20-1 INTRODUCTION

Foundations supporting reciprocating engines, compressors, radar towers, punch presses, turbines, large electric motors and generators, etc. are subject to vibrations caused by unbalanced machine forces as well as the static weight of the machine. If these vibrations are excessive, they may damage the machine or cause it not to function properly. Further, the vibrations may adversely affect the building or persons working near the machinery unless the frequency and amplitude of the vibrations are controlled.

The design of foundations for control of vibrations was often on the basis of increasing the mass (or weight) of the foundation and/or strengthening the soil beneath the foundation base by using piles. This procedure generally works; however, the early designers recognized that this often resulted in considerable overdesign. Not until the 1950s did a few foundation engineers begin to use vibration analyses, usually based on a theory of a surface load on an elastic half-space. In the 1960s the lumped mass approach was introduced, the elastic half-space theory was refined, and both methods were validated.

The principal difficulty in vibration analysis now consists in determining the necessary soil values of shear modulus \( G' \) and Poisson’s ratio \( \mu \) for input into the differential equation solution that describes vibratory motion. The general methods for design of foundations, both shallow and deep, that are subject to vibration (but not earthquakes) and for the determination of the required soil variables will be taken up in some detail in the following sections.

20-2 ELEMENTS OF VIBRATION THEORY

A solid block base rests in the ground as in Fig. 20-1. The ground support is shown replaced by a single soil spring. This is similar to the beam-on-elastic-foundation case except the beam uses several springs and the foundation base here only uses one. Also this spring is
for dynamic loading and will be computed differently from the beam problem. As we shall later see, the spring will be frequency-dependent as well.

Here the soil spring has compressed under the static weight of the block an amount

$$z_s = \frac{W}{K_z} \quad (a)$$

At this time $z_s$ and the spring $K_z$ are both static values. Next we give the block a quick solid shove in the $z$ direction with a quick release, at which time the block begins to move up and down (it vibrates). Now the $z_s$ and soil $K_z$ are dynamic values.

We probably could not see the movement, but it could be measured with sensitive electronic measuring equipment. After some time the block comes to rest at a slightly larger displacement $z_s'$ as shown in Fig. 20-1. The larger displacement is from the vibration producing a state change in the soil (a slightly reduced void ratio or more dense particle packing).

We can write the differential equation to describe this motion [given in any elementary dynamics or mechanical vibration textbook (as Den Hartog (1952)], using the terms shown in Fig. 20-1a in a form of $F = ma$ to give, in one dimension,

$$m\ddot{z} + K_z z = 0 \quad (b)$$

Solving by the methods given in differential equation textbooks after dividing through by the mass term $m$ and defining $\omega_n^2 = K_z/m$, we can obtain the period of vibration $T$ as

$$T = \frac{2\pi}{\omega_n}$$

and the natural frequency $f_n$ as any one of the following

$$f_n = \omega_n \frac{2\pi}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{K_z}{m}} = \frac{1}{2\pi} \sqrt{\frac{K_z g}{W}} = \frac{1}{2\pi} \sqrt{\frac{g}{z_s}} \quad (20-1)$$

From Eq. (b) it would appear that the vibration will continue forever; we know from experience that this is not so. There must be some damping present, so we will add a damping device termed a dashpot (analog = automobile shock absorbers) to the idealized model. To maintain symmetry we will add half the dashpot to each edge of the base as in Fig. 20-1b.
Dashpots are commonly described as developing a restoring force that is proportional to the velocity \( \dot{z} \) of the mass being damped. With this concept for the dashpot force a vertical force summation gives the following differential equation

\[
m\ddot{z} + c\dot{z} + Kz = 0 \tag{c}
\]

Solving this equation, we obtain the general form of the instantaneous dynamic displacement \( z \) as

\[
z = C_1 e^{\beta_1 t} + C_2 e^{\beta_2 t} \tag{d}
\]

where

\[
\begin{align*}
\{ \beta_1 \} &= \frac{-c \pm \sqrt{c^2 + 4Kz}}{2m} \\
\{ \beta_2 \} &= \frac{-c \pm \sqrt{(\frac{c}{2m})^2 - \frac{Kz}{m}}
\end{align*}
\tag{e}
\]

From the \( \beta \) values we note that the \( \sqrt{\ } \) term is one of the following:

**Case 1. No damping** (< 0)—gives Eq. (20-1) when \( c = 0 \).

**Case 2. Overdamped** (> 0) with \( c^2 > 4Kz m \) (see Fig. 20-2).

**Case 3. Critically damped** (= 0) with \( c^2 = 4Kz m \). We define the critical damping as

\[
c_{z,c} = 2m \omega_n = 2 \sqrt{Kz m}
\]

**Figure 20-2** Plot of time-displacement curves for three types of damped movement. The plot is relative since the natural frequency is constant and \( \omega t \) is to the same scale. The variable in the three plots is the damping factor.
Case 4. Underdamped \((< 0)\) when \(c_z^2 < 4K_z m\). This is the usual case in foundation vibrations since case 1 is impossible (even the spring \(K_z\) will have some internal damping) and vibrations rapidly dissipate in cases 2 and 3 as shown in Fig. 20-2.

