t \tag{4.48}$$

EXAMPLE 4.4

The so-called *cutting-of-flux* equation states that the voltage v induced in a wire of length l (in the frame of the wire) moving with respect to a constant magnetic field with flux density of

magnitude B is given by

$$v = l \mathrm{v}_\perp B$$

where $\mathrm{v}_\perp$ is the component of the wire velocity perpendicular to the direction of the magnetic flux density.

Consider the two-pole elementary three-phase machine of Fig. 4.32. Assume the rotor-produced air-gap flux density to be of the form

$$B_{\mathrm{ag}}(\theta_\mathrm{r}) = B_{\mathrm{peak}} \sin \theta_\mathrm{r}$$

and the rotor to rotate at constant angular velocity ω_e. (Note that since this is a two-pole machine, $\omega_\mathrm{m} = \omega_\mathrm{e}$). Show that if one assumes that the armature-winding coil sides are in the air gap and not in the slots, the voltage induced in a full-pitch, N-turn concentrated armature phase coil can be calculated from the cutting-of-flux equation and that it is identical to that calculated using Eq. 4.48. Let the average air-gap radius be r and the air-gap length be g ($g \ll r$).

■ **Solution**

We begin by noting that the cutting-of-flux equation requires that the conductor be moving and the magnetic field to be nontime varying. Thus in order to apply it to calculating the stator magnetic field, we must translate our reference frame to the rotor.

In the rotor frame, the magnetic field is constant and the stator coil sides, when moved to the center of the air gap at radius r, appear to be moving with velocity $\omega_{\mathrm{me}}r$ which is perpendicular to the radially-directed air-gap flux. If the rotor and phase-coil magnetic axes are assumed to be aligned at time $t = 0$, the location of a coil side as a function of time will be given by $\theta_\mathrm{r} = -\omega_{\mathrm{me}}t$. The voltage induced in one side of one turn can therefore be calculated as

$$e_1 = l \mathrm{v}_\perp B_{\mathrm{ag}}(\theta_\mathrm{r}) = l \omega_{\mathrm{me}} r B_{\mathrm{peak}} \sin(-\omega_{\mathrm{me}}t)$$

There are N turns per coil and two sides per turn. Thus the total coil voltage is given by

$$e = 2Ne_1 = -2Nl\omega_{\mathrm{me}}r B_{\mathrm{peak}} \sin \omega_{\mathrm{me}}t$$

From Eq. 4.48, the voltage induced in the full-pitched, 2-pole stator coil is given by

$$e = -\omega_{\mathrm{me}}N\Phi_\mathrm{p} \sin \omega_{\mathrm{me}}t$$

Substituting $\Phi_\mathrm{p} = 2B_{\mathrm{peak}}lr$ from Eq. 4.44 gives

$$e = -\omega_{\mathrm{me}}N(2B_{\mathrm{peak}}lr) \sin \omega_{\mathrm{me}}t$$

which is identical to the voltage determined using the cutting-of-flux equation.

In the normal steady-state operation of ac machines, we are usually interested in the rms values of voltages and currents rather than their instantaneous values. From Eq. 4.48 the maximum value of the induced voltage is

$$E_{\max} = \omega_{\mathrm{me}}k_\mathrm{w}N_{\mathrm{ph}}\Phi_\mathrm{p} = 2\pi f_{\mathrm{me}}k_\mathrm{w}N_{\mathrm{ph}}\Phi_\mathrm{p} \tag{4.49}$$

Its rms value is

$$E_{rms} = \frac{2\pi}{\sqrt{2}} f_{me} k_w N_{ph} \Phi_p = \sqrt{2}\, \pi f_{me} k_w N_{ph} \Phi_p \qquad (4.50)$$

where f_{me} is the electrical speed of the rotor measured in Hz, which is also equal to the electrical frequency of the generated voltage. Note that these equations are identical in form to the corresponding emf equations for a transformer. Relative motion of a coil and a constant-amplitude spatial flux-density wave in a rotating machine produces voltage in the same fashion as does a time-varying flux in association with stationary coils in a transformer. Rotation introduces the element of time variation and transforms a space distribution of flux density into a time variation of voltage.

The voltage induced in a single winding is a single-phase voltage. For the production of a set of balanced, three-phase voltages, it follows that three windings displaced 120 electrical degrees in space must be used, as shown in elementary form in Fig. 4.12. The machine of Fig. 4.12 is shown to be Y-connected and hence each winding voltage is a phase-neutral voltage. Thus, Eq. 4.50 gives the rms line-neutral voltage produced in this machine when N_{ph} is the total series turns per phase. For a Δ-connected machine, the voltage winding voltage calculated from Eq. 4.50 would be the machine line-line voltage.

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EXAMPLE 4.5

A two-pole, three-phase, Y-connected 60-Hz round-rotor synchronous generator has a field winding with N_f distributed turns and winding factor k_f. The armature winding has N_a turns per phase and winding factor k_a. The air-gap length is g, and the mean air-gap radius is r. The armature-winding active length is l. The dimensions and winding data are

$$N_f = 68 \text{ series turns} \qquad k_f = 0.945$$
$$N_a = 18 \text{ series turns/phase} \qquad k_a = 0.933$$
$$r = 0.53 \text{ m} \qquad g = 4.5 \text{ cm}$$
$$l = 3.8 \text{ m}$$

The rotor is driven by a steam turbine at a speed of 3600 r/min. For a field current of $I_f = 720$ A dc, compute (a) the peak fundamental mmf $(F_{ag1})_{peak}$ produced by the field winding, (b) the peak fundamental flux density $(B_{ag1})_{peak}$ in the air gap, (c) the fundamental flux per pole Φ_p, and (d) the rms value of the open-circuit voltage generated in the armature.

■ **Solution**

a. From Eq. 4.8

$$(F_{ag1})_{peak} = \frac{4}{\pi}\left(\frac{k_f N_f}{poles}\right) I_f = \frac{4}{\pi}\left(\frac{0.945 \times 68}{2}\right) 720$$

$$= \frac{4}{\pi}(32.1)720 = 2.94 \times 10^4 \text{ A} \cdot \text{turns/pole}$$

b. Using Eq. 4.12, we get

$$(B_{ag1})_{peak} = \frac{\mu_0 (F_{ag1})_{peak}}{g} = \frac{4\pi \times 10^{-7} \times 2.94 \times 10^4}{4.5 \times 10^{-2}} = 0.821 \text{ T}$$

Because of the effect of the slots containing the armature winding, most of the air-gap flux is confined to the stator teeth. The flux density in the teeth at a pole center is higher than the value calculated in part (b), probably by a factor of about 2. In a detailed design this flux density must be calculated to determine whether the teeth are excessively saturated.

c. From Eq. 4.44

$$\Phi_p = 2(B_{ag1})_{peak} lr = 2(0.821)(3.8)(0.53) = 3.31 \text{ Wb}$$

d. From Eq. 4.50 with $f_{me} = 60$ Hz

$$E_{rms,line-neutral} = \sqrt{2}\, \pi f_{me} k_a N_a \Phi_p = \sqrt{2}\, \pi (60)(0.933)(18)(3.31)$$

$$= 14.8 \text{ kV rms}$$

The line-line voltage is thus

$$E_{rms,line-line} = \sqrt{3}\, (14.8 \text{ kV}) = 25.7 \text{ kV rms}$$

Practice Problem 4.4

The rotor of the machine of Example 4.5 is to be rewound. The new field winding will have a total of 76 series turns and a winding factor of 0.925. (a) Calculate the field current which will result in a peak air-gap flux density of 0.83 T. (b) Calculate the corresponding rms line-line open-circuit voltage which will result if this modified machine is operated at this value of field current and 3600 rpm.

Solution

    a. $I_f = 696$ A

    b. $E_{rms,line-line} = 26.0$ kV rms

4.6.2   DC Machines

In a dc machine, although the ultimate objective is the generation of dc voltage, ac voltages are produced in the armature-winding coils as these coils rotate through the dc flux distribution of the stationary field winding. The armature-winding alternating voltage must therefore be rectified. Mechanical rectification is provided by the commutator as has been discussed in Section 4.2.2.

Consider the single N-turn armature coil of the elementary, two-pole dc machine of Fig. 4.17. The simple two-segment commutator provides full-wave rectification of the coil voltage. Although the spatial distribution of the air-gap flux in dc machines is typically far from sinusoidal, we can approximate the magnitude of the generated voltage by assuming a sinusoidal distribution. As we have seen, such a flux distribution will produce a sinusoidal ac voltage in the armature coil. The rectification action of
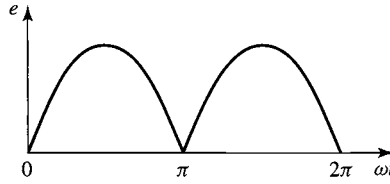
Figure 4.33 Voltage between the brushes in the elementary dc machine of Fig. 4.17.

the commutator will produce a dc voltage across the brushes as in Fig. 4.33. The average, or dc, value of this voltage can be found from taking the average of Eq. 4.48,

$$E_a = \frac{1}{\pi} \int_0^\pi \omega_{me} N \Phi_p \sin(\omega_{me}t) \, d(\omega_{me}t) = \frac{2}{\pi} \omega_{me} N \Phi_p \qquad (4.51)$$

For dc machines it is usually more convenient to express the voltage E_a in terms of the mechanical speed ω_m (rad/sec) or n (r/min). Substitution of Eq. 4.46 in Eq. 4.51 for a multipole machine then yields

$$E_a = \left(\frac{\text{poles}}{\pi}\right) N \Phi_p \omega_m = \text{poles } N \Phi_p \left(\frac{n}{30}\right) \qquad (4.52)$$

The single-coil dc winding implied here is, of course, unrealistic in the practical sense, and it will be essential later to examine the action of commutators more carefully. Actually, Eq. 4.52 gives correct results for the more practical distributed ac armature windings as well, provided N is taken as the total number of turns in series between armature terminals. Usually the voltage is expressed in terms of the total number of active conductors C_a and the number m of parallel paths through the armature winding. Because it takes two coil sides to make a turn and $1/m$ of these are connected in series, the number of series turns is $N_a = C_a/(2m)$. Substitution in Eq. 4.52 then gives

$$E_a = \left(\frac{\text{poles}}{2\pi}\right) \left(\frac{C_a}{m}\right) \Phi_p \omega_m = \left(\frac{\text{poles}}{60}\right) \left(\frac{C_a}{m}\right) \Phi_p n \qquad (4.53)$$

4.7 TORQUE IN NONSALIENT-POLE MACHINES

The behavior of any electromagnetic device as a component in an electromechanical system can be described in terms of its electrical-terminal equations and its displacement and electromechanical torque. The purpose of this section is to derive the voltage and torque equations for an idealized elementary machine, results which can be readily extended later to more complex machines. We derive these equations from two viewpoints and show that basically they stem from the same ideas.

The first viewpoint is essentially the same as that of Section 3.6. The machine will be regarded as a circuit element whose inductances depend on the angular position

of the rotor. The flux linkages λ and magnetic field coenergy will be expressed in terms of the currents and inductances. The torque can then be found from the partial derivative of the energy or coenergy with respect to the rotor position and the terminal voltages from the sum of the resistance drops Ri and the Faraday-law voltages $d\lambda/dt$. The result will be a set of nonlinear differential equations describing the dynamic performance of the machine.

