C.11 Bang-bang Control

11.1 Introduction

This chapter deals with the control with restrictions: is bounded and might well be possible to have discontinuities.

To illustrate some of the basic concepts involved when controls are bounded and allowed to have discontinuities we start with a simple physical problem: Derive a controller such that a car move a distance \( a \) with minimum time.

The motion equation of the car

\[
\frac{d^2x}{dt^2} = u
\]  

(11.1)

where

\[
u = u(t), -\alpha \leq u \leq \beta
\]  

(11.2)

represents the applied acceleration or deceleration (braking) and \( x \) the distance traveled. The problem can be stated as minimize

\[
T = \int_0^T dt
\]  

(11.3)

subject to (10.11) and (10.12) and boundary conditions

\[
x(0) = 0, \dot{x}(0) = 0, x(T) = a, \dot{x}(T) = 0
\]  

(11.4)

The methods we developed in the last chapter would be appropriate for this problem except that they cannot cope with inequality constraints of the form (11.2). We can change this constraint into an equality constraint by introducing another control variable, \( v \), where

\[
v^2 = (u + \alpha)(\beta - u)
\]  

(11.5)

Since \( v \) is real, \( u \) must satisfy (11.2). We introduce the usual state variable notation \( x_1 = x \) so that

\[
\dot{x}_1 = x_2, x_1(0) = 0, x_2(T) = a
\]  

(11.6)

\[
\dot{x}_2 = u, x_2(0) = 0, x_3(T) = 0
\]  

(11.7)

We now form the augmented functional

\[
\int_0^T \left[ 1 + p_1(x_2 - \dot{x}_1) + p_2(u - \dot{x}_3) + \mu[v^2 - (u + \alpha)(\beta - u)] \right] dt
\]  

(11.8)

where \( p_1, p_2, \eta \) are Lagrange multipliers associated with the constraints (11.6), (11.7) and (11.5) respectively. The Euler equations for the state variables \( x_1, x_2 \) and control variables \( u \) and \( v \) are

\[
\frac{\partial F}{\partial x_1} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}_1} \right) = 0 \Rightarrow \dot{p}_1 = 0
\]  

(11.9)

\[
\frac{\partial F}{\partial x_2} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}_2} \right) = 0 \Rightarrow \dot{p}_2 = -p_1
\]  

(11.10)

\[
\frac{\partial F}{\partial u} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{u}} \right) = 0 \Rightarrow p_2 = \mu(\beta - \alpha - 2u)
\]  

(11.11)

\[
\frac{\partial F}{\partial v} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{v}} \right) = 0 \Rightarrow 2v\mu = 0
\]  

(11.12)

(11.12) \Rightarrow v = 0 \text{ or } \mu = 0 \text{. We will consider these two cases.}

(i) \( \mu = 0 \)

\[
\Rightarrow \begin{cases} 
   p_1 = 0 \\
   p_2 = 0 
\end{cases} \Rightarrow \text{be impossible.}
\]

(ii) \( v = 0 \)

(11.5): \( v^2 = (u + \alpha)(\beta - u) \Rightarrow u = -\alpha \text{ or } u = \beta \)

Hence

\[
\dot{x}_2 = \begin{cases} 
   \beta & 0 \leq t \leq \tau \\
   -\alpha & \tau < t \leq T 
\end{cases}
\]  

(11.13)

the switch taking place at time \( \tau \). Integrating using boundary conditions on \( x_2 \)

\[
x_2 = \begin{cases} 
   \beta t & 0 \leq t \leq \tau \\
   -\alpha(t - \tau) & \tau < t \leq T 
\end{cases}
\]  

(11.14)

Both distance, \( x_1 \), and velocity, \( x_2 \), are continuous at \( t = \tau \), we must have

(11.14) \Rightarrow \beta \tau = -\alpha(t - \tau)

(11.15) \Rightarrow \frac{1}{2} \beta \tau^2 = a - \frac{1}{2} \alpha(t - \tau)^2

Eliminating \( T \) gives the switching time as

\[
\tau = \sqrt{\frac{2aa}{\beta(\alpha + \beta)}}
\]  

(11.15)

and the final time is

\[
T = \sqrt{\frac{2a(\alpha + \beta)}{a\beta}}
\]  

(11.16)

The problem now is completely solved and the optimal control is specified by
For simplicity, we consider that the Hamiltonian is a vector function of the state vector and define the Hamiltonian

\[ H = f_0 + \sum_{i=1}^{n} p_i f_i \]  

(11.21)

This type of control is called **bang-bang control**.