From case 3 we can define a damping ratio \(D_z\) as

\[ D_z = \frac{c_z}{c_{zc}} \quad \text{(here } i = z) \quad (20-2) \]

and the damped circular frequency \(\omega_d\) from a reordering of the \(\sqrt{}\) term of Eq. (e)

\[ \omega_d = \sqrt{-1} \sqrt{\frac{K_z}{m} - \left(\frac{c_z\omega_n}{c_{zc}}\right)^2} \quad (20-3) \]

Since \(\omega_n^2 = K_f/m\) and \(c_z^2 = D_z^2 c_{zc}^2\), we can obtain an alternative form as

\[ \omega_d = \omega_n \sqrt{1 - D_z^2} \quad (20-3a) \]

The \(\sqrt{-1}\) disappears, since \(D_z \leq 1\).

In the general vibrating base case, however, we have a base load consisting of a large weight colliding with an anvil as in a punch press, a piece of rotating machinery, or an operating engine. The engine in turn may drive a piece of equipment such as a compressor or pump. Any of these latter can have the effect of an unbalanced force (or several forces) rotating about an axis such as a crankshaft (see Fig. 20-3). From elementary dynamics a mass \(m_e\) connected to a shaft with an arm of \(\bar{y}\) rotating at a circular frequency of \(\omega\) produces a force at any instant in time of

\[ F_i = m_e \bar{y} \omega^2 \]

If the operating frequency is \(\omega_o\) it is evident that the force \(F_i\) is varying from zero to the maximum at the operating speed, after which it is a constant. It is also evident that along the particular axis of interest the foregoing force will vary as

\[ F = F_o \sin \omega t \quad \text{or as} \quad F = F_o \cos \omega t \]

In these cases we rewrite Eq. (c) as

\[ m \ddot{z} + c_z \dot{z} + K_z z = F(t) \quad (f) \]

Using the same methods as for Eqs. (b) and (c), we obtain the following for the case of \(F = F_o \sin \omega t\)—the maximum \(z\) occurs at \(\omega t = \pm \pi/2\) radians \(\rightarrow F_o \times 1 = F_o\):

\[ z = \frac{F_o}{\sqrt{(K_z - m\omega^2)^2 + c_z^2 \omega^2}} \quad (20-4a) \]

After factoring \(K_z\) and making substitutions for \(m\) and \(c_z\), we obtain the following:

\[ z = \frac{F_o/K_z}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2D_z\omega/\omega_n)^2}} \quad (20-4b) \]

Note that \(K_z\) is a static soil spring for \(\omega = 0\) and is a dynamic value when \(\omega > 0\)—in other words, \(K_z = f(\omega)\). If the radical in the denominator of Eq. (20-4b) is written as

\[ A = (1 - a^2)^2 + (2D_z a)^2 \]
then by setting the derivative \( dA/da = 0 \) we obtain the maximum value of dynamic \( z \) possible in the form (and using \( z_s = \) static displacement = \( F_0/K_z \)) as

\[
 z_{\text{max}} = \frac{z_s}{2D_z \sqrt{1 - D_z^2}} \quad (20-5)
\]

The resonant frequency \( f_r \) is obtained as

\[
f_r = f_n \sqrt{1 - 2D_z^2} \quad (20-6)
\]

where \( f_n \) is defined by Eq. (20-1). Here resonance is somewhat below the natural base frequency \( f_n \). Use \( F_o = \) constant or \( F_o = m_e\bar{y}\omega_o^2 \) in these equations. When \( F_o = m_e\bar{y}\omega^2 \) the resonance frequency can be computed as

\[
f_r' = \frac{f_n}{\sqrt{1 - 2D_z^2}} \quad (20-6a)
\]

which gives a resonance frequency above the natural frequency \( f_n \).

It is sometimes useful to obtain curves of relative displacement versus a frequency ratio as in Fig. 20-4. In this case we can rewrite Eq. (20-4b) to obtain an amplitude ratio of

\[
\frac{z}{z_s} = N
\]
Constant-amplitude exciting force (never larger than $F_0$)

$N = \frac{z}{z_0}$

Frequency-dependent exciting force $f_0$ varies with $\omega^2$

Figure 20-4  Plots of relative amplitude ratios versus frequency ratios.
where \( N \) equals \( 1/\text{(square root terms)} \) and is a magnification factor. When the exciting force \( F_0 \) is frequency dependent (Fig. 20-3c) the amplitude ratio is

\[
\frac{z}{z_s} = N \left( \frac{f}{f_n} \right)^2 = N'
\]

It is about as easy, however, to program Eq. (20-4a), simply vary \( F_0 = m \ddot{y} \omega^2 \), and directly compute the \( z/z_s \) ratio—particularly since both \( K_z \) and \( c_z \) are frequency-dependent. The values of \( N \) and \( N' \) for a range of \( f/f_n = 0 \) to \( 3 \) and for several values of damping ratio \( D = 0, 0.1, \text{etc.} \) are shown in Fig. 20-4. The most significant feature is that \( N \) ranges from 1 to a peak at \( f/f_n \) slightly less than 1 and approaches zero at large \( f/f_n \) where a frequency-dependent force produces an \( N' \) that starts at zero, peaks slightly beyond \( f/f_n = 1 \), and then flattens toward 1 at large \( f/f_n \).

For vibration analyses we can directly use Eq. (20-4a) if we have values of soil spring \( K_z \) and damping coefficient \( c_z \) and can reasonably identify the block mass \( (W/g) \) that includes the base and all permanent attachments. We also must have a value of \( F_0 \). We do not usually use the force \( F = F_0 \sin \omega t \) since we are interested in the maximum \( z \) and at some instant in time \( \sin \omega t = 1 \) so \( F = F_0 \), but we do have to be aware the vibration displacement oscillates at \( \pm z \) from \( z_s \).