The second viewpoint regards the machine as two groups of windings producing magnetic flux in the air gap, one group on the stator, and the other on the rotor. By making suitable assumptions regarding these fields (similar to those used to derive analytic expressions for the inductances), simple expressions can be derived for the flux linkages and the coenergy in the air gap in terms of the field quantities. The torque and generated voltage can then be found from these expressions. In this fashion, torque can be expressed explicitly as the tendency for two magnetic fields to align, in the same way that permanent magnets tend to align, and generated voltage can be expressed in terms of the relative motion between a field and a winding. These expressions lead to a simple physical picture of the normal steady-state behavior of rotating machines.

4.7.1 Coupled-Circuit Viewpoint

Consider the elementary smooth-air-gap machine of Fig. 4.34 with one winding on the stator and one on the rotor and with θ_m being the mechanical angle between the axes of the two windings. These windings are distributed over a number of slots so that their mmf waves can be approximated by space sinusoids. In Fig. 4.34a the coil sides s, $-s$ and r, $-r$ mark the positions of the centers of the belts of conductors comprising the distributed windings. An alternative way of drawing these windings is shown in Fig. 4.34b, which also shows reference directions for voltages and currents. Here it is assumed that current in the arrow direction produces a magnetic field in the air gap in the arrow direction, so that a single arrow defines reference directions for both current and flux.
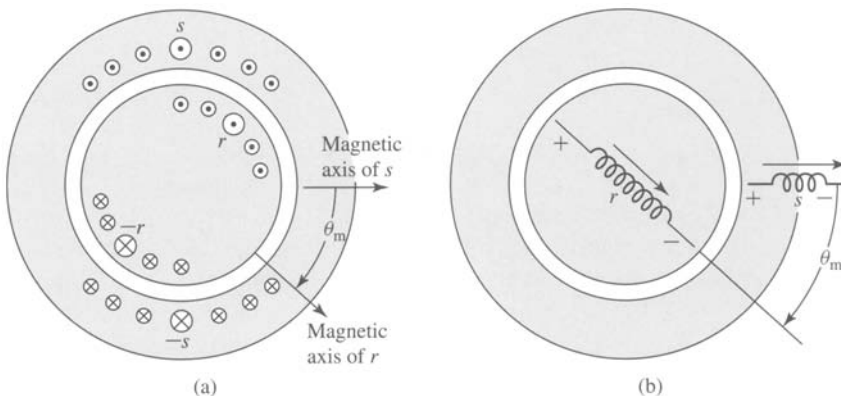

Figure 4.34 Elementary two-pole machine with smooth air gap: (a) winding distribution and (b) schematic representation.

The stator and rotor are concentric cylinders, and slot openings are neglected. Consequently, our elementary model does not include the effects of salient poles, which are investigated in later chapters. We also assume that the reluctances of the stator and rotor iron are negligible. Finally, although Fig. 4.34 shows a two-pole machine, we will write the derivations that follow for the general case of a multipole machine, replacing θ_m by the electrical rotor angle

$$\theta_{me} = \left(\frac{poles}{2}\right)\theta_m \tag{4.54}$$

Based upon these assumptions, the stator and rotor self-inductances L_{ss} and L_{rr} can be seen to be constant, but the stator-to-rotor mutual inductance depends on the electrical angle θ_{me} between the magnetic axes of the stator and rotor windings. The mutual inductance is at its positive maximum when $\theta_{me} = 0$ or 2π, is zero when $\theta_{me} = \pm\pi/2$, and is at its negative maximum when $\theta_{me} = \pm\pi$. On the assumption of sinusoidal mmf waves and a uniform air gap, the space distribution of the air-gap flux wave is sinusoidal, and the mutual inductance will be of the form

$$\mathcal{L}_{sr}(\theta_{me}) = L_{sr}\cos(\theta_{me}) \tag{4.55}$$

where the script letter $\mathcal{L}$ denotes an inductance which is a function of the electrical angle θ_{me}. The italic capital letter L denotes a constant value. Thus L_{sr} is the magnitude of the mutual inductance; its value when the magnetic axes of the stator and rotor are aligned ($\theta_{me} = 0$). In terms of the inductances, the stator and rotor flux linkages λ_s and λ_r are

$$\lambda_s = L_{ss}i_s + \mathcal{L}_{sr}(\theta_{me})i_r = L_{ss}i_s + L_{sr}\cos(\theta_{me})i_r \tag{4.56}$$

$$\lambda_r = \mathcal{L}_{sr}(\theta_{me})i_s + L_{rr}i_r = L_{sr}\cos(\theta_{me})i_s + L_{rr}i_r \tag{4.57}$$

where the inductances can be calculated as in Appendix B. In matrix notation

$$\begin{bmatrix} \lambda_s \\ \lambda_r \end{bmatrix} = \begin{bmatrix} L_{ss} & \mathcal{L}_{sr}(\theta_{me}) \\ \mathcal{L}_{sr}(\theta_{me}) & L_{rr} \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix} \tag{4.58}$$

The terminal voltages v_s and v_r are

$$v_s = R_s i_s + \frac{d\lambda_s}{dt} \tag{4.59}$$

$$v_r = R_r i_r + \frac{d\lambda_r}{dt} \tag{4.60}$$

where R_s and R_r are the resistances of the stator and rotor windings respectively.

When the rotor is revolving, θ_{me} must be treated as a variable. Differentiation of Eqs. 4.56 and 4.57 and substitution of the results in Eqs. 4.59 and 4.60 then give

$$v_s = R_s i_s + L_{ss}\frac{di_s}{dt} + L_{sr}\cos(\theta_{me})\frac{di_r}{dt} - L_{sr}i_r\sin(\theta_{me})\frac{d\theta_{me}}{dt} \tag{4.61}$$

$$v_r = R_r i_r + L_{rr}\frac{di_r}{dt} + L_{sr}\cos(\theta_{me})\frac{di_r}{dt} - L_{sr}i_s\sin(\theta_{me})\frac{d\theta_{me}}{dt} \tag{4.62}$$

where

$$\frac{d\theta_{me}}{dt} = \omega_{me} = \left(\frac{\text{poles}}{2}\right)\omega_m \tag{4.63}$$

is the instantaneous speed in electrical radians per second. In a two-pole machine (such as that of Fig. 4.34), θ_{me} and ω_{me} are equal to the instantaneous shaft angle θ_m and the shaft speed ω_m respectively. In a multipole machine, they are related by Eqs. 4.54 and 4.46. The second and third terms on the right-hand sides of Eqs. 4.61 and 4.62 are $L(di/dt)$ induced voltages like those induced in stationary coupled circuits such as the windings of transformers. The fourth terms are caused by mechanical motion and are proportional to the instantaneous speed. These are the speed voltage terms which correspond to the interchange of power between the electric and mechanical systems.

The electromechanical torque can be found from the coenergy. From Eq. 3.70

$$W'_{fld} = \frac{1}{2}L_{ss}i_s^2 + \frac{1}{2}L_{rr}i_r^2 + L_{sr}i_s i_r \cos\theta_{me}$$

$$= \frac{1}{2}L_{ss}i_s^2 + \frac{1}{2}L_{rr}i_r^2 + L_{sr}i_s i_r \cos\left(\left(\frac{\text{poles}}{2}\right)\theta_m\right) \tag{4.64}$$

Note that the coenergy of Eq. 4.64 has been expressed specifically in terms of the shaft angle θ_m because the torque expression of Eq. 3.68 requires that the torque be obtained from the derivative of the coenergy with respect to the spatial angle θ_m and not with respect to the electrical angle θ_{me}. Thus, from Eq. 3.68

$$T = \frac{\partial W'_{fld}(i_s, i_r, \theta_m)}{\partial\theta_m}\bigg|_{i_s,i_r} = -\left(\frac{\text{poles}}{2}\right)L_{sr}i_s i_r \sin\left(\frac{\text{poles}}{2}\theta_m\right)$$

$$= -\left(\frac{\text{poles}}{2}\right)L_{sr}i_s i_r \sin\theta_{me} \tag{4.65}$$

where T is the electromechanical torque acting to accelerate the rotor (i.e., a positive torque acts to increase θ_m). The negative sign in Eq. 4.65 means that the electromechanical torque acts in the direction to bring the magnetic fields of the stator and rotor into alignment.

Equations 4.61, 4.62, and 4.65 are a set of three equations relating the electrical variables v_s, i_s, v_r, i_r and the mechanical variables T and θ_m. These equations, together with the constraints imposed on the electrical variables by the networks connected to the terminals (sources or loads and external impedances) and the constraints imposed on the rotor (applied torques and inertial, frictional, and spring torques), determine the performance of the device and its characteristics as a conversion device between the external electrical and mechanical systems. These are nonlinear differential equations and are difficult to solve except under special circumstances. We are not specifically concerned with their solution here; rather we are using them merely as steps in the development of the theory of rotating machines.

EXAMPLE 4.6

Consider the elementary two-pole, two-winding machine of Fig. 4.34. Its shaft is coupled to a mechanical device which can be made to absorb or deliver mechanical torque over a wide range of speeds. This machine can be connected and operated in several ways. For this example, let us consider the situation in which the rotor winding is excited with direct current I_r and the stator winding is connected to an ac source which can either absorb or deliver electric power. Let the stator current be

$$i_s = I_s \cos \omega_e t$$

where $t = 0$ is arbitrarily chosen as the moment when the stator current has its peak value.

a. Derive an expression for the magnetic torque developed by the machine as the speed is varied by control of the mechanical device connected to its shaft.

b. Find the speed at which average torque will be produced if the stator frequency is 60 Hz.

c. With the assumed current-source excitations, what voltages are induced in the stator and rotor windings at synchronous speed ($\omega_m = \omega_e$)?

■ Solution

a. From Eq. 4.65 for a two-pole machine

$$T = -L_{sr} i_s i_r \sin \theta_m$$

For the conditions of this problem, with $\theta_m = \omega_m t + \delta$

$$T = -L_{sr} I_s I_r \cos \omega_e t \sin (\omega_m t + \delta)$$

where ω_m is the clockwise angular velocity impressed on the rotor by the mechanical drive and δ is the angular position of the rotor at $t = 0$. Using a trigonometric identity,[2] we have

$$T = -\frac{1}{2} L_{sr} I_s I_r \{ \sin [(\omega_m + \omega_e)t + \delta] + \sin [(\omega_m - \omega_e)t + \delta] \}$$

The torque consists of two sinusoidally time-varying terms of frequencies $\omega_m + \omega_e$ and $\omega_m - \omega_e$. As shown in Section 4.5, ac current applied to the two-pole, single-phase stator winding in the machine of Fig. 4.34 creates two flux waves, one traveling in the positive θ_m direction with angular velocity ω_e and the second traveling in the negative θ_m direction also with angular velocity ω_e. It is the interaction of the rotor with these two flux waves which results in the two components of the torque expression.

b. Except when $\omega_m = \pm\omega_e$, the torque averaged over a sufficiently long time is zero. But if $\omega_m = \omega_e$, the rotor is traveling in synchronism with the positive-traveling stator flux wave, and the torque becomes

$$T = -\frac{1}{2} L_{sr} I_s I_r [\sin (2\omega_e t + \delta) + \sin \delta]$$

---

[2] $\sin \alpha \cos \beta = \frac{1}{2}[\sin (\alpha + \beta) + \sin (\alpha - \beta)]$

The first sine term is a double-frequency component whose average value is zero. The second term is the average torque

$$T_{avg} = -\frac{1}{2}L_{sr}I_s I_r \sin \delta$$

A nonzero average torque will also be produced when $\omega_m = -\omega_e$ which merely means rotation in the counterclockwise direction; the rotor is now traveling in synchronism with the negative-traveling stator flux wave. The negative sign in the expression for T_{avg} means that a positive value of T_{avg} acts to reduce δ.

