C.12 Applications of Optimal Control

12.6 Rocket Trajectories

The governing equation of rocket motion is

$$m \frac{d\mathbf{v}}{dt} = m\mathbf{F} + \mathbf{F}_{ext}$$  \hspace{1cm} (8.22)

where

- $m$: rocket’s mass
- $\mathbf{v}$: rocket’s velocity
- $\mathbf{F}$: thrust produced by the rocket motor
- $\mathbf{F}_{ext}$: external force

It can be seen that

$$|\mathbf{F}| = \frac{c}{m} \beta$$  \hspace{1cm} (8.23)

where

- $c$: relative exhaust speed
- $\beta = -\frac{dm}{dt}$: burning rate

Let $\phi$ be the thrust attitude angle, that is the angle between the rocket axis and the horizontal, as in the Fig. 8.4, then the equations of motion are

$$\frac{dv_1}{dt} = \frac{c\beta}{m} \cos \phi$$  \hspace{1cm} (8.24)

$$\frac{dv_2}{dt} = \frac{c\beta}{m} \sin \phi - g$$

Here $\mathbf{v} = (v_1, v_2)$ and the external force is the gravity force only $\mathbf{F}_{ext} = (0, -mg)$.

![Fig. 8.4 Rocket Flight Path](image)

The minimum fuel problem is to choose the controls, $\beta$ and $\phi$, so as to take the rocket from initial position to a prescribed height, say, $\mathbf{y}$, in such a way as to minimize the fuel used. The fuel consumed is

$$\int_{0}^{T} \beta \, dt$$  \hspace{1cm} (8.25)

where $T$ is the time at which $\mathbf{y}$ is reached.

Problem

Minimizing the fuel (cost function)

$$J = \int_{0}^{T} \beta \, dt$$  \hspace{1cm} (12.41)

subject to the differential constraints

$$\frac{dv_1}{dt} = \frac{c\beta}{m} \cos \phi$$  \hspace{1cm} (12.42)

$$\frac{dv_2}{dt} = \frac{c\beta}{m} \sin \phi - g$$  \hspace{1cm} (12.43)

and also

$$\frac{dx}{dt} = v_1$$  \hspace{1cm} (12.44)

$$\frac{dy}{dt} = v_2$$  \hspace{1cm} (12.45)

The boundary conditions

$$t = 0 : x = 0, y = 0, v_1 = 0, v_2 = 0$$

$$t = T : x \text{ not specified}, y = \mathbf{y}, v_1 = \mathbf{v}, v_2 = 0$$

Thus we have four state variables, namely $x, y, v_1, v_2$ and two controls $\beta$ (the rate at which the exhaust gases are emitted) and $\phi$ (the thrust attitude angle). In practice we must have bounds on $\beta$, that is,

$$0 \leq \beta \leq \bar{\beta}$$  \hspace{1cm} (12.46)

so that $\beta = 0$ corresponds to the rocket motor being shut down and $\beta = \bar{\beta}$ corresponds to the motor at full power.

The Hamilton for the system is

$$H = \beta + p_1 v_1 + p_2 v_2 + p_3 \frac{c\beta}{m} \cos \phi + p_4 \left( \frac{c\beta}{m} \sin \phi - g \right)$$  \hspace{1cm} (12.47)

where $p_1, p_2, p_3, p_4$ are the adjoint variables associated with $x, y, v_1, v_2$ respectively.

$$(12.47) \Rightarrow$$

$$H = p_1 v_1 + p_2 v_2 - p_4 g + \bar{\beta} \left( 1 + p_3 \frac{c}{m} \cos \phi + p_4 \frac{c}{m} \sin \phi \right)$$