Carefully note that within the terms \( f \) and \( \omega \) in the frequency ratios are the frequencies of the machine that are developed by the unbalanced forces and depend on revolutions per minute or cycles per second (Hz); and \( f_n, \omega_n, \text{and} \omega_d \) are the natural \((n)\) and damped \((d)\) system frequencies.

### 20-3 THE GENERAL CASE OF A VIBRATING BASE

Figure 20-5 illustrates the general case of a foundation with 6 d.o.f. of vibration/excitation modes possible. From this figure we can have

<table>
<thead>
<tr>
<th>Translations:</th>
<th>3 modes along the ( x, y, \text{and} \ z ) axes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotations:</td>
<td>3 modes about the ( x, y, \text{and} \ z ) axes</td>
</tr>
</tbody>
</table>

The rotations about the \( x \) and \( y \) axes are usually termed \textit{rocking modes} and the \( z \) axis rotation is termed \textit{yawing}.

**Figure 20-5** Rectangular foundation block with 6 degrees of freedom.
There are three procedures currently used to analyze these vibration modes:

a. Elastic half-space theory [outlined by Sung (1953)]
b. Analog methods [as given by Richart et al. (1970)]
c. Lumped mass or lumped parameter method (as given in the preceding section)

After an extensive literature survey and review of the several methods the author decided that the lumped mass approach is at least as reliable and substantially more general than any of the alternative procedures. Current state-of-art allows adjustments to the spring and damping constants for frequency. The same soil data are required as for any of the alternative procedures, and the method is rather simple, for it is only necessary to program Eq. (20-4a) to increment the frequency of the engine/machine to obtain the corresponding displacement amplitudes and see if any are too large for the particular equipment. Of course it is also necessary to obtain certain data, as subsequently noted, as input along with soil parameters. It is particularly helpful to use a computer program to do most of the work because this type of problem is computationally intensive—particularly when making parametric studies.

By direct analogy of Eq. (f) we can write differential equations as follows:

For sliding modes:

\[
\begin{align*}
m\ddot{x} + c_x\dot{x} + K_x x &= F_x(t) \\
m\ddot{y} + c_y\dot{y} + K_y y &= F_y(t)
\end{align*}
\]

For rocking modes:

\[
I_{\theta_i}\ddot{\theta} + c_{\theta_i}\dot{\theta} + K_{\theta_i}\theta = F_{\theta_i}(t)
\]

Since the differential equation is similar in form for all cases we have a general solution in Eq. (20-4a) with appropriate entries for \(K_z, c_z\), and \(m\) as follows:

<table>
<thead>
<tr>
<th>Axis</th>
<th>Spring</th>
<th>Damping</th>
<th>Mass, (m_i) =</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(K_x)</td>
<td>(c_x)</td>
<td>(m_x = m)</td>
</tr>
<tr>
<td>(y)</td>
<td>(K_y)</td>
<td>(c_y)</td>
<td>(m_y = m)</td>
</tr>
<tr>
<td>(z)</td>
<td>(K_z)</td>
<td>(c_z)</td>
<td>(m_z = m = \frac{W}{g})</td>
</tr>
</tbody>
</table>

| \(x\) | \(K_{\theta_x}\) | \(c_{\theta_x}\) | \(I_{\theta_x}\) |
| \(y\) | \(K_{\theta_y}\) | \(c_{\theta_y}\) | \(I_{\theta_y}\) |
| \(z\) | \(K_{\theta_z}\) | \(c_{\theta_z}\) | \(I_{\theta_z}\) |

where \(W = \text{weight of base + all machinery and other attachments that will vibrate with the base; } g = \text{gravitation constant (9.807 or 32.2).}\)

Values of \(I_{\theta_i}\) can be computed using formulas given in Table 20-1 or from methods given in most dynamics textbooks. The mass \(m\) used in these equations is the same for all translation modes. Most mass moments of inertia will be composites in which the transfer formula will be required; however, the total mass \(m\) will be the same in all the modes.
The soil spring ($K_i$) and damping ($c_i$) terms can be computed by a number of procedures, all giving slight to major computed differences in vibration displacements. Fortunately the spring and damping effects are under the square root of Eq. (20-4) so the estimation effect is somewhat reduced. We would not like, however, to compute a displacement of, say, 0.001 mm and have the value be 0.01 mm, which results in the machine supported by the base becoming damaged from excessive base movements.

**20-4 SOIL SPRINGS AND DAMPING CONSTANTS**

Barkan (1962) is a frequently cited source for soil springs. Other references such as Richart et al. (1970) and Novak and Beredugo (1972) also give methods to compute spring values. Dobry and Gazetas (1986) made an extensive literature survey for methods to compute the spring and damping constants and plotted the values from the several sources versus a dimensionless frequency factor $a_o$ and produced a series of best-fit curves. They then compared

**TABLE 20-1**

**Mass moments of inertia $I_{\theta i}$ for shapes most likely to be used for a vibrating base**

Method of derivation is found in most dynamics textbooks. Units are mass×length² (for SI = kN·m·s²). Use transfer formula to transfer to parallel axes for composite sections.