This machine is an idealized single-phase synchronous machine. With a stator frequency of 60 Hz, it will produce nonzero average torque for speeds of $\pm\omega_m = \omega_e = 2\pi 60$ rad/sec, corresponding to speeds of ± 3600 r/min as can be seen from Eq. 4.41.

c. From the second and fourth terms of Eq. 4.61 (with $\theta_e = \theta_m = \omega_m t + \delta$), the voltage induced in the stator when $\omega_m = \omega_e$ is

$$e_s = -\omega_e L_{ss}I_s \sin \omega_e t - \omega_e L_{sr}I_r \sin (\omega_e t + \delta)$$

From the third and fourth terms of Eq. 4.62, the voltage induced in the rotor is

$$e_r = -\omega_0 L_{sr}I_s[\sin \omega_e t \cos (\omega_e t + \delta) + \cos \omega_s t \sin (\omega_e t + \delta)]$$

$$= -\omega_e L_{sr}I_s \sin (2\omega_e t + \delta)$$

The backwards-rotating component of the stator flux induces a double-frequency voltage in the rotor, while the forward-rotating component, which is rotating in sychronism with the rotor, appears as a dc flux to the rotor, and hence induces no voltage in the rotor winding.

Now consider a uniform-air-gap machine with several stator and rotor windings. The same general principles that apply to the elementary model of Fig. 4.34 also apply to the multiwinding machine. Each winding has its own self-inductance as well as mutual inductances with other windings. The self-inductances and mutual inductances between pairs of windings on the same side of the air gap are constant on the assumption of a uniform gap and negligible magnetic saturation. However, the mutual inductances between pairs of stator and rotor windings vary as the cosine of the angle between their magnetic axes. Torque results from the tendency of the magnetic field of the rotor windings to line up with that of the stator windings. It can be expressed as the sum of terms like that of Eq. 4.65.

EXAMPLE 4.7

Consider a 4-pole, three-phase synchronous machine with a uniform air gap. Assume the armature-winding self- and mutual inductances to be constant

$$L_{aa} = L_{bb} = L_{cc}$$

$$L_{ab} = L_{bc} = L_{ca}$$

Similarly, assume the field-winding self-inductance L_f to be constant while the mutual inductances between the field winding and the three armature phase windings will vary with the angle θ_m between the magnetic axis of the field winding and that of phase a

$$\mathcal{L}_{af} = L_{af} \cos 2\theta_m$$

$$\mathcal{L}_{bf} = L_{af} \cos (2\theta_m - 120°)$$

$$\mathcal{L}_{cf} = L_{af} \cos (2\theta_m + 120°)$$

Show that when the field is excited with constant current I_f and the armature is excited by balanced-three-phase currents of the form

$$i_a = I_a \cos (\omega_e t + \delta)$$

$$i_b = I_a \cos (\omega_e t - 120° + \delta)$$

$$i_c = I_a \cos (\omega_e t + 120° + \delta)$$

the torque will be constant when the rotor travels at synchronous speed ω_s as given by Eq. 4.40.

■ Solution

The torque can be calculated from the coenergy as described in Section 3.6. This particular machine is a four-winding system and the coenergy will consist of four terms involving 1/2 the self-inductance multiplied by the square of the corresponding winding current as well as product-terms consisting of the mutual inductances between pairs of windings multiplied by the corresponding winding currents. Noting that only the terms involving the mutual inductances between the field winding and the three armature phase windings will contain terms that vary with θ_m, we can write the coenergy in the form

$$
\begin{aligned}
W'_{fld}(i_a, i_b, i_c, i_f, \theta_m) &= (\text{constant terms}) + \mathcal{L}_{af} i_a i_f + \mathcal{L}_{bf} i_b i_f + \mathcal{L}_{cf} i_c i_f \\
&= (\text{constant terms}) + L_{af} I_a I_f [\cos 2\theta_m \cos (\omega_e t + \delta) \\
&\quad + \cos (2\theta_m - 120°) \cos (\omega_e t - 120° + \delta) \\
&\quad + \cos (2\theta_m + 120°) \cos (\omega_e t + 120° + \delta)] \\
&= (\text{constant terms}) + \frac{3}{2} L_{af} I_a I_f \cos (2\theta_m - \omega_e t - \delta)
\end{aligned}
$$

The torque can now be found from the partial derivative of W'_{fld} with respect to θ_m

$$
\begin{aligned}
T &= \left. \frac{\partial W'_{fld}}{\partial \theta_m} \right|_{i_a, i_b, i_c, i_f} \\
&= -3 L_{af} I_a I_f \sin (2\theta_m - \omega_e t - \delta)
\end{aligned}
$$

From this expression, we see that the torque will be constant when the rotor rotates at synchronous velocity ω_s such that

$$\theta_m = \omega_s t = \left(\frac{\omega_e}{2} \right) t$$

in which case the torque will be equal to

$$T = 3 L_{af} I_a I_f \sin \delta$$

Note that unlike the case of the single-phase machine of Example 4.6, the torque for this three-phase machine operating at synchronous velocity under balanced-three-phase conditions is constant. As we have seen, this is due to the fact that the stator mmf wave consists of a single rotating flux wave, as opposed to the single-phase case in which the stator phase current produces both a forward- and a backward-rotating flux wave. This backwards flux wave is not in synchronism with the rotor and hence is responsible for the double-frequency time-varying torque component seen in Example 4.6.

For the four-pole machine of Example 4.7, find the synchronous speed at which a constant torque will be produced if the rotor currents are of the form

$$i_a = I_a \cos{(\omega_e t + \delta)}$$

$$i_b = I_a \cos{(\omega_e t + 120° + \delta)}$$

$$i_c = I_a \cos{(\omega_e t - 120° + \delta)}$$

Solution

$$\omega_s = -(\omega_e/2)$$

In Example 4.7 we found that, under balanced conditions, a four-pole synchronous machine will produce constant torque at a rotational angular velocity equal to half of the electrical excitation frequency. This result can be generalized to show that, under balanced operating conditions, a multiphase, multipole synchronous machine will produce constant torque at a rotor speed, at which the rotor rotates in synchronism with the rotating flux wave produced by the stator currents. Hence, this is known as the *synchronous speed* of the machine. From Eqs. 4.40 and 4.41, the synchronous speed is equal to $\omega_s = (2/\text{poles})\omega_e$ in rad/sec or $n_s = (120/\text{poles})f_e$ in r/min.

4.7.2 Magnetic Field Viewpoint

In the discussion of Section 4.7.1 the characteristics of a rotating machine as viewed from its electric and mechanical terminals have been expressed in terms of its winding inductances. This viewpoint gives little insight into the physical phenomena which occur within the machine. In this section, we will explore an alternative formulation in terms of the interacting magnetic fields.

As we have seen, currents in the machine windings create magnetic flux in the air gap between the stator and rotor, the flux paths being completed through the stator and rotor iron. This condition corresponds to the appearance of magnetic poles on both the stator and the rotor, centered on their respective magnetic axes, as shown in Fig. 4.35a for a two-pole machine with a smooth air gap. Torque is produced by the tendency of the two component magnetic fields to line up their magnetic axes. A useful physical picture is that this is quite similar to the situation of two bar magnets pivoted at their centers on the same shaft; there will be a torque, proportional to the angular displacement of the bar magnets, which will act to align them. In the machine
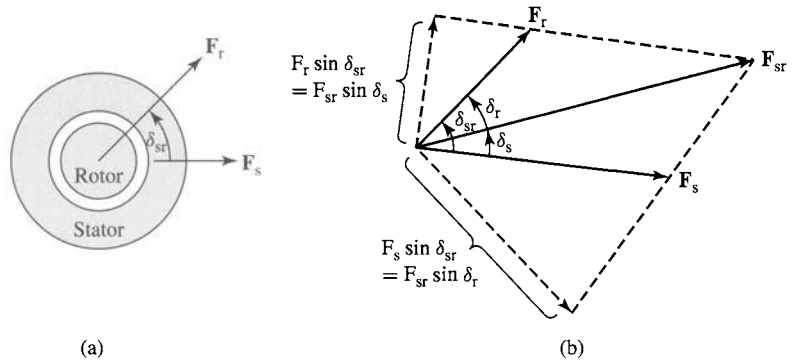
Figure 4.35 Simplified two-pole machine: (a) elementary model and (b) vector diagram of mmf waves. Torque is produced by the tendency of the rotor and stator magnetic fields to align. Note that these figures are drawn with δ_{sr} positive, i.e., with the rotor mmf wave $\mathbf{F_r}$ leading that of the stator $\mathbf{F_s}$.

of Fig. 4.35a, the resulting torque is proportional to the product of the amplitudes of the stator and rotor mmf waves and is also a function of the angle δ_{sr} measured from the axis of the stator mmf wave to that of the rotor. In fact, we will show that, for a smooth-air-gap machine, the torque is proportional to $\sin \delta_{sr}$.

In a typical machine, most of the flux produced by the stator and rotor windings crosses the air gap and links both windings; this is termed the *mutual flux,* directly analogous to the mutual or magnetizing flux in a transformer. However, some of the flux produced by the rotor and stator windings does not cross the air gap; this is analogous to the leakage flux in a transformer. These flux components are referred to as the *rotor leakage flux* and the *stator leakage flux.* Components of this leakage flux include slot and toothtip leakage, end-turn leakage, and space harmonics in the air-gap field.

Only the mutual flux is of direct concern in torque production. The leakage fluxes do affect machine performance however, by virtue of the voltages they induce in their respective windings. Their effect on the electrical characteristics is accounted for by means of leakage inductances, analogous to the use of inclusion of leakage inductances in the transformer models of Chapter 2.

When expressing torque in terms of the winding currents or their resultant mmf's, the resulting expressions do not include terms containing the leakage inductances. Our analysis here, then, will be in terms of the resultant mutual flux. We shall derive an expression for the magnetic coenergy stored in the air gap in terms of the stator and rotor mmfs and the angle δ_{sr} between their magnetic axes. The torque can then be found from the partial derivative of the coenergy with respect to angle δ_{sr}.

For analytical simplicity, we will assume that the radial length g of the air gap (the radial clearance between the rotor and stator) is small compared with the radius of the rotor or stator. For a smooth-air-gap machine constructed from electrical steel with high magnetic permeability, it is possible to show that this will result in air-gap flux which is primarily radially directed and that there is relatively little difference

between the flux density at the rotor surface, at the stator surface, or at any intermediate radial distance in the air gap. The air-gap field then can be represented as a radial field H_{ag} or B_{ag} whose intensity varies with the angle around the periphery. The line integral of H_{ag} across the gap then is simply $H_{ag}g$ and equals the *resultant air-gap mmf* $\mathcal{F}_{sr}$ produced by the stator and rotor windings; thus

$$H_{ag}g = \mathcal{F}_{sr} \tag{4.66}$$

where the script $\mathcal{F}$ denotes the mmf wave as a function of the angle around the periphery.