If we assuming that $\left( 1 + p_3 \frac{c}{m} \cos \phi + p_4 \frac{c}{m} \sin \phi \right) \neq 0$, we see that $H$ is linear in the control $\beta$, so that $\beta$ is bang-bang. That is $\beta = 0$ or $\beta = \bar{\beta}$. We must clearly start with $\beta = \bar{\beta}$ so that, with one switch in $\beta$, we have
\[ \beta = \begin{cases} \bar{\beta} & \text{for } 0 \leq t < t_1 \\ 0 & \text{for } t_1 \leq t \leq T \end{cases} \] (12.48)

Now \( H \) must also be maximized with respect to the second control, \( \phi \), that is, \( \partial H / \partial \phi = 0 \) giving

\[- p_3 \frac{c \beta}{m} \sin \phi + p_4 \frac{c \beta}{m} \cos \phi = 0\]

which yields \( \tan \phi = \frac{p_4}{p_3} \).

The adjoint variables satisfy the equations

\[ \dot{p}_1 = - \frac{\partial H}{\partial x} = 0 \quad \Rightarrow \quad p_1 = A \]
\[ \dot{p}_2 = - \frac{\partial H}{\partial y} = 0 \quad \Rightarrow \quad p_2 = B \]
\[ \dot{p}_3 = - \frac{\partial H}{\partial v_1} = -p_1 = -A \quad \Rightarrow \quad p_3 = C - At \]
\[ \dot{p}_4 = - \frac{\partial H}{\partial v_2} = -p_2 \quad \Rightarrow \quad p_4 = D - Bt \]

where, \( A, B, C, D \) are constant. Thus

\[ \tan \phi = \frac{D - Bt}{C - At} \]

Since \( x \) is not specified at \( t = T \), the transversality condition is \( p_1(T) = 0 \), that is, \( A = 0 \), and so

\[ \tan \phi = a - bt \] (12.49)

where \( a = D / C, b = B / C \).

The problem, in principle, I now solved. To complete it requires just integration and algebraic manipulation. With \( \phi \) given by (12.49) and \( \beta \) given by (12.48), we can integrate (12.42) to (12.45). This will bring four further constants of integration; together with \( a, b \) and the switchover time \( t_1 \) we have seven unknown constants. These are determined from the seven end-point conditions at \( t = 0 \) and \( t = T \). A typical trajectory is shown in Fig. 12.5.

12.7 Servo Problem

The problem here is to minimize

\[ \int_{0}^{T} dt \]

subject to

\[ \frac{d^2 \theta}{dt^2} + \alpha \frac{d \theta}{dt} + \omega^2 \theta = u \] (12.51)

Boundary conditions

\[ t = 0 : \theta = \theta_0, \dot{\theta} = \dot{\theta}_0 \]
\[ t = T : \theta = 0, \dot{\theta} = 0 \]

Constraint on control: \( |u| \leq 1 \).

Introduce the state variables: \( x_1 = \theta, x_2 = \dot{\theta} \). Then (12.51) becomes

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = ax_2 - \omega^2 x_1 + u \] (12.52)

As usual, we form the Hamiltonian

\[ H = 1 + p_1 y_2 + p_2 (-ax_2 - \omega^2 x_1 + u) \] (12.53)

where \( p_1, p_2 \) are adjoint variables satisfying

\[ \dot{p}_1 = - \frac{\partial H}{\partial x} = \omega^2 p_2 \] (12.54)
\[ \dot{p}_2 = - \frac{\partial H}{\partial y} = -p_1 + ap_2 \] (12.55)

Since \( H \) is linear in \( u \), and \( |u| \leq 1 \), we again immediately see that the control \( u \) is bang-bang, that is, \( u = \pm 1 \), in fact

\[ u = \begin{cases} +1 & \text{if } p_2 < 0 \\ -1 & \text{if } p_2 > 0 \end{cases} \]

To ascertain the number of switches, we must solve for \( p_2 \).

From (12.54) and (12.55), we see that

\[ \dot{p}_2 = -p_1 + a p_2 \]

that is,

\[ \dot{p}_2 - a \dot{p}_2 + \omega^2 p_2 = 0 \] (12.56)

The solution of (12.56) gives us the switching time.