<table>
<thead>
<tr>
<th>Shape</th>
<th>$I_{\theta x}$</th>
<th>$I_{\theta y}$</th>
<th>$I_{\theta z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slender rod</td>
<td>$\frac{1}{12}mL^2$</td>
<td>$\frac{1}{12}mL^2$</td>
<td></td>
</tr>
<tr>
<td>Circular cylinder</td>
<td>$\frac{1}{2}mr^2$</td>
<td>$\frac{1}{12}m(3r^2 + L^2)$</td>
<td>$\frac{1}{4}mL^2$ (base)</td>
</tr>
<tr>
<td>Thin disk</td>
<td>$\frac{1}{2}mr^2$</td>
<td>$\frac{1}{2}mr^2$</td>
<td></td>
</tr>
</tbody>
</table>

The soil spring ($K_i$) and damping ($c_i$) terms can be computed by a number of procedures, all giving slight to major computed differences in vibration displacements. Fortunately the spring and damping effects are under the square root of Eq. (20-4) so the estimation effect is somewhat reduced. We would not like, however, to compute a displacement of, say, 0.001 mm and have the value be 0.01 mm, which results in the machine supported by the base becoming damaged from excessive base movements.
Transfer formula:

\[ I'_{\theta i} = I_{\theta i} + m d^2 \]

predicted vibrations from these curves with measured values and found very good agreement in all cases. The dimensionless frequency parameter \( a_o \) is defined for a round base as

\[ a_o = \frac{\omega r_o}{V_s} = \omega r_o \sqrt{\frac{\rho}{G'}} \]

where the shear wave velocity in the soil is defined by Eq. (20-15) with \( \rho \) = density of soil and \( G' \) = shear modulus defined in Sec. 20-5. The corresponding \( a_o \) used in the curves of spring versus \( a_o \) and damping versus \( a_o \) for rectangular bases is

\[ a_o = \frac{\omega B}{V_s} \]
Figure 20-6  Factor $J_a = A/4L^2$ for several geometric shapes. Note axis orientation in all cases, and length $= 2L$ and width $= 2B$.

Carefully note that the base width defined in Fig. 20-6 is $2B$, so the $B$ value used in this equation is half the base width (analogous to $r_o$ for a round base). Here $\omega$ is the frequency of the machine and not the system natural frequency.

In most cases the foundation base being dynamically excited is not round with a radius $r_o$ but, rather, is rectangular—often with an $L/B$ of 2 to 5. The solutions generally published prior to those of Dobry and Gazetas required converting the rectangular (or other) shape to an equivalent round base, equivalent being defined as a round base with the same area in plan as in the actual base. Solution quality deteriorated as the $L/B$ ratio increased as would be expected since the equivalent round foundation would become a poor model at larger $L/B$ ratios. Observe that in using the Dobry and Gazetas method we would make a better model by converting a round base to an equivalent square than by converting a square to an equivalent round base.

This method uses a base of dimensions $2B \times 2L$ as shown on Fig. 20-6. Note very carefully that the circumscribed base width is $2B$ and the length is $2L$. This gives the plan area directly as $2B \times 2L$ only for solid rectangles. For all other bases one must obtain the circumscribed dimensions and then compute the actual base area by any practical means (perhaps by using the sum of several components consisting of squares, triangles, etc.).

Similarly in this method it may be necessary, in computing certain of the soil springs, to use the plan moment of inertia about the $x$, $y$, or $z$ axes. This computation gives for a solid rectangle

$$I_x = 1.333LB^3 \quad I_y = 1.333BL^3 \quad \text{and} \quad I_z = I_x + I_y = J$$
From Fig. 20-6 we see that $2L$ is always parallel to the $x$ axis, giving $L/B \geq 1$. The factor $1.333 = \frac{16}{12}$ since we use $2B \times 2L$. When the circumscribed dimensions are not completely filled in, it is necessary to compute the moment of inertia about any axis ($I_x$ is about the $x$ axis, etc.) using the sum of the component parts and the transfer of axes formula as necessary.

Another constant used by this procedure is

$$J_a = \frac{\text{Area}}{4L^2} \quad \text{(dimensionless)} \quad (20-8)$$

with several values of $J_a$ shown on Fig. 20-6.

In using the method it is necessary first to compute the static spring values using the curve fit values given in Table 20-2 and, for damping, values from Table 20-4 to obtain

Spring: $K_i$ \hspace{1cm} Damping: $c_i$

These values are then multiplied by frequency-dependent factors $\eta_i$ obtained from Fig. 20-7a, b, c (as appropriate) and by $\lambda_i$ factors from curve-fitted equations in Table 20-4 (done by the author for computer programming convenience). These factors then give the dynamic springs and damping coefficients as

Spring: $\overline{K}_i = \eta_i K_i$ \hspace{1cm} Damping: $\overline{c}_i = \lambda_i c_i$

### TABLE 20-2

**Dynamic spring $K_i$ for use in Eq. (20-4a). Obtain $S_i$ factors from Table 20-3**

Values shown are "static" values that must be multiplied by a factor $\eta$, which may be obtained from Fig. 20-7.* Note that $2L = \text{base length}$ and $2B = \text{base width}$ with $L$ parallel to $x$ axis and $B$ parallel to $y$ axis.