The mmf waves of the stator and rotor are spatial sine waves with δ_{sr} being the phase angle between their magnetic axes in electrical degrees. They can be represented by the space vectors $\mathbf{F}_s$ and $\mathbf{F}_r$ drawn along the magnetic axes of the stator- and rotor-mmf waves respectively, as in Fig. 4.35b. The resultant mmf $\mathbf{F}_{sr}$ acting across the air gap, also a sine wave, is their vector sum. From the trigonometric formula for the diagonal of a parallelogram, its peak value is found from

$$F_{sr}^2 = F_s^2 + F_r^2 + 2F_s F_r \cos \delta_{sr} \tag{4.67}$$

in which the F's are the peak values of the mmf waves. The resultant radial H_{ag} field is a sinusoidal space wave whose peak value $H_{ag,peak}$ is, from Eq. 4.66,

$$(H_{ag})_{peak} = \frac{F_{sr}}{g} \tag{4.68}$$

Now consider the magnetic field coenergy stored in the air gap. From Eq. 3.49, the coenergy density at a point where the magnetic field intensity is H is $(\mu_0/2)H^2$ in SI units. Thus, the coenergy density averaged over the volume of the air gap is $\mu_0/2$ times the average value of H_{ag}^2. The average value of the square of a sine wave is half its peak value. Hence,

$$\text{Average coenergy density} = \frac{\mu_0}{2} \left(\frac{(H_{ag})_{peak}^2}{2} \right) = \frac{\mu_0}{4} \left(\frac{F_{sr}}{g} \right)^2 \tag{4.69}$$

The total coenergy is then found as

$$W_{fld}' = (\text{average coenergy density})(\text{volume of air gap})$$

$$= \frac{\mu_0}{4} \left(\frac{F_{sr}}{g} \right)^2 \pi Dlg = \frac{\mu_0 \pi Dl}{4g} F_{sr}^2 \tag{4.70}$$

where l is the axial length of the air gap and D is its average diameter.

From Eq. 4.67 the coenergy stored in the air gap can now be expressed in terms of the peak amplitudes of the stator- and rotor-mmf waves and the space-phase angle between them; thus

$$W_{fld}' = \frac{\mu_0 \pi Dl}{4g} \left(F_s^2 + F_r^2 + 2F_s F_r \cos \delta_{sr} \right) \tag{4.71}$$

Recognizing that holding mmf constant is equivalent to holding current constant, an expression for the electromechanical torque T can now be obtained in terms of the

interacting magnetic fields by taking the partial derivative of the field coenergy with respect to angle. For a two-pole machine

$$T = \left.\frac{\partial W'_{\text{fld}}}{\partial \delta_{\text{sr}}}\right|_{F_s, F_r} = -\left(\frac{\mu_0 \pi D l}{2g}\right) F_s F_r \sin \delta_{\text{sr}} \qquad (4.72)$$

The general expression for the torque for a multipole machine is

$$T = -\left(\frac{\text{poles}}{2}\right)\left(\frac{\mu_0 \pi D l}{2g}\right) F_s F_r \sin \delta_{\text{sr}} \qquad (4.73)$$

In this equation, δ_{sr} is the electrical space-phase angle between the rotor and stator mmf waves and the torque T acts in the direction to accelerate the rotor. Thus when δ_{sr} is positive, the torque is negative and the machine is operating as a generator. Similarly, a negative value of δ_{sr} corresponds to positive torque and, correspondingly, motor action.

This important equation states that the torque is proportional to the peak values of the stator- and rotor-mmf waves F_s and F_r and to the sine of the electrical space-phase angle δ_{sr} between them. The minus sign means that the fields tend to align themselves. Equal and opposite torques are exerted on the stator and rotor. The torque on the stator is transmitted through the frame of the machine to the foundation.

One can now compare the results of Eq. 4.73 with that of Eq. 4.65. Recognizing that F_s is proportional to i_s and F_r is proportional to i_r, one sees that they are similar in form. In fact, they must be equal, as can be verified by substitution of the appropriate expressions for F_s, F_r (Section 4.3.1), and L_{sr} (Appendix B). Note that these results have been derived with the assumption that the iron reluctance is negligible. However, the two techniques are equally valid for finite iron permeability.

On referring to Fig. 4.35b, it can be seen that $F_r \sin \delta_{\text{sr}}$ is the component of the F_r wave in electrical space quadrature with the F_s wave. Similarly $F_s \sin \delta_{\text{sr}}$ is the component of the F_s wave in quadrature with the F_r wave. Thus, the torque is proportional to the product of one magnetic field and the component of the other in quadrature with it, much like the cross product of vector analysis. Also note that in Fig. 4.35b

$$F_s \sin \delta_{\text{sr}} = F_{\text{sr}} \sin \delta_r \qquad (4.74)$$

and

$$F_r \sin \delta_{\text{sr}} = F_{\text{sr}} \sin \delta_s \qquad (4.75)$$

where, as seen in Fig. 4.35, δ_r is the angle measured from the axis of the resultant mmf wave to the axis of the rotor mmf wave. Similarly, δ_s is the angle measured from the axis of the stator mmf wave to the axis of the resultant mmf wave.

The torque, acting to accelerate the rotor, can then be expressed in terms of the resultant mmf wave F_{sr} by substitution of either Eq. 4.74 or Eq. 4.75 in Eq. 4.73; thus

$$T = -\left(\frac{\text{poles}}{2}\right)\left(\frac{\mu_0 \pi D l}{2g}\right) F_s F_{\text{sr}} \sin \delta_s \qquad (4.76)$$

$$T = -\left(\frac{\text{poles}}{2}\right)\left(\frac{\mu_0 \pi D l}{2g}\right) F_r F_{\text{sr}} \sin \delta_r \qquad (4.77)$$

Comparison of Eqs. 4.73, 4.76, and 4.77 shows that the torque can be expressed in terms of the component magnetic fields due to *each* current acting alone, as in Eq. 4.73, or in terms of the *resultant* field and *either* of the components, as in Eqs. 4.76 and 4.77, *provided that we use the corresponding angle between the axes of the fields.* Ability to reason in any of these terms is a convenience in machine analysis.

In Eqs. 4.73, 4.76, and 4.77, the fields have been expressed in terms of the peak values of their mmf waves. When magnetic saturation is neglected, the fields can, of course, be expressed in terms of the peak values of their flux-density waves or in terms of total flux per pole. Thus the peak value B_{ag} of the field due to a sinusoidally distributed mmf wave in a uniform-air-gap machine is $\mu_0 F_{ag,peak}/g$, where $F_{ag,peak}$ is the peak value of the mmf wave. For example, the resultant mmf F_{sr} produces a resultant flux-density wave whose peak value is $B_{sr} = \mu_0 F_{sr}/g$. Thus, $F_{sr} = g B_{sr}/\mu_0$ and substitution in Eq. 4.77 gives

$$T = -\left(\frac{poles}{2}\right)\left(\frac{\pi Dl}{2}\right) B_{sr} F_r \sin \delta_r \qquad (4.78)$$

One of the inherent limitations in the design of electromagnetic apparatus is the saturation flux density of magnetic materials. Because of saturation in the armature teeth the peak value B_{sr} of the resultant flux-density wave in the air gap is limited to about 1.5 to 2.0 T. The maximum permissible value of the winding current, and hence the corresponding mmf wave, is limited by the temperature rise of the winding and other design requirements. Because the resultant flux density and mmf appear explicitly in Eq. 4.78, this equation is in a convenient form for design purposes and can be used to estimate the maximum torque which can be obtained from a machine of a given size.

EXAMPLE 4.8

An 1800-r/min, four-pole, 60-Hz synchronous motor has an air-gap length of 1.2 mm. The average diameter of the air-gap is 27 cm, and its axial length is 32 cm. The rotor winding has 786 turns and a winding factor of 0.976. Assuming that thermal considerations limit the rotor current to 18 A, estimate the maximum torque and power output one can expect to obtain from this machine.

■ **Solution**

First, we can determine the maximum rotor mmf from Eq. 4.8

$$(F_r)_{max} = \frac{4}{\pi}\left(\frac{k_r N_r}{poles}\right)(I_r)_{max} = \frac{4}{\pi}\left(\frac{0.976 \times 786}{4}\right)18 = 4395 \text{ A}$$

Assuming that the peak value of the resultant air-gap flux is limited to 1.5 T, we can estimate the maximum torque from Eq. 4.78 by setting δ_r equal to $-\pi/2$ (recognizing that negative values of δ_r, with the rotor mmf lagging the resultant mmf, correspond to positive, motoring torque)

$$T_{max} = \left(\frac{poles}{2}\right)\left(\frac{\pi Dl}{2}\right) B_{sr}(F_r)_{max}$$

$$= \left(\frac{4}{2}\right)\left(\frac{\pi \times 0.27 \times 0.32}{2}\right) 1.5 \times 4400 = 1790 \text{ N} \cdot \text{m}$$

For a synchronous speed of 1800 r/min, $\omega_m = n_s (\pi/30) = 1800 (\pi/30) = 60\pi$ rad/sec, and thus the corresponding power can be calculated as $P_{max} = \omega_m T_{max} = 337$ kW.

Practice Problem 4.6

Repeat Example 4.8 for a two-pole, 60-Hz synchronous motor with an air-gap length of 1.3 mm, an average air-gap diameter of 22 cm and an axial length of 41 cm. The rotor winding has a 900 turns and a winding factor of 0.965. The maximum rotor current is 22 A.

Solution

$T_{max} = 2585$ N $\cdot$ m and $P_{max} = 975$ kW

Alternative forms of the torque equation arise when it is recognized that the resultant flux per pole is

$$\Phi_p = \text{(average value of B over a pole)(pole area)} \tag{4.79}$$

and that the average value of a sinusoid over one-half wavelength is $2/\pi$ times its peak value. Thus

$$\Phi_p = \frac{2}{\pi} B_{peak} \left(\frac{\pi Dl}{\text{poles}} \right) = \left(\frac{2Dl}{\text{poles}} \right) B_{peak} \tag{4.80}$$

where B_{peak} is the peak value of the corresponding flux-density wave. For example, using the peak value of the resultant flux B_{sr} and substitution of Eq. 4.80 into Eq. 4.78 gives

$$T = -\frac{\pi}{2} \left(\frac{\text{poles}}{2} \right)^2 \Phi_{sr} F_r \sin \delta_r \tag{4.81}$$

where Φ_{sr} is the resultant flux per pole produced by the combined effect of the stator and rotor mmf's.

To recapitulate, we now have several forms in which the torque of a uniform-air-gap machine can be expressed in terms of its magnetic fields. *All are merely statements that the torque is proportional to the product of the magnitudes of the interacting fields and to the sine of the electrical space angle between their magnetic axes.* The negative sign indicates that the electromechanical torque acts in a direction to decrease the displacement angle between the fields. In our preliminary discussion of machine types, Eq. 4.81 will be the preferred form.