### 11.2 Pontryagin’s Principle (early 1960s)

**Problem:** We are seeking extremum values of the functional

\[ J = \int_{0}^{T} f_0(x, u, t) dt \]  

(11.18)

subject to state equations

\[ \dot{x}_i = f_i(x, u, t) \quad (i = 1, 2, \ldots, n) \]  

(11.19)

initial conditions \( x = x_0 \) and final conditions on \( x_1, x_2, \ldots, x_q \) (\( q \leq n \)) and subject to \( u \in U \), the admissible control region.

For example, in the previous problem, the admissible control region is defined by

\[ U = \{ u : -\alpha \leq u \leq \beta \} \]

As in section 10.4, we form the augmented functional

\[ J^* = \int_{0}^{T} \left[ f_0 + \sum_{i=1}^{n} p_i f_i - \dot{x}_i \right] dt \]  

(11.20)

and define the Hamiltonian

\[ H = f_0 + \sum_{i=1}^{n} p_i f_i \]  

(11.21)

For simplicity, we consider that the Hamiltonian is a function of the state vector \( x \), control vector \( u \), and adjoint vector \( p \), that is, \( H = H(x, u, p) \).

We can express \( J^* \) as

\[ J^* = \int_{0}^{T} \left[ H - \sum_{i=1}^{n} p_i \dot{x}_i \right] dt \]  

(11.22)

and evaluating the Euler equations for \( x_i \), we obtain as in section 10.4 the adjoint equations

\[ \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, 2, \ldots, n) \]  

(11.23)

The Euler equations for the control variables, \( u_i \), do not follow as in section 10.4 as it is possible that there are discontinuities in \( u_i \), and so we cannot assume that the partial derivatives \( \partial H / \partial u_i \) exist. On the other hand, we can apply the free end point condition (9.23): \( \frac{\partial F}{\partial x_i} = 0 \) to obtain

\[ p_k(T) = 0 \quad k = q + 1, \ldots, n \]  

(11.24)

that is, the adjoint variable is zero at every end point where the corresponding state variable is not specified. As before, we refer to (11.24) as **transversality conditions**.

Our difficulty now lies in obtaining the analogous equation to \( \partial H / \partial u_i = 0 \) for continuous controls. For the moment, let us assume that we can differentiate \( H \) with respect to \( u \), and consider a small variation \( \delta u \) in the control \( u \) such that \( u + \delta u \) still belong to \( U \), the admissible control region. Corresponding to the small change in \( u \), there will be small change in \( x \), say \( \delta x \), and in \( p \), say \( \delta p \). The change in the value of \( J^* \) will be \( \delta J^* \), where

\[ \delta J^* = \int_{0}^{T} \left[ H - \sum_{i=1}^{n} p_i \dot{x}_i \right] dt \]

The small change operator, \( \delta \), obeys the same sort of properties as the differential operator \( d/dx \).

Assuming we can interchange the small change operator, \( \delta \), and integral sign, we obtain

\[ \delta J^* = \int_{0}^{T} \left( \delta (H - \sum_{i=1}^{n} p_i \dot{x}_i) \right) dt \]

\[ = \int_{0}^{T} \left( \delta H - \sum_{i=1}^{n} \delta (p_i \dot{x}_i) \right) dt \]

\[ = \int_{0}^{T} \left( \delta H - \sum_{i=1}^{n} \dot{x}_i \delta p_i - \sum_{i=1}^{n} p_i \delta \dot{x}_i \right) dt \]