<table>
<thead>
<tr>
<th>For rectangular bases</th>
<th>Round base</th>
<th>Strip</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vertical mode</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_z = S_z \frac{2LG'}{1 - \mu}$</td>
<td>$K_z = \frac{4G'B}{1 - \mu}$</td>
<td>$K_z = \frac{0.8G'(2L)}{1 - \mu}$</td>
</tr>
<tr>
<td><strong>Horizontal mode</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallel to $y$ axis</td>
<td>$K_y = S_y \frac{2LG'}{2 - \mu}$</td>
<td>$K_y = \frac{8G'B}{2 - \mu}$</td>
</tr>
<tr>
<td>Parallel to $x$ axis</td>
<td>$K_x = S_y \frac{2LG'}{2 - \mu}$</td>
<td>$K_x = K_y$</td>
</tr>
<tr>
<td>$K_x = S_y \frac{2LG'}{2 - \mu} - 0.21LG \left(1 - \frac{B}{L}\right)$</td>
<td>($n_x = 1$ for $K_s$ so $\overline{K}_x = K_x$)</td>
<td></td>
</tr>
<tr>
<td><strong>Rocking mode</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>About $x$ axis</td>
<td>$K_{\theta x} = S_{\theta x} \frac{G'}{1 - \mu} (I_{\theta x})^{0.75} \left(\frac{B}{L}\right)^{-0.25}$</td>
<td>$K_{\theta x} = \frac{8G'B^3}{3(1 - \mu)}$</td>
</tr>
<tr>
<td>About $y$ axis</td>
<td>$K_{\theta y} = S_{\theta y} \frac{G'}{1 - \mu} (I_{\theta y})^{0.75}$</td>
<td>$K_{\theta y} = K_{\theta x}$</td>
</tr>
<tr>
<td><strong>Torsion mode</strong></td>
<td>$K_t = S_t G'(J)^{0.75}$</td>
<td>$K_t = \frac{16G'B^3}{3}$</td>
</tr>
</tbody>
</table>

*After Dobry and Gazetas (1986).
Vertical \( \eta \) factors. Note these are dependent on Poisson's ratio \( \mu \). Use \( \mu = 0.5 \) for saturated clay and \( \mu = 0.33 \) for all other soil.

Fig. 20-7 The \( \eta \) factors to convert static springs of Table 20-2 to dynamic values as \( \bar{K}_i = \eta K_i \). Curves condensed from Dobry and Gazetas (1985).

These dynamic spring \( \bar{K}_i \) and damping \( \bar{c}_i \) are based on a perfectly elastic soil with zero material damping. Experimental evidence indicates that even at very small strains soil exhibits a material (or hysteretic) damping. This is usually specified using a frequency-independent damping ratio \( D_i \) (see Eq. 20-2) that is used to adjust \( \bar{K}_i \) and \( \bar{c}_i \) further according to Lysmer as cited by Dobry and Gazetas (1986) as follows:

\[
K_i = \bar{K}_i - \omega \bar{c}_i D_i \quad (20-9)
\]
\[
c_i = \bar{c}_i + \frac{2\bar{K}_i D_i}{\omega} \quad (20-10)
\]

These values of \( K_i \) and \( c_i \) are used in Eq. (20-4) or its variations depending on whether there is translation or rocking.

Values of material damping \( D_i \) are considered in Sec. 20-5.3.

From the discussion to this point it is evident that the only really practical way to solve vibration problems is to use a computer program. In a computer program it is necessary to do the following:

1. Allow input of the problem parameters (base data, soil data, and dynamic force data).
2. Compute the static spring and damping and the dynamic factors \( \eta_i \) and \( \lambda_i \) as appropriate for that mode. Use Fig. 20-7 for the \( \eta_i \) factors.
3. Compute the dynamic spring and damping values using Eqs. (20-9) and (20-10).
4. Solve Eq. (20-4a) for the displacement amplitude or Eq. (20-4b) for the magnification factor to apply to the static displacement.

5. Output sufficient results so a spot check for correctness can be made.

To accomplish these requirements, FADDYNF1 (B-11) is provided on your program diskette. This program directly uses the equations for a rectangular base given in Tables 20-2 and 20-3. A curve-fitting (regression-type) analysis was used to obtain a best fit of the curves of Fig. 20-7 with the coefficients directly programmed for the $\eta_i$ values. A similar curve-fitting scheme was used to produce Table 20-4 from the cited reference (without providing the figures in the text to conserve space). The coefficients were then programmed in the several subroutines in the program so that $\lambda_i$ could be obtained. A linear interpolation between curves is used for intermediate values of $L/B$. 

Figure 20-7 (continued)
TABLE 20-3
$S_i$ factors for computing $K_i$ of Table 20-2

<table>
<thead>
<tr>
<th>Mode</th>
<th>Applicable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical:</td>
<td></td>
</tr>
<tr>
<td>$S_z = 0.8$</td>
<td>$J_a \leq 0.02$</td>
</tr>
<tr>
<td>$S_z = 0.73 + 1.54(J_a)^{0.75}$</td>
<td>$J_a &gt; 0.02$</td>
</tr>
<tr>
<td>Horizontal:</td>
<td></td>
</tr>
<tr>
<td>$S_y = 2.24$</td>
<td>$J_a \leq 0.16$</td>
</tr>
<tr>
<td>$S_y = 4.5(J_a)^{0.38}$</td>
<td>$J_a &gt; 0.16$</td>
</tr>
<tr>
<td>Rocking:</td>
<td></td>
</tr>
<tr>
<td>$S_{ax} = 2.54$</td>
<td>$B/L \leq 0.4$</td>
</tr>
<tr>
<td>$S_{ax} = 3.2(B/L)^{0.25}$</td>
<td>$B/L &gt; 0.4$</td>
</tr>
<tr>
<td>$S_{ay} = 3.2$</td>
<td>All $B/L$</td>
</tr>
<tr>
<td>Torsion:</td>
<td></td>
</tr>
<tr>
<td>$S_t = 3.8 + 10.7(1 - B/L)^{10}$</td>
<td>All $B/L$</td>
</tr>
</tbody>
</table>

$J_a = \text{area} / (4L^2)$ where $2L =$ length of base; area = $2B \times 2L$ for solid rectangle.