One further remark can be made concerning the torque equations and the thought process leading to them. There was no restriction in the derivation that the mmf wave or flux-density wave remain stationary in space. They may remain stationary, or they may be traveling waves, as discussed in Section 4.5. As we have seen, if the magnetic fields of the stator and rotor are constant in amplitude and travel around the air gap at the same speed, a steady torque will be produced by the tendency of the stator and rotor fields to align themselves in accordance with the torque equations.

4.8 LINEAR MACHINES

In general, each of the machine types discussed in this book can be produced in linear versions in addition to the rotary versions which are commonly found and which are discussed extensively in the following chapters. In fact, for clarity of discussion, many of the machine types discussed in this book are drawn in their developed (Cartesian coordinate) form, such as in Fig. 4.19b.

Perhaps the most widely known use of linear motors is in the transportation field. In these applications, linear induction motors are used, typically with the ac "stator" on the moving vehicle and with a conducting stationary "rotor" constituting the rails. In these systems, in addition to providing propulsion, the induced currents in the rail may be used to provide levitation, thus offering a mechanism for high-speed transportation without the difficulties associated with wheel-rail interactions on more conventional rail transport.

Linear motors have also found application in the machine tool industry and in robotics where linear motion (required for positioning and in the operation of manipulators) is a common requirement. In addition, reciprocating linear machines are being constructed for driving reciprocating compressors and alternators.

The analysis of linear machines is quite similar to that of rotary machines. In general, linear dimensions and displacements replace angular ones, and forces replace torques. With these exceptions, the expressions for machine parameters are derived in an analogous fashion to those presented here for rotary machines, and the results are similar in form.

Consider the linear winding shown in Fig. 4.36. This winding, consisting of N turns per slot and carrying a current i, is directly analogous to the rotary winding shown in developed form in Fig. 4.25. In fact, the only difference is that the angular position θ_a is replaced by the linear position z.

The fundamental component of the mmf wave of Fig. 4.36 can be found directly from Eq. 4.13 simply by recognizing that this winding has a wavelength equal to β and that the fundamental component of this mmf wave varies as $\cos(2\pi z/\beta)$. Thus replacing the angle θ_a in Eq. 4.13 by $2\pi z/\beta$, we can find the fundamental component of the mmf wave directly as

$$H_{\text{ag}1} = \frac{4}{\pi}\left(\frac{Ni}{2g}\right)\cos\left(\frac{2\pi z}{\beta}\right) \tag{4.82}$$

If an actual machine has a distributed winding (similar to its rotary counterpart, shown in Fig. 4.20) consisting of a total of N_{ph} turns distributed over p periods in z (i.e., over a length of $p\beta$), the fundamental component of H_{ag} can be found by analogy with Eq. 4.15

$$H_{\text{ag}1} = \frac{4}{\pi}\left(\frac{k_{\text{w}}N_{\text{ph}}i}{2pg}\right)\cos\left(\frac{2\pi z}{\beta}\right) \tag{4.83}$$

where k_{w} is the winding factor.

In a fashion analogous to the discussion of Section 4.5.2, a three-phase linear winding can be made from three windings such as those of Fig. 4.31, with each phase
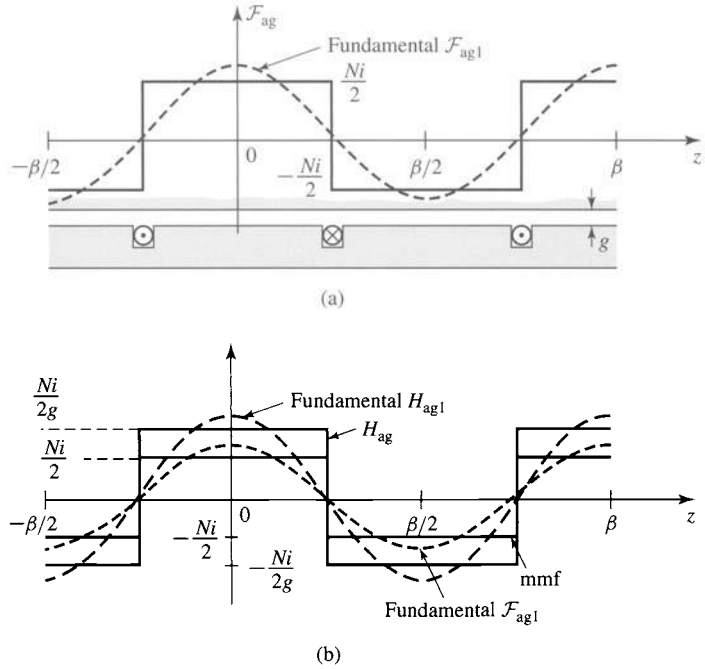
Figure 4.36   The mmf and H field of a concentrated full-pitch linear winding.

displaced in position by a distance $\beta/3$ and with each phase excited by balanced three-phase currents of angular frequency ω_e

$$i_a = I_m \cos \omega_e t \qquad (4.84)$$

$$i_b = I_m \cos (\omega_e t - 120°) \qquad (4.85)$$

$$i_c = I_m \cos (\omega_e t + 120°) \qquad (4.86)$$

Following the development of Eqs. 4.26 through 4.38, we can see that there will be a single positive-traveling mmf which can be written directly from Eq. 4.38 simply by replacing θ_a by $2\pi z/\beta$ as

$$\mathcal{F}^+(z, t) = \frac{3}{2} F_{max} \cos \left(\frac{2\pi z}{\beta} - \omega_e t \right) \qquad (4.87)$$

where F_{max} is given by

$$F_{max} = \frac{4}{\pi} \left(\frac{k_w N_{ph}}{2p} \right) I_m \qquad (4.88)$$

From Eq. 4.87 we see that the result is an mmf which travels in the z direction with a linear velocity

$$v = \frac{\omega_e \beta}{2\pi} = f_e \beta \tag{4.89}$$

where f_e is the exciting frequency in hertz.

A three-phase linear ac motor has a winding with a wavelength of $\beta = 0.5$ m and an air gap of 1.0 cm in length. A total of 45 turns, with a winding factor $k_w = 0.92$, are distributed over a total winding length of $3\beta = 1.5$ m. Assume the windings to be excited with balanced three-phase currents of peak amplitude 700 A and frequency 25 Hz. Calculate (a) the amplitude of the resultant mmf wave, (b) the corresponding peak air-gap flux density, and (c) the velocity of this traveling mmf wave.

■ **Solution**

a. From Eqs. 4.87 and 4.88, the amplitude of the resultant mmf wave is

$$F_{peak} = \frac{3}{2} \frac{4}{\pi} \left(\frac{k_w N_{ph}}{2p} \right) I_m$$

$$= \frac{3}{2} \frac{4}{\pi} \left(\frac{0.92 \times 45}{2 \times 3} \right) 700$$

$$= 8.81 \times 10^3 \text{ A/m}$$

b. The peak air-gap flux density can be found from the result of part (a) by dividing by the air-gap length and multiplying by μ_0:

$$B_{peak} = \frac{\mu_0 F_{peak}}{g}$$

$$= \frac{(4\pi \times 10^{-7})(8.81 \times 10^3)}{0.01}$$

$$= 1.11 \text{ T}$$

c. Finally, the velocity of the traveling wave can be determined from Eq. 4.89:

$$v = f_e \beta = 25 \times 0.5 = 12.5 \text{ m/s}$$

A three-phase linear synchronous motor has a wavelength of 0.93 m. It is observed to be traveling at speed of 83 km/hr. Calculate the frequency of the electrical excitation required under this operating condition.

Solution

$$f = 24.8 \text{ Hz}$$

Linear machines are not discussed specifically in this book. Rather, the reader is urged to recognize that the fundamentals of their performance and analysis correspond directly to those of their rotary counterparts. One major difference between these two machine types is that linear machines have *end effects,* corresponding to the magnetic fields which "leak" out of the air gap ahead of and behind the machine. These effects are beyond the scope of this book and have been treated in detail in the published literature.[3]

4.9 MAGNETIC SATURATION

The characteristics of electric machines depend heavily upon the use of magnetic materials. These materials are required to form the magnetic circuit and are used by the machine designer to obtain specific machine characteristics. As we have seen in Chapter 1, magnetic materials are less than ideal. As their magnetic flux is increased, they begin to saturate, with the result that their magnetic permeabilities begin to decrease, along with their effectiveness in contributing to the overall flux density in the machine.

Both electromechanical torque and generated voltage in all machines depend on the winding flux linkages. For specific mmf's in the windings, the fluxes depend on the reluctances of the iron portions of the magnetic circuits and on those of the air gaps. Saturation may therefore appreciably influence the characteristics of the machines.

Another aspect of saturation, more subtle and more difficult to evaluate without experimental and theoretical comparisons, concerns its influence on the basic premises from which the analytic approach to machinery is developed. Specifically, relations for the air-gap mmf are typically based on the assumption of negligible reluctance in the iron. When these relations are applied to practical machines with varying degrees of saturation in the iron, significant errors in the analytical results can be expected. To improve these analytical relationships, the actual machine can be replaced for these considerations by an equivalent machine, one whose iron has negligible reluctance but whose air-gap length is increased by an amount sufficient to absorb the magnetic-potential drop in the iron of the actual machine.

Similarly, the effects of air-gap nonuniformities such as slots and ventilating ducts are also incorporated by increasing the effective air-gap length. Ultimately, these various approximate techniques must be verified and confirmed experimentally. In cases where such simple techniques are found to be inadequate, detailed analyses, such as those employing finite-element or other numerical techniques, can be used. However, it must be recognized that the use of these techniques represents a significant increase in modeling complexity.

Saturation characteristics of rotating machines are typically presented in the form of an *open-circuit characteristic,* also called a *magnetization curve* or *saturation*

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[3] See, for example, S. Yamamura, *Theory of Linear Induction Motors,* 2d ed., Halsted Press, 1978. Also, S. Nasar and I. Boldea, *Linear Electric Motors: Theory, Design and Practical Applications,* Prentice-Hall, 1987.
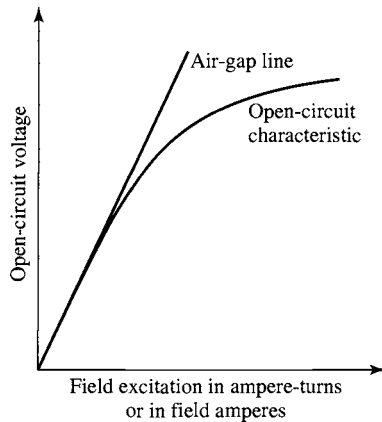
Figure 4.37 Typical open-circuit characteristic and air-gap line.

curve. An example is shown in Fig. 4.37. This characteristic represents the magnetization curve for the particular iron and air geometry of the machine under consideration. For a synchronous machine, the open-circuit saturation curve is obtained by operating the machine at constant speed and measuring the open-circuit armature voltage as a function of the field current. The straight line tangent to the lower portion of the curve is the *air-gap line,* corresponding to low levels of flux within the machine. Under these conditions the reluctance of the machine iron is typically negligible, and the mmf required to excite the machine is simply that required to overcome the reluctance of the air gap. If it were not for the effects of saturation, the air-gap line and open-circuit characteristic would coincide. Thus, the departure of the curve from the air-gap line is an indication of the degree of saturation present. In typical machines the ratio at rated voltage of the total mmf to that required by the air gap alone usually is between 1.1 and 1.25.