Using chain rule for partial differentiation

\[ \partial H = \sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \dot{u}_j + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \dot{x}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \dot{p}_i \]

so that

\[ \delta J^* = \int_{0}^{T} \left[ \sum_{j=1}^{m} \frac{\partial H}{\partial u_j} \delta u_j + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \delta x_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \delta p_i \right] dt \]

\[ + \int_{0}^{T} \left( \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \dot{p}_i - \dot{p}_i \delta x_i - \dot{x}_i \delta p_i - p_i \delta \dot{x}_i \right) dt \]

since \( \dot{p}_i = -\partial H / \partial x_i \). Also, from (11.21) \( \frac{\partial H}{\partial p_i} = f_i \).
using (11.19): \[ \dot{x}_i = f(x, u, t), (i = 1, 2, \ldots, n) \]. Thus

\[
\delta J^* = \int_0^T \left[ \sum_{j=1}^m \frac{\partial H}{\partial u_j} \delta u_j - \sum_{i=1}^n \left( p_i \delta x_i + p_i \delta \dot{x}_i \right) \right] dt
\]

\[
= \int_0^T \left[ \sum_{j=1}^m \frac{\partial H}{\partial u_j} \delta u_j - \sum_{i=1}^n \frac{d}{dt} \left( p_i \delta x_i \right) \right] dt
\]

We can now integrate the second part of the integrand to yield

\[
\delta J^* = \int_0^T \sum_{j=1}^m \frac{\partial H}{\partial u_j} \delta u_j dt
\]

(11.25)

At \( t = 0 \) : \( x_i (i = 1, \ldots, n) \) are specified \( \Rightarrow \delta x_i (0) = 0 \)

At \( t = T \) : \( x_i (i = 1, \ldots, q) \) are fixed \( \Rightarrow \delta x_i (T) = 0 \)

For \( i = q+1, \ldots, n \), from the transversality conditions, (11.24)

\[ p_i (T) \delta x_i (T) = 0 \]

for \( i = 1, 2, \ldots, q+1, \ldots, n \). We now have

\[
\delta J^* = \int_0^T \sum_{j=1}^m \frac{\partial H}{\partial u_j} \delta u_j dt
\]

where \( \delta u_j \) is the small variation in the \( j^{th} \) component of the control vector \( u \). Since all these variations are independent, and we require \( \delta J^* = 0 \) for a turning point when the controls are continuous, we conclude that

\[
\frac{\partial H}{\partial u_j} = 0 \quad (j = 1, 2, \ldots, m)
\]

(11.26)

But this is only valid when the controls are continuous and not constrained. In our present case when \( u \in U \), the admissible control region and discontinuities in \( u \) are allowed. The arguments presented above follow through in the same way, except that (\( \partial H/\partial u_j \) ) must be replaced by

\[
H(x; u_1, u_2, \ldots, u_j + \delta u_j, \ldots, u_m; p) - H(x, u, p)
\]

We thus obtain

\[
\delta J^* = \int_0^T \sum_{j=1}^m \left[ H(x; u_1, \ldots, u_j + \delta u_j, \ldots, u_m; p) - H(x, u, p) \right] dt
\]

In order for \( u \) to be a minimizing control, we must have \( \delta J^* \geq 0 \) for all admissible controls \( u + \delta u \). This implies that

\[
H(x; u_1, \ldots, u_j + \delta u_j, \ldots, u_m; p) \geq H(x, u, p)
\]

(11.27)

for all admissible \( \delta u_j \) and for \( j = 1, \ldots, m \). So we have established that on the optimal control \( H \) is minimized with respect to the control variables, \( u_1, u_2, \ldots, u_m \). This is known as Pontryagin's minimum principle.

We first illustrate its use by examining a simple problem. We required to minimize

\[
J = \int_0^T dt
\]

subject to \( \dot{x}_1 = x_2, \dot{x}_2 = u \) where \( -\alpha \leq u \leq \beta \) and \( x_1 (T) = \alpha, x_2 (T) = 0, x_1 (0) = x_2 (0) = 0 \).