The program should generally be limited to $L/B \leq 5$. One should check the range of $L/B$ given either in Fig. 20-7 or in Table 20-4 since certain vibration modes may allow a larger $L/B$. The range of $a_o$ should be limited to between zero and 1.5 (usual range provided in most literature sources), which should cover nearly all likely base designs. For $a_o$ to exceed 1.5 one would have a very high speed machine (large $\omega$) and/or a small ground shear wave velocity $V_s$. In these cases some kind of soil strengthening or the use of piles may be necessary if vibration control is critical.

20-5 SOIL PROPERTIES FOR DYNAMIC BASE DESIGN

The soil spring constants shown in Table 20-2 directly depend on the dynamic soil shear modulus $G'$ and Poisson's ratio $\mu$. The unit weight $\gamma_s$ is needed to compute the soil density $\rho$ as

$$\rho = \frac{\gamma_s}{9.807 \text{ kN} \cdot \text{s}^2/\text{m}^4} \quad \text{(SI and $\gamma_s$ in kN/m}^3)$$

$$\rho = \frac{\gamma_s}{32.2 \text{ k} \cdot \text{s}^2/\text{ft}^4} \quad \text{(Fps and $\gamma_s$ in k/ft}^3)$$

It is usual to estimate $\mu$ in the range of 0.3 to 0.5 as done in Chap. 5 for foundation settlements. We note that the dynamic coefficients $\eta_x$ and $\eta_y$ also depend on $\mu$; however, here only two values—0.333 and 0.50—can be used (as programmed in the computer program). These two values are probably sufficient for most problems since $\mu$ is estimated and not directly measured.

The unit weight of a cohesive soil can be directly measured using the procedures outlined in Example 2-1. In other cases it can generally be estimated with sufficient precision using Table 3-4 or simply be taken as between 17 and 20 kN/m$^3$ (or 110 to 125 lb/ft$^3$). Larger values for $\gamma_s$ can be justified for a dynamic analysis as no one would place a base on loose soil. The soil would be either densified or stiffened with admixtures, soil-cement piles, or stone columns; or the base would be placed on piles.

20-5.1 Laboratory Determination of $G'$

The shear modulus can be estimated from resonant-column tests. These involve a laboratory apparatus consisting of a specially constructed triaxial cell capable of providing a very small
TABLE 20-4
Damping constants for computing the damping coefficient $c_i$
Obtained using curve fitting to enlarged figures from Dobry and Gazetas (1986).
Values programmed in computer program. $A =$ actual base area

For vertical damping in range of $0 \leq a_o \leq 1.5$

\[ \lambda_z = \frac{c_z}{\rho V_{LA} A} = X_1 + (a_o R) X_2 + (a_o R)^2 X_3 + X_4 \exp(-a_o R) \]

<table>
<thead>
<tr>
<th>$L/B = R$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9716</td>
<td>-0.0500</td>
<td>0.0520</td>
<td>-0.0660</td>
</tr>
<tr>
<td>2</td>
<td>1.2080</td>
<td>-0.1640</td>
<td>0.0385</td>
<td>-0.2515</td>
</tr>
<tr>
<td>4</td>
<td>1.0900</td>
<td>-0.0025</td>
<td>0.0012</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>1.2285</td>
<td>-0.0359</td>
<td>0.0024</td>
<td>0.1515</td>
</tr>
<tr>
<td>10</td>
<td>1.3112</td>
<td>-0.0285</td>
<td>0.0011</td>
<td>0.4388</td>
</tr>
</tbody>
</table>

For $R > 10$ use $\lambda_z = \lambda_{z10}(1 + 0.001 R)$ For $a_o > 1.5$ use $c_z = \rho V_{LA} A$

For sliding damping parallel to $y$ axis in range of $0 \leq a_o \leq 1.5$

\[ \lambda_y = \frac{c_y}{\rho V_y A} = X_1 + (a_o R) X_2 + (a_o R)^2 X_3 + X_4 \exp(-a_o R) \]

<table>
<thead>
<tr>
<th>$L/B = R$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5720</td>
<td>-0.6140</td>
<td>0.2118</td>
<td>-0.7062</td>
</tr>
<tr>
<td>2</td>
<td>1.0200</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>1.7350</td>
<td>-0.2915</td>
<td>0.0288</td>
<td>-0.4950</td>
</tr>
<tr>
<td>10</td>
<td>1.8040</td>
<td>-0.1273</td>
<td>0.0051</td>
<td>0.7960</td>
</tr>
</tbody>
</table>

For $R > 10$ use $\lambda_y(R) = \lambda_{y10}(1 + 0.0025 R)$ For $a_o > 1.5$ use $c_y = \rho V_y A$

For sliding damping parallel to $x$ axis use the following

- $0 \leq L/B \leq 3$ use $c_x = \lambda_{y1}(1) \rho V_y A$
- $L/B > 3$ use $c_x = \rho V_y A$