At the design stage, the open-circuit characteristic can be calculated from design data techniques such as finite-element analyses. A typical finite-element solution for the flux distribution around the pole of a salient-pole machine is shown in Fig. 4.38. The distribution of the air-gap flux found from this solution, together with the fundamental and third-harmonic components, is shown in Fig. 4.39.

In addition to saturation effects, Fig. 4.39 clearly illustrates the effect of a nonuniform air gap. As expected, the flux density over the pole face, where the air gap is small, is much higher than that away from the pole. This type of detailed analysis is of great use to a designer in obtaining specific machine properties.

As we have seen, the magnetization curve for an existing synchronous machine can be determined by operating the machine as an unloaded generator and measuring the values of terminal voltage corresponding to a series of values of field current. For an induction motor, the machine is operated at or close to synchronous speed (in which case very little current will be induced in the rotor windings), and values of the magnetizing current are obtained for a series of values of impressed stator voltage.
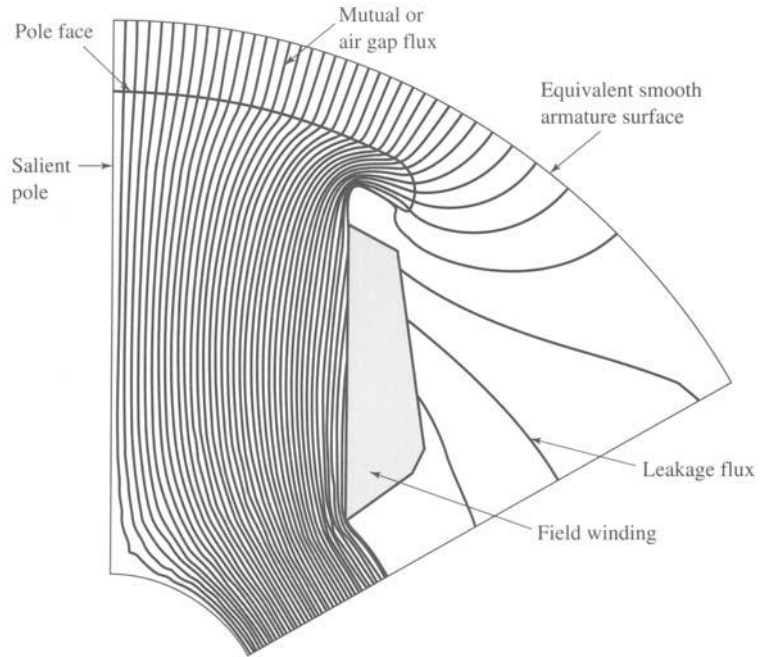
Figure 4.38   Finite-element solution for the flux distribution around a salient pole. (*General Electric Company.*)
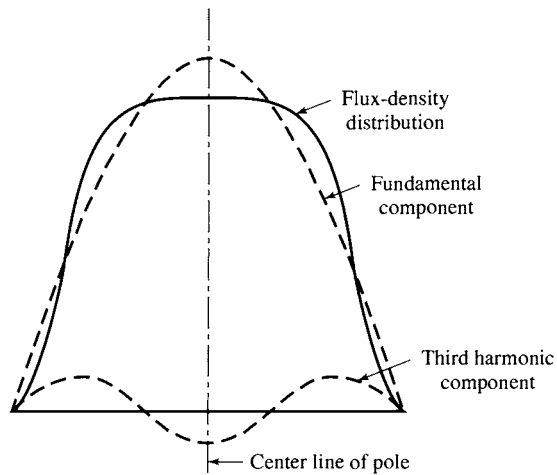

Figure 4.39   Flux-density wave corresponding to Fig. 4.38 with its fundamental and third-harmonic components.

It should be emphasized, however, that saturation in a fully loaded machine occurs as a result of the total mmf acting on the magnetic circuit. Since the flux distribution under load generally differs from that of no-load conditions, the details of the machine saturation characteristics may vary from the open-circuit curve of Fig. 4.37.

4.10 LEAKAGE FLUXES

In Section 2.4 we showed that in a two-winding transformer the flux created by each winding can be separated into two components. One component consists of flux which links both windings, and the other consists of flux which links only the winding creating the flux. The first component, called *mutual flux,* is responsible for coupling between the two coils. The second, known as *leakage flux,* contributes only to the self-inductance of each coil.

Note that the concept of mutual and leakage flux is meaningful only in the context of a multiwinding system. For systems of three or more windings, the bookkeeping must be done very carefully. Consider, for example, the three-winding system of Fig. 4.40. Shown schematically are the various components of flux created by a current in winding 1. Here φ_{123} is clearly mutual flux that links all three windings, and φ_{1l} is clearly leakage flux since it links only winding 1. However, φ_{12} is mutual flux with respect to winding 2 yet is leakage flux with respect to winding 3, while φ_{13} mutual flux with respect to winding 3 and leakage flux with respect to winding 2.

Electric machinery often contains systems of multiple windings, requiring careful bookkeeping to account for the flux contributions of the various windings. Although the details of such analysis are beyond the scope of this book, it is useful to discuss these effects in a qualitative fashion and to describe how they affect the basic machine inductances.

Air-Gap Space-Harmonic Fluxes In this chapter we have seen that although single distributed coils create air-gap flux with a significant amount of space-harmonic
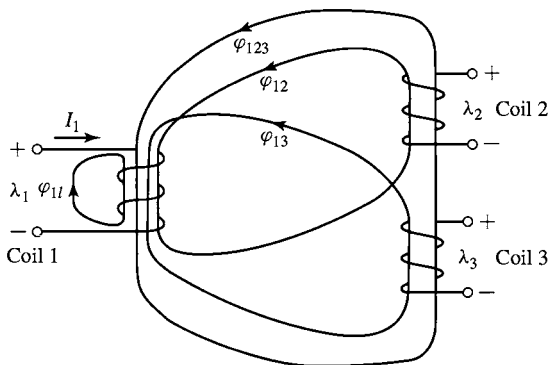

Figure 4.40 Three-coil system showing components of mutual and leakage flux produced by current in coil 1.

content, it is possible to distribute these windings so that the space-fundamental component is emphasized while the harmonic effects are greatly reduced. As a result, we can neglect harmonic effects and consider only space-fundamental fluxes in calculating the self and mutual-inductance expressions of Eqs. B.26 and B.27.

Though often small, the space-harmonic components of air-gap flux do exist. In dc machines they are useful torque-producing fluxes and therefore can be counted as mutual flux between the rotor and stator windings. In ac machines, however, they may generate time-harmonic voltages or asynchronously rotating flux waves. These effects generally cannot be rigorously accounted for in most standard analyses. Nevertheless, it is consistent with the assumptions basic to these analyses to recognize that these fluxes form a part of the leakage flux of the individual windings which produce them.

Slot-Leakage Flux   Figure 4.41 shows the flux created by a single coil side in a slot. Notice that in addition to flux which crosses the air gap, contributing to the air-gap flux, there are flux components which cross the slot. Since this flux links only the coil that is producing it, it also forms a component of the leakage inductance of the winding producing it.

End-Turn Fluxes   Figure 4.42 shows the stator end windings on an ac machine. The magnetic field distribution created by end turns is extremely complex. In general these fluxes do not contribute to useful rotor-to-stator mutual flux, and thus they, too, contribute to leakage inductance.

From this discussion we see that the self-inductance expression of Eq. B.26 must, in general, be modified by an additional term L_l, which represents the winding leakage inductance. This leakage inductance corresponds directly to the leakage inductance of a transformer winding as discussed in Chapter 1. Although the leakage inductance is usually difficult to calculate analytically and must be determined by approximate or empirical techniques, it plays an important role in machine performance.
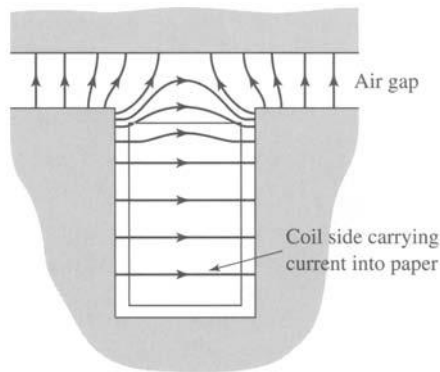

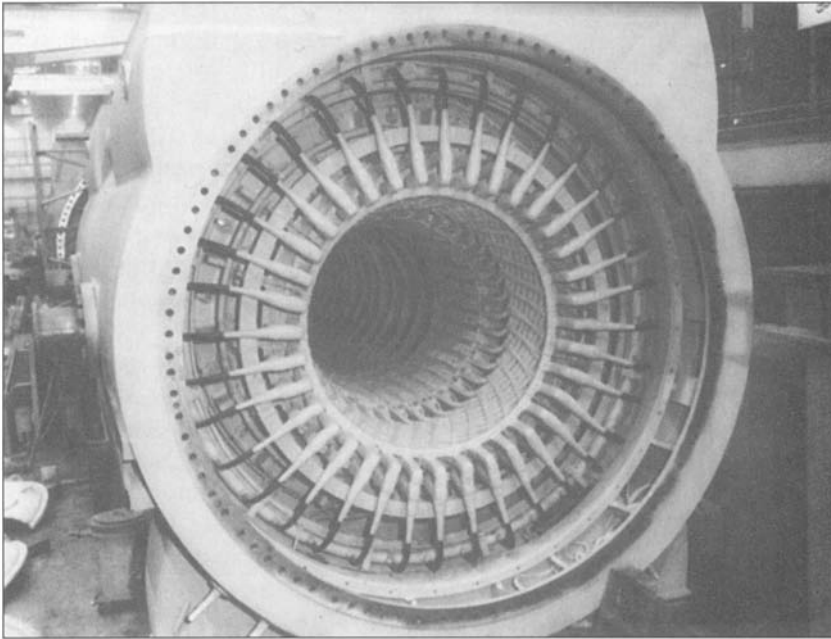**Figure 4.41** Flux created by a single coil side in a slot.

Figure 4.42 End view of the stator of a 26-kV 908-MVA 3600 r/min turbine generator with water-cooled windings. Hydraulic connections for coolant flow are provided for each winding end turn. (*General Electric Company.*)

4.11 SUMMARY

This chapter presents a brief and elementary description of three basic types of rotating machines: synchronous, induction, and dc machines. In all of them the basic principles are essentially the same. Voltages are generated by the relative motion of a magnetic field with respect to a winding, and torques are produced by the interaction of the magnetic fields of the stator and rotor windings. The characteristics of the various machine types are determined by the methods of connection and excitation of the windings, but the basic principles are essentially similar.