Introducing adjoint variables \( p_1 \) and \( p_2 \), the Hamiltonian is given by

\[
H = 1 + p_1 x_2 + p_2 u
\]

We must minimize \( H \) with respect to \( u \), and where \( u \in U = [-\alpha, \beta] \), the admissible control region. Since \( H \) is linear in \( u \), it clearly attains its minimum on the boundary of the control region, that is, either at \( u = -\alpha \) or \( u = \beta \). This illustrated in Fig.11.3. In fact we can write the optimal control as

\[
u = \begin{cases} 
-\alpha & \text{if } p_2 > 0 \\
\beta & \text{if } p_2 < 0 
\end{cases}
\]

But \( p_2 \) will vary in time, and satisfies the adjoint equations,

\[
\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0
\]

\[
\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1
\]

Thus \( p_1 = A \), a constant, and \( p_2 = -At + B \), where \( B \) is constant. Since \( p_2 \) is a linear function of \( t \), there will be most one switch in the control, since \( p_2 \) has at most one zero, and from the physical situation there must be at least one switch. So we conclude that

(i) \( \text{the control } u = -\alpha \text{ or } u = \beta \), that is, bang-bang control;
(ii) \( \text{there is one and only one switch in the control.} \)

Again, it is clear from the basic problem that initially \( u = \beta \), followed by \( u = -\alpha \) at the appropriate time.

11.3 Switching Curves

In the last section we met the idea of a switch in a control. The time (and position) of switching from one extremum value of the control to another does of course depend on the initial starting point in the phase plane. By considering a specific example we shall show how these switching positions define a switching curve.
Suppose a system is described by the state variables \(x_1, x_2\) where
\[
\dot{x}_1 = -x_1 + u \quad (11.28) \\
\dot{x}_2 = u \quad (11.29)
\]
Here \(u\) is the control variable which is subject to the constraints \(-1 \leq u \leq 1\). Given that at \(t = 0, x_1 = a, x_2 = b\), we wish to find the optimal control which takes the system to \(x_1 = 0, x_2 = 0\) in minimum time; that is, we wish to minimize
\[
J = \int_0^T dt \quad (11.30)
\]
while moving from \((a, b)\) to \((0,0)\) in the \(x_1 - x_2\) phase plane and subject to (11.28), (11.29) and
\[
-1 \leq u \leq 1 \quad (11.31)
\]
Following the procedure outline in section 11.2, we introduce the adjoint variables \(p_1\) and \(p_2\) and the Hamiltonian
\[
H = 1 + p_1(-x_1 + u) + p_2u \\
= u(p_1 + p_2) + 1 - p_1x_1
\]
Since \(H\) is linear in \(u\), and \(|u| \leq 1\), \(H\) is minimized with respect to \(u\) by taking
\[
u = \begin{cases} 
+1 & \text{if } p_1 + p_2 < 0 \\
-1 & \text{if } p_1 + p_2 > 0 
\end{cases}
\]
So the control is bang-bang and the number of switches will depend on the sign changes in \(p_1 + p_2\). As the adjoint equations, (11.23), are
\[
p_1 = \frac{\partial H}{\partial x_1} = p_1 \\
p_2 = -\frac{\partial H}{\partial x_2} = 0 \\
\Rightarrow \quad p_1 = Ae^t, \quad p_2 = B, \quad A \text{ and } B \text{ are constant}
\]
and \(p_1 + p_2 = Ae^t + B\), and this function has at most one sign change.

So we know that from any initial point \((a, b)\), the optimal control will be bang-bang, that is, \(u = \pm 1\), with at most one switch in the control. Now suppose \(u = k\), when \(k = \pm 1\), then the state equations for the system are
\[
\dot{x}_1 = -x_1 + k \\
\dot{x}_2 = k
\]
We can integrate each equation to give
\[
x_1 = Ae^{-t} + k, \quad A \text{ and } B \text{ are constants} \\
x_2 = kt + B
\]
The \(x_1 - x_2\) plane trajectories are found by eliminating \(t\), giving
\[
x_2 = -k \log |(x_1 - k)/A|
\]
Now if \(u = 1\), that is, \(k = 1\), then the trajectories are of the form
\[
x_2 = - \log |x_1 - 1| + C \\
(11.32)
\]
that is
\[
x_2 = - \log |x_1 - 1| + C
\]
where \(C\) is constant. The curves for different values of \(C\) are illustrated in Fig. 11.4.