For rocking damping use

\[ \lambda_{r1} = \frac{c_i}{\rho V_{LA} I} = a_o X_1 + a_o^2 X_2 + a_o^3 X_3 + a_o^4 X_4 \]

For rocking damping about $x$ axis

<table>
<thead>
<tr>
<th>$L/B = R$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 2</td>
<td>0.0337</td>
<td>1.1477</td>
<td>-1.0369</td>
<td>0.2849</td>
</tr>
<tr>
<td>5</td>
<td>1.0757</td>
<td>-0.4492</td>
<td>-0.1621</td>
<td>0.1550</td>
</tr>
<tr>
<td>$\geq 10$</td>
<td>1.6465</td>
<td>-1.5247</td>
<td>0.8516</td>
<td>-0.2046</td>
</tr>
</tbody>
</table>

\[ \lambda_{rx} = \text{rocking about } x \text{ axis} \]

For rocking damping about $y$ axis

<table>
<thead>
<tr>
<th>$L/B = R$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0337</td>
<td>1.1477</td>
<td>-1.0369</td>
<td>0.2849</td>
</tr>
<tr>
<td>2</td>
<td>0.2383</td>
<td>1.6257</td>
<td>-1.6804</td>
<td>0.4895</td>
</tr>
<tr>
<td>3</td>
<td>0.6768</td>
<td>1.5620</td>
<td>-2.0227</td>
<td>0.6382</td>
</tr>
<tr>
<td>4 and 5</td>
<td>1.4238</td>
<td>0.5046</td>
<td>-1.5762</td>
<td>0.6052</td>
</tr>
</tbody>
</table>

\[ \lambda_{ry} = \text{rocking about } y \text{ axis} (\text{same as } \lambda_{rx}) \]

For $R \geq 100$ ($\approx \infty$) $\lambda_{ry} = 1$

(continued on next page)
TABLE 20-4 (continued)
Damping constants for computing the damping coefficient $c_i$

For torsion damping use

$$\lambda_i = \frac{c_i}{\rho V_p l} = a_0 X_1 + a_1 X_2 + a_2 X_3 + X_4 \tan^{-1} \frac{R}{a_0}$$

<table>
<thead>
<tr>
<th>$L/B = R$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0452</td>
<td>0.5277</td>
<td>-0.1843</td>
<td>0.0214</td>
</tr>
<tr>
<td>2</td>
<td>0.8945</td>
<td>0.2226</td>
<td>-0.0042</td>
<td>-0.0612</td>
</tr>
<tr>
<td>3</td>
<td>1.6330</td>
<td>-0.8238</td>
<td>0.1156</td>
<td>-0.0962</td>
</tr>
<tr>
<td>4</td>
<td>2.6028</td>
<td>-2.0521</td>
<td>0.5312</td>
<td>-0.1070</td>
</tr>
<tr>
<td>$&gt;100$</td>
<td>$\lambda_i = 1.0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

amplitude vibration to a soil specimen. The technique is described in some detail in Cunny and Fry (1973) and in ASTM D 4015.

The value of dynamic shear modulus $G'$ can be estimated using empirical equations presented by Hardin and Black (1968) as

$$G' = \frac{6900(2.17 - e)^2}{1 + e} \sqrt{\sigma_o} \quad \text{(kPa)}$$

(20-11)

for round-grained sands, where the void ratio $e < 0.80$.

For angular-grained materials, with $e > 0.6$, and clays of modest activity the estimate of $G'$ is

$$G' = \frac{3230(2.97 - e^2)}{1 + e} \sqrt{\sigma_o} \quad \text{(kPa)}$$

(20-12)

Hardin and Drnevich (1972) included the overconsolidation ratio (OCR) into Eq. (20-12) to obtain

$$G' = \frac{3230(2.97 - e^2)}{1 + e} \sqrt{\sigma_o} \text{OCR}^M \quad \text{(kPa)}$$

(20-12a)

where, in Eqs. (20-11) through Eq. (20-13),

- $e$ = void ratio in situ or in laboratory test sample
- $\sigma_o$ = mean effective stress $= \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ for laboratory sample
  $= \frac{\sigma_1(1 + 2K_o)}{3}$ in situ, kPa

A more general form of Eq. (20-12a) is the following:

$$G' = C_o \frac{(2.97 - e^2)}{F(e)} \text{OCR}^M \sigma_o^n \quad \text{(kPa)}$$

(20-13)

where terms not previously defined for Eqs. (20-11) and (20-12) are

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_o$</td>
<td>3230</td>
<td>440–1450, but use 770</td>
</tr>
<tr>
<td>$n$</td>
<td>0.5</td>
<td>0.51–0.73, but use 0.65</td>
</tr>
<tr>
<td>$F(e)$</td>
<td>$1 + e^i$</td>
<td>$1 + e$</td>
</tr>
</tbody>
</table>

† Hardin and Blandford (1989) suggest $F(e) = 0.3 + 0.7e^2$. 
TABLE 20-5
Representative values of shear modulus $G'$