The basic analytical tools for studying rotating machines are expressions for the generated voltages and for the electromechanical torque. Taken together, they express the coupling between the electric and mechanical systems. To develop a reasonably quantitative theory without the confusion arising from too much detail, we have made several simplifying approximations. In the study of ac machines we have assumed sinusoidal time variations of voltages and currents and sinusoidal space waves of air-gap flux density and mmf. On examination of the mmf produced by distributed ac windings we found that the space-fundamental component is the most important. On the other hand, in dc machines the armature-winding mmf is more nearly a sawtooth wave. For our preliminary study in this chapter, however, we have assumed sinusoidal mmf distributions for both ac and dc machines. We examine this assumption more

thoroughly for dc machines in Chapter 7. Faraday's law results in Eq. 4.50 for the rms voltage generated in an ac machine winding or Eq. 4.53 for the average voltage generated between brushes in a dc machine.

On examination of the mmf wave of a three-phase winding, we found that balanced three-phase currents produce a constant-amplitude air-gap magnetic field rotating at synchronous speed, as shown in Fig. 4.31 and Eq. 4.39. The importance of this fact cannot be overstated, for it means that it is possible to operate such machines, either as motors or generators, under conditions of constant torque (and hence constant electrical power as is discussed in Appendix A), eliminating the double-frequency, time-varying torque inherently associated with single-phase machines. For example, imagine a multimegawatt single-phase 60-Hz generator subjected to multimegawatt instantaneous power pulsation at 120 Hz! The discovery of rotating fields led to the invention of the simple, rugged, reliable, self-starting polyphase induction motor, which is analyzed in Chapter 6. (A single-phase induction motor will not start; it needs an auxiliary starting winding, as shown in Chapter 9.)

In single-phase machines, or in polyphase machines operating under unbalanced conditions, the backward-rotating component of the armature mmf wave induces currents and losses in the rotor structure. Thus, the operation of polyphase machines under balanced conditions not only eliminates the second-harmonic component of generated torque, it also eliminates a significant source of rotor loss and rotor heating. It was the invention of polyphase machines operating under balanced conditions that made possible the design and construction of large synchronous generators with ratings as large as 1000 MW.

Having assumed sinusoidally-distributed magnetic fields in the air gap, we then derived expressions for the magnetic torque. The simple physical picture for torque production is that of two magnets, one on the stator and one on the rotor, as shown schematically in Fig. 4.35a. The torque acts in the direction to align the magnets. To get a reasonably close quantitative analysis without being hindered by details, we assumed a smooth air gap and neglected the reluctance of the magnetic paths in the iron parts, with a mental note that this assumption may not be valid in all situations and a more detailed model may be required.

In Section 4.7 we derived expressions for the magnetic torque from two viewpoints, both based on the fundamental principles of Chapter 3. The first viewpoint regards the machine as a set of magnetically-coupled circuits with inductances which depend on the angular position of the rotor, as in Section 4.7.1. The second regards the machine from the viewpoint of the magnetic fields in the air gap, as in Section 4.7.2. It is shown that the torque can be expressed as the product of the stator field, the rotor field, and the sine of the angle between their magnetic axes, as in Eq. 4.73 or any of the forms derived from Eq. 4.73. The two viewpoints are complementary, and ability to reason in terms of both is helpful in reaching an understanding of how machines work.

This chapter has been concerned with basic principles underlying rotating-machine theory. By itself it is obviously incomplete. Many questions remain unanswered. How do we apply these principles to the determination of the characteristics of synchronous, induction, and dc machines? What are some of the practical problems

that arise from the use of iron, copper, and insulation in physical machines? What are some of the economic and engineering considerations affecting rotating-machine applications? What are the physical factors limiting the conditions under which a machine can operate successfully? Appendix D discusses some of these problems. Taken together, Chapter 4 along with Appendix D serve as an introduction to the more detailed treatments of rotating machines in the following chapters.

4.12 PROBLEMS

4.1 The rotor of a six-pole synchronous generator is rotating at a mechanical speed of 1200 r/min.

   a. Express this mechanical speed in radians per second.

   b. What is the frequency of the generated voltage in hertz and in radians per second?

   c. What mechanical speed in revolutions per minute would be required to generate voltage at a frequency of 50 Hz?

4.2 The voltage generated in one phase of an unloaded three-phase synchronous generator is of the form $v(t) = V_0 \cos \omega t$. Write expressions for the voltage in the remaining two phases.

4.3 A three-phase motor is used to drive a pump. It is observed (by the use of a stroboscope) that the motor speed decreases from 898 r/min when the pump is unloaded to 830 r/min as the pump is loaded.

   a. Is this a synchronous or an induction motor?

   b. Estimate the frequency of the applied armature voltage in hertz.

   c. How many poles does this motor have?

4.4 The object of this problem is to illustrate how the armature windings of certain machines, i.e., dc machines, can be approximately represented by uniform current sheets, the degree of correspondence growing better as the winding is distributed in a greater number of slots around the armature periphery. For this purpose, consider an armature with eight slots uniformly distributed over 360 electrical degrees (corresponding to a span of one pole pair). The air gap is of uniform length, the slot openings are very small, and the reluctance of the iron is negligible.

      Lay out 360 electrical degrees of the armature with its slots in developed form in the manner of Fig. 4.23a and number the slots 1 to 8 from left to right. The winding consists of eight single-turn coils, each carrying a direct current of 10 A. Coil sides placed in any of the slots 1 to 4 carry current directed into the paper; those placed in any of the slots 5 to 8 carry current out of the paper.

   a. Consider that all eight slots are placed with one side in slot 1 and the other in slot 5. The remaining slots are empty. Draw the rectangular mmf wave produced by these slots.

   b. Next consider that four coils have one side in slot 1 and the other in slot 5, while the remaining four coils have one side in slot 3 and the other in

slot 7. Draw the component rectangular mmf waves produced by each group of coils, and superimpose the components to give the resultant mmf wave.

c. Now consider that two coils are placed in slots 1 and 5, two in slots 2 and 6, two in 3 and 7, and two in 4 and 8. Again superimpose the component rectangular waves to produce the resultant wave. Note that the task can be systematized and simplified by recognizing that the mmf wave is symmetric about its axis and takes a step at each slot which is directly proportional to the number of ampere-conductors in that slot.

d. Let the armature now consist of 16 slots per 360 electrical degrees with one coil side per slot. Draw the resultant mmf wave.

4.5 A three-phase Y-connected ac machine is initially operating under balanced three-phase conditions when one of the phase windings becomes open-circuited. Because there is no neutral connection on the winding, this requires that the currents in the remaining two windings become equal and opposite. Under this condition, calculate the relative magnitudes of the resultant positive- and negative-traveling mmf waves.

4.6 What is the effect on the rotating mmf and flux waves of a three-phase winding produced by balanced-three-phase currents if two of the phase connections are interchanges?

4.7 In a balanced two-phase machine, the two windings are displaced 90 electrical degrees in space, and the currents in the two windings are phase-displaced 90 electrical degrees in time. For such a machine, carry out the process leading to an equation for the rotating mmf wave corresponding to Eq. 4.39 (which is derived for a three-phase machine).

4.8 This problem investigates the advantages of short-pitching the stator coils of an ac machine. Figure 4.43a shows a single full-pitch coil in a two-pole machine. Figure 4.43b shows a fractional-pitch coil for which the coil sides are β radians apart, rather than π radians ($180°$) as is the case for the full-pitch coil.

For an air-gap radial flux distribution of the form

$$B_r = \sum_{n \text{ odd}} B_n \cos n\theta$$

where $n = 1$ corresponds to the fundamental space harmonic, $n = 3$ to the third space harmonic, and so on, the flux linkage of each coil is the integral of B_r over the surface spanned by that coil. Thus for the nth space harmonic, the ratio of the maximum fractional-pitch coil flux linkage to that of the full-pitch coil is

$$\frac{\int_{-\beta/2}^{\beta/2} B_n \cos n\theta \, d\theta}{\int_{-\pi/2}^{\pi/2} B_n \cos n\theta \, d\theta} = \frac{\int_{-\beta/2}^{\beta/2} \cos n\theta \, d\theta}{\int_{-\pi/2}^{\pi/2} \cos n\theta \, d\theta} = |\sin (n\beta/2)|$$

It is common, for example, to fractional-pitch the coils of an ac machine by 30 electrical degrees ($\beta = 5\pi/6 = 150°$). For $n = 1, 3, 5$ calculate the fractional reduction in flux linkage due to short pitching.
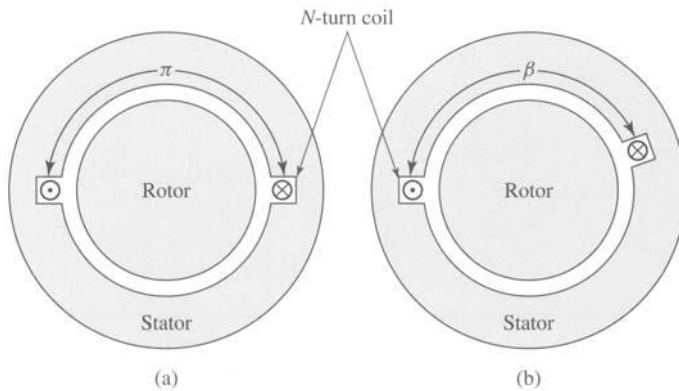
Figure 4.43 Problem 4.8: (a) full-pitch coil and
(b) fractional-pitch coil.

4.9 A six-pole, 60-Hz synchronous machine has a rotor winding with a total of
138 series turns and a winding factor $k_r = 0.935$. The rotor length is 1.97 m,
the rotor radius is 58 cm, and the air-gap length $= 3.15$ cm.
 a. What is the rated operating speed in r/min?
 b. Calculate the rotor-winding current required to achieve a peak
 fundamental air-gap flux density of 1.23 T.
 c. Calculate the corresponding flux per pole.

4.10 Assume that a phase winding of the synchronous machine of Problem 4.9
consists of one full-pitch, 11-turn coil per pole pair, with the coils connected
in series to form the phase winding. If the machine is operating at rated speed
and under the operating conditions of Problem 4.9, calculate the rms
generated voltage per phase.

4.11 The synchronous machine of Problem 4.9 has a three-phase winding with 45
series turns per phase and a winding factor $k_w = 0.928$. For the flux condition
and rated speed of Problem 4.9, calculate the rms-generated voltage per phase.

4.12 The three-phase synchronous machine of Problem 4.9 is to be moved to an
application which requires that its operating frequency be reduced from 60 to
50 Hz. This application requires that, for the operating condition considered
in Problem 4.9, the rms generated voltage equal 13.0 kV line-to-line. As a
result, the machine armature must be rewound with a different number of
turns. Assuming a winding factor of $k_w = 0.928$, calculate the required
number of series turns per phase.

4.13 Figure 4.44 shows a two-pole rotor revolving inside a smooth stator which
carries a coil of 110 turns. The rotor produces a sinusoidal space distribution
of flux at the stator surface; the peak value of the flux-density wave being
0.85 T when the current in the rotor is 15 A. The magnetic circuit is linear.
The inside diameter of the stator is 11 cm, and its axial length is 0.17 m. The
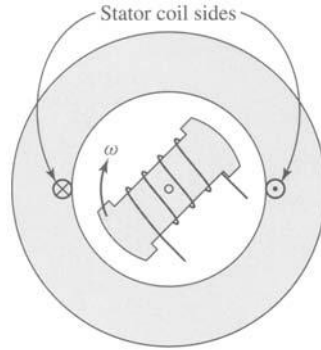rotor is driven at a speed of 50 r/sec.

