

C.8 Introduction to Optimal Control

8.1 Control and Optimal Control

Take an example: the problem of rocket launching a satellite into an orbit about the earth.

- A control problem would be that of choosing the thrust angle and rate of emission of the exhaust gases so that the rocket takes the satellite into its prescribed orbit.
- An associated optimal control problem is to choose the controls to affect the transfer with, for example, minimum expenditure of fuel, or in minimum time.

8.2 Examples

8.2.1 Economic Growth
8.2.2 Resource Depletion
8.2.3 Exploited Population
8.2.4 Advertising Policies
8.2.5 Rocket Trajectories

The governing equation of rocket motion is

\[ m \frac{dv}{dt} = mF + F_{ext} \] (8.22)

where

- \( m \): rocket’s mass
- \( v \): rocket’s velocity
- \( F \): thrust produced by the rocket motor
- \( F_{ext} \): external force

It can be seen that

\[ |F| = \frac{c}{m} \beta \] (8.23)

where

- \( c \): relative exhaust speed
- \( \beta = \frac{dn}{dt} \): burning rate

Let \( \phi \) be the thrust attitude angle, that is the angle between the rocket axis and the horizontal, as in the Fig. 8.4, then the equations of motion are

\[ \begin{align*}
\frac{dv_1}{dt} &= \frac{c \beta}{m} \cos \phi \\
\frac{dv_2}{dt} &= \frac{c \beta}{m} \sin \phi - g
\end{align*} \] (8.24)

Here \( v = (v_1, v_2) \) and the external force is the gravity force only \( F_{ext} = (0, -mg) \).

The minimum fuel problem is to choose the controls, \( \beta \) and \( \phi \), so as to take the rocket from initial position to a prescribed height, say, \( \bar{y} \), in such a way as to minimize the fuel used. The fuel consumed is

\[ \int_0^T \beta \, dt \] (8.25)

where \( T \) is the time at which \( \bar{y} \) is reached.

8.2.6 Servo Problem

A control surface on an aircraft is to be kept at rest at a position. Disturbances move the surface and if not corrected it would behave as a damped harmonic oscillator, for example

\[ \ddot{\theta} + a \dot{\theta} + w^2 \theta = 0 \] (8.26)

where \( \theta \) is the angle from the desired position (that is, \( \theta = 0 \)).

The disturbance gives initial values \( \theta = \theta_0, \dot{\theta} = \dot{\theta}_0 \), but a servomechanism applies a restoring torque, so that (8.26) is modified to

\[ \ddot{\theta} + a \dot{\theta} + w^2 \theta + u = 0 \] (8.27)

The problem is to choose \( u(t) \), which will be bounded by \( |u(t)| \leq c \), so as to bring the system to \( \theta = 0, \dot{\theta} = 0 \) in minimum time. We can write this time as

\[ \int_0^T dt \] (8.28)

and so this integral is required to be minimized.

8.3 Functionals

All the above examples involve finding extremum values of integrals, subject to varying constraints. The integrals are all of the form

\[ J = \int_{t_0}^{t_1} F(x, \dot{x}, t) \, dt \] (8.29)

where \( F \) is a given function of function \( x(t) \), its derivative \( \dot{x}(t) \) and the independent variable \( t \). The path \( x(t) \) is defined for \( t_0 \leq t \leq t_1 \)

- given a path \( x = x_1(t) \) ⇒ give the value \( J = J_1 \)
- given a path \( x = x_2(t) \) ⇒ give the value \( J = J_2 \)

in general, \( J_1 \neq J_2 \), and we call integrals of the form (8.29) functional.
8.4 The Basic Optimal Control Problem

Consider the system of the general form \( \dot{x} = f(x, u, t) \), where \( f = (f_1, f_2, \ldots, f_n)' \) and if the system is linear

\[
\dot{x} = A x + B u \tag{8.33}
\]

The basic control is to choose the control vector \( u \in U \) such that the state vector \( x \) is transferred from \( x_0 \) to a terminal point at time \( T \) where some of the state variables are specified. The region \( U \) is called the admissible control region.

If the transfer can be accomplished, the problem in optimal control is to effect the transfer so that the functional

\[
J = \int_0^T f_0(x, u, t) dt \tag{8.34}
\]

is maximized (or minimized).