Follow the same procedure for \(u = -1\), giving
\[
x_2 = \log |x_1 + 1| + C \\
(11.33)
\]
and the curves are illustrated in Fig. 11.5.
The basic problem is to reach the origin from an arbitrary initial point. All the possible trajectories are illustrated in Figs. 11.4 and 11.5, and we can see that these trajectories are only two possible paths which reach the origin, namely AO in Fig. 11.4 and BO in Fig. 11.5.

Combining the two diagrams we develop the Fig. 11.6. The curve AOB is called switching curve. For initial points below AOB, we take $u = +1$ until the switching curve is reached, followed by $u = -1$ until the origin is reached. Similarly for the points above AOB, $u = -1$ until the switching curve is reached, followed by $u = +1$ until the origin is reached. So we have solved the problem of finding the optimal trajectory from an arbitrary starting point.

Thus the switching curve has equation

$$x_2 = \begin{cases} \log(1 + x_1) & \text{for } x_1 > 0 \\ -\log(1 - x_1) & \text{for } x_1 < 0 \end{cases}$$  \hspace{1cm} (11.34)

### 11.4 Transversality conditions

To illustrate how the transversality conditions ($p_j(T) = 0$ if $x_j$ is not specified) are used, we consider the problem of finding the optimum control $u$ when $|u| \leq 1$ for the system described by

$$\dot{x}_1 = x_2$$  \hspace{1cm} (11.35)
$$\dot{x}_2 = u$$  \hspace{1cm} (11.36)

which takes the system from an arbitrary initial point $x_1(0) = a, x_2(0) = b$ to any point on the $x_2$ axis, that is, $x_1(T) = 0$ but $x_2(T)$ is not given, and minimize

$$J = \int_0^T \dot{p} \, dt$$  \hspace{1cm} (11.37)

subject to (11.35), (11.36), the above boundary conditions on $x_1$ and $x_2$, and such that

$$-1 \leq u \leq 1$$  \hspace{1cm} (11.38)

Following the usual procedure, we form the Hamiltonian

$$H = 1 + p_1 x_2 + p_2 u$$  \hspace{1cm} (11.39)

We minimize $H$ with respect to $u$, where $-1 \leq u \leq 1$, which gives

$$u = \begin{cases} +1 & \text{if } p_2 < 0 \\ -1 & \text{if } p_2 > 0 \end{cases}$$

The adjoint variables satisfy

$$\begin{cases} \dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0 \\ \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1 \end{cases} \Rightarrow \begin{cases} p_1 = A \\ p_2 = -At + B \end{cases}$$

Since $x_2(T)$ is not specified, the transversality condition becomes $p_2(T) = 0$. Hence $0 = -At + B$ and $p_2 = A(T - t)$. For $0 < t < T$, there is no change in the sign of $p_2$, and hence no switch in $u$. Thus either $u = +1$ or $u = -1$, but with no switches. We have

$$x_2^2 = 2(x_1 - B) \quad \text{when } u = 1$$  \hspace{1cm} (11.40)
$$x_2^2 = -2(x_1 - B) \quad \text{when } u = -1$$  \hspace{1cm} (11.41)

These trajectories are illustrated in Fig. 11.7, the direction of the arrows being determined from $u = 2x_2$. We first consider initial points $(a, b)$ for which $a > 0$. For points above the curve OA, there is only one trajectory,
\( u = -1 \) which reaches the \( x_2 \)-axis, and this must be the optimal curve. For points below the curve OA, there are two possible curves, as shown in Fig. 11.8.

From (11.36): \( \dot{x}_2 = u \), that is \( \dot{x}_2 = \pm 1 \), and the integrating between 0 and \( T \) gives

\[ x_2(T) - x_2(0) = \pm T \]

that is

\[ T = |x_2(T) - x_2(0)| \quad (11.42) \]

Hence the modulus of the difference in final and initial values of \( x_2 \) given the time taken. This is shown in the diagram as \( T_+ \) for \( u = +1 \) and \( T_- \) for \( u = -1 \). The complete set of optimal trajectories is illustrated in Fig. 11.9.

Fig. 11.9 Optimal Trajectories to reach \( x_2 \)-axis in minimum time

11.5 Extension to the Boltza problem