Figure 4.44 Elementary
generator for Problem 4.13.

a. The rotor is excited by a current of 15 A. Taking zero time as the instant
    when the axis of the rotor is vertical, find the expression for the
    instantaneous voltage generated in the open-circuited stator coil.

b. The rotor is now excited by a 50-Hz sinusoidal alternating currrent whose
    peak value is 15 A. Consequently, the rotor current reverses every half
    revolution; it is timed to be at its maximum just as the axis of the rotor is
    vertical (i.e., just as it becomes aligned with that of the stator coil).
    Taking zero time as the instant when the axis of the rotor is vertical, find
    the expression for the instantaneous voltage generated in the
    open-circuited stator coil. This scheme is sometimes suggested as a dc
    generator without a commutator; the thought being that if alternative half
    cycles of the alternating voltage generated in part (a) are reversed by
    reversal of the polarity of the field (rotor) winding, then a pulsating direct
    voltage will be generated in the stator. Discuss whether or not this scheme
    will work.

4.14 A three-phase two-pole winding is excited by balanced three-phase 60-Hz
currents as described by Eqs. 4.23 to 4.25. Although the winding distribution
has been designed to minimize harmonics, there remains some third and fifth
spatial harmonics. Thus the phase-a mmf can be written as

$$\mathcal{F}_a = i_a(A_1 \cos \theta_a + A_3 \cos 3\theta_a + A_5 \cos 5\theta_a)$$

Similar expressions can be written for phases b (replace θ_a by $\theta_a - 120°$) and
c (replace θ_a by $\theta_a + 120°$). Calculate the total three-phase mmf. What is the
angular velocity and rotational direction of each component of the mmf?

4.15 The nameplate of a dc generator indicates that it will produce an output
voltage of 24 V dc when operated at a speed of 1200 r/min. By what factor
must the number of armature turns be changed such that, for the same
field-flux per pole, the generator will produce an output voltage of 18 V dc at
a speed of 1400 r/min?

4.16 The armature of a two-pole dc generator has a total of 320 series turns. When operated at a speed of 1800 r/min, the open-circuit generated voltage is 240 V. Calculate Φ_p, the air-gap flux per pole.

4.17 The design of a four-pole, three-phase, 230-V, 60-Hz induction motor is to be based on a stator core of length 21 cm and inner diameter 9.52 cm. The stator winding distribution which has been selected has a winding factor $k_w = 0.925$. The armature is to be Y-connected, and thus the rated phase voltage will be $230/\sqrt{3}$ V.

   a. The designer must pick the number of armature turns so that the flux density in the machine is large enough to make efficient use of the magnetic material without being so large as to result in excessive saturation. To achieve this objective, the machine is to be designed with a peak fundamental air-gap flux density of 1.25 T. Calculate the required number of series turns per phase.

   b. For an air-gap length of 0.3 mm, calculate the self-inductance of an armature phase based upon the result of part (a) and using the inductance formulas of Appendix B. Neglect the reluctance of the rotor and stator iron and the armature leakage inductance.

4.18 A two-pole, 60-Hz, three-phase, laboratory-size synchronous generator has a rotor radius of 5.71 cm, a rotor length of 18.0 cm, and an air-gap length of 0.25 mm. The rotor field winding consists of 264 turns with a winding factor of $k_r = 0.95$. The Y-connected armature winding consists of 45 turns per phase with a winding factor $k_w = 0.93$.

   a. Calculate the flux per pole and peak fundamental air-gap flux density which will result in an open-circuit, 60-Hz armature voltage of 120 V rms/phase (line-to-neutral).

   b. Calculate the dc field current required to achieve the operating condition of part (a).

   c. Calculate the peak value of the field-winding to armature-phase-winding mutual inductance.

4.19 Write a MATLAB script which calculates the required total series field- and armature-winding turns for a three-phase, Y-connected synchronous motor given the following information:

   Rotor radius, R (meters)   Rotor length, l (meters)
   Air-gap length, g (meters)   Number of poles, poles
   Electrical frequency, f_e Peak fundamental air-gap flux density, B_{peak}
   Field-winding factor, k_f Armature-winding factor, k_w
   Rated rms open-circuit line-to-line terminal voltage, V_{rated}
   Field-current at rated-open-circuit terminal voltage, I_f

4.20 A four-pole, 60-Hz synchronous generator has a rotor length of 5.2 m, diameter of 1.24 m, and air-gap length of 5.9 cm. The rotor winding consists of a series connection of 63 turns per pole with a winding factor of $k_r = 0.91$. The peak value of the fundamental air-gap flux density is limited to 1.1 T and

the rotor winding current to 2700 A. Calculate the maximum torque (N·m) and power output (MW) which can be supplied by this machine.

4.21 Thermal considerations limit the field-current of the laboratory-size synchronous generator of Problem 4.18 to a maximum value of 2.4 A. If the peak fundamental air-gap flux density is limited to a maximum of 1.3 T, calculate the maximum torque (N·m) and power (kW) which can be produced by this generator.

4.22 Figure 4.45 shows in cross section a machine having a rotor winding f and two identical stator windings a and b whose axes are in quadrature. The self-inductance of each stator winding is L_{aa} and of the rotor is L_{ff}. The air gap is uniform. The mutual inductance between a stator winding depends on the angular position of the rotor and may be assumed to be of the form

$$M_{af} = M \cos \theta_0 \quad M_{bf} = M \sin \theta_0$$

where M is the maximum value of the mutual inductance. The resistance of each stator winding is R_a.

a. Derive a general expression for the torque T in terms of the angle θ_0, the inductance parameters, and the instantaneous currents i_a, I_b, and i_f. Does this expression apply at standstill? When the rotor is revolving?

b. Suppose the rotor is stationary and constant direct currents $I_a = I_0$, $I_b = I_0$, and $I_f = 2I_0$ are supplied to the windings in the directions indicated by the dots and crosses in Fig. 4.45. If the rotor is allowed to move, will it rotate continuously or will it tend to come to rest? If the latter, at what value of θ_0?
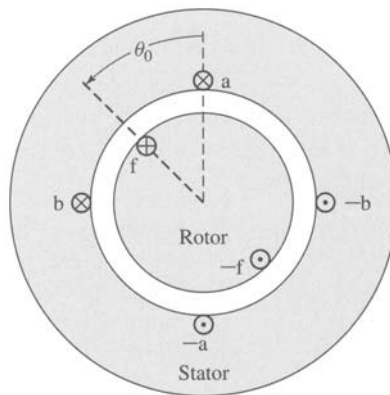

Figure 4.45 Elementary cylindrical-rotor, two-phase synchronous machine for Problem 4.22.

c. The rotor winding is now excited by a constant direct current I_f while the stator windings carry balanced two-phase currents

$$i_a = \sqrt{2}I_a \cos \omega t \quad i_b = \sqrt{2}I_a \sin \omega t$$

The rotor is revolving at synchronous speed so that its instantaneous angular position is given by $\theta_0 = \omega t - \delta$, where δ is a phase angle describing the position of the rotor at $t = 0$. The machine is an elementary two-phase synchronous machine. Derive an expression for the torque under these conditions.

d. Under the conditions of part (c), derive an expression for the instantaneous terminal voltages of stator phases a and b.

4.23 Consider the two-phase synchronous machine of Problem 4.22. Derive an expression for the torque acting on the rotor if the rotor is rotating at constant angular velocity, such that $\theta_0 = \omega t + \delta$, and the phase currents become unbalanced such that

$$i_a = \sqrt{2}I_a \cos \omega t \quad i_b = \sqrt{2}(I_a + I') \sin \omega t$$

What are the instantaneous and time-averaged torque under this condition?

4.24 Figure 4.46 shows in schematic cross section a salient-pole synchronous machine having two identical stator windings a and b on a laminated steel core. The salient-pole rotor is made of steel and carries a field winding f connected to slip rings.

Because of the nonuniform air gap, the self- and mutual inductances are functions of the angular position θ_0 of the rotor. Their variation with θ_0 can be approximated as:

$$L_{aa} = L_0 + L_2 \cos 2\theta_0 \quad L_{bb} = L_0 - L_2 \cos 2\theta_0 \quad M_{ab} = L_2 \sin 2\theta_0$$
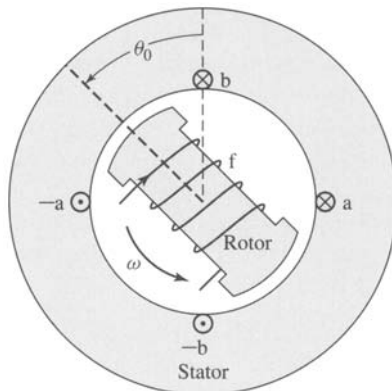

Figure 4.46 Schematic two-phase, salient-pole synchronous machine for Problem 4.24.

where L_0 and L_2 are positive constants. The mutual inductance between the rotor and the stator windings are functions of θ_0

$$M_{af} = M \cos \theta_0 \quad M_{bf} = M \sin \theta_0$$

where M is also a positive constant. The self-inductance of the field winding, L_{ff}, is constant, independent of θ_0.

Consider the operating condition in which the field winding is excited by direct current I_f and the stator windings are connected to a balanced two-phase voltage source of frequency ω. With the rotor revolving at synchronous speed, its angular position will be given by $\theta_0 = \omega t$.

Under this operating condition, the stator currents will be of the form

$$i_a = \sqrt{2}I_a \cos(\omega t + \delta) \quad i_b = \sqrt{2}I_a \sin(\omega t + \delta)$$

a. Derive an expression for the electromagnetic torque acting on the rotor.

b. Can the machine be operated as a motor and/or a generator? Explain.

c. Will the machine continue to run if the field current I_f is reduced to zero? Support you answer with an expression for the torque and an explanation as to why such operation is or is not possible.

4.25 A three-phase linear ac motor has an armature winding of wavelength 25 cm. A three-phase balanced set of currents at a frequency of 100 Hz is applied to the armature.

a. Calculate the linear velocity of the armature mmf wave.

b. For the case of a synchronous rotor, calculate the linear velocity of the rotor.

c. For the case of an induction motor operating at a slip of 0.045, calculate the linear velocity of the rotor.

4.26 The linear-motor armature of Problem 4.25 has a total active length of 7 wavelengths, with a total of 280 turns per phase with a winding factor $k_w = 0.91$. For an air-gap length of 0.93 cm, calculate the rms magnitude of the balanced three-phase currents which must be supplied to the armature to achieve a peak space-fundamental air-gap flux density of 1.45 T.

4.27 A two-phase linear permanent-magnet synchronous motor has an air-gap of length 1.0 mm, a wavelength of 12 cm, and a pole width of 4 cm. The rotor is 5 wavelengths in length. The permanent magnets on the rotor are arranged to produce an air-gap magnetic flux distribution that is uniform over the width of a pole but which varies sinusoidally in space in the direction of rotor travel. The peak density of this air-gap flux is 0.97 T.

a. Calculate the net flux per pole.

b. Each armature phase consists of 10 turns per pole, with all the poles connected in series. Assuming that the armature winding extends many wavelengths past either end of the rotor, calculate the peak flux linkages of the armature winding.

c. If the rotor is traveling at a speed of 6.3 m/sec, calculate the rms voltage induced in the armature winding.