<table>
<thead>
<tr>
<th>Material</th>
<th>ksi</th>
<th>MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clean dense quartz sand</td>
<td>1.8–3</td>
<td>12–20</td>
</tr>
<tr>
<td>Micaceous fine sand</td>
<td>2.3</td>
<td>16</td>
</tr>
<tr>
<td>Berlin sand ($\epsilon = 0.53$)</td>
<td>2.5–3.5</td>
<td>17–24</td>
</tr>
<tr>
<td>Loamy sand</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>Dense sand-gravel</td>
<td>10$^+$</td>
<td>70$^+$</td>
</tr>
<tr>
<td>Wet soft silty clay</td>
<td>1.3–2</td>
<td>9–15</td>
</tr>
<tr>
<td>Dry soft silty clay</td>
<td>2.5–3</td>
<td>17–21</td>
</tr>
<tr>
<td>Dry silty clay</td>
<td>4–5</td>
<td>25–35</td>
</tr>
<tr>
<td>Medium clay</td>
<td>2–4</td>
<td>12–30</td>
</tr>
<tr>
<td>Sandy clay</td>
<td>2–4</td>
<td>12–30</td>
</tr>
</tbody>
</table>

Values for the OCR exponent $M$ in Eqs. (20-12a) and (20-13) are related to the plasticity index $I_p$ of the soil as follows:

<table>
<thead>
<tr>
<th>$I_p$, %</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>0.18</td>
<td>0.30</td>
<td>0.41</td>
<td>0.48</td>
<td></td>
</tr>
</tbody>
</table>

Anderson et al. (1978) and others indicate that Eqs. (20-12) and (20-12a) are likely to underpredict $G'$ in situ by a factor from 1.3 to 2.5 since they do not include the stiffening effects from cementation and anisotropy. On the other hand, Kim and Novak (1981) found for several Canadian clays and silts that Eq. (20-12a) overpredicted $G'$ by a factor of about 2. Typical values of $G'$ as found by several researchers are given in Table 20-5 as a guide or for preliminary estimates of vibration amplitudes.

One cannot use static triaxial test values of $E_s$ to compute dynamic values of $G'$, since the strain $\epsilon_d$ for the dynamic $G'$ is on the order of 0.002 to 0.00001 (or less) where triaxial strains $\epsilon_{tr}$ are usually recorded (and plotted) in the range of 0.01$^+$.

20-5.2 In Situ Determination of Dynamic Shear Modulus $G'$

In an elastic, homogeneous soil mass dynamically stressed at a point near the surface, three elastic waves travel outward at different speeds. These are as follows:

- Compression (or $P$) wave
- Shear (or secondary $S$) wave—usually wave of interest
- Surface (or Rayleigh) wave

The velocity of the Rayleigh wave is about 10 percent less than that of the shear wave. For surface measurements it is often used in lieu of the shear wave owing to the complex waveform displayed on the pickup unit from these nearly simultaneous wave arrivals. The wave peaks on the waveform are used to indicate wave arrival so the time of travel from shock source to detection unit can be computed. Compression and shear wave velocities are related to the dynamic elastic constants of the soil according to Theory of Elasticity as follows:

Compression: $V_c = \sqrt{\frac{E_s(1-\mu)}{\rho(1+\pi)(1-2\mu)}} \quad (20-14)$
Shear: \[ V_s = \frac{\sqrt{G'}}{\rho} \] (20-15)

The relationship between shear modulus \( G' \) and stress-strain modulus \( E_s \) is the same as for static values and is given by Eq. (b) of Sec. 2-14, repeated here for convenience with a slight rearrangement:

\[ E_s = 2(1 + \mu)G' \]

Dividing Eq. (20-15) by Eq. (2-14), squaring, substituting, and simplifying, we obtain

\[ \left( \frac{V_s}{V_c} \right)^2 = \frac{1 - 2\mu}{2(1 - \mu)} \] (20-16)

From Eq. (20-16) we see the shear wave ranges from

\[ 0 \leq V_s \leq 0.707V_c \]

depending on Poisson’s ratio \( \mu \). From this it is evident the compression waves will arrive at the detection unit some time before the shear and surface waves arrive.

The shear modulus can be obtained by making field measurements of the shear wave velocity \( V_s \) and by using Eq. (20-15) to find

\[ G' = \rho V_s^2 \]

In addition to the direct measurement of the Rayleigh surface shear wave and using Eq. (20-15) to compute \( G' \), one can obtain the shear wave velocity \( V_s \) in situ using any one of a number of tests such as the up-hole, down-hole, cross-hole, bottom-hole, in-hole, and seismic cone penetration [see Robertson and Addo (1991), which includes a large reference list describing the tests in some detail]. More recently (ca. 1984) a modification of the surface wave method, termed the spectral analysis of surface wave (SASW) method, has been suggested.

The cross-hole and down-hole methods for the in situ shear wave velocity \( V_s \) are described in considerable detail by Woods (1986, with large number of references).

**CROSS-HOLE METHOD.** In the cross-hole method (see Fig. 20-8a) two boreholes a known distance apart are drilled to some depth, preferably on each side of the base location so that the shear wave can be measured between the two holes and across the base zone.

At a depth of about \( B \) a sensor device is located in the side or bottom of one hole and a shock-producing device (or small blast) in the other. A trigger is supplied with the shock so that the time for the induced shear wave can be observed at the pickup unit. The time of travel \( T_h \) of the known distance \( D_h \) between the two holes gives the shear wave velocity \( V_s \) (in units of \( D_h \)) as

\[ V_s = \frac{D_h}{T_h} \]

**DOWN-HOLE METHOD.** The down-hole method is similar to the cross-hole but has the advantage of only requiring one boring, as shown on Fig. 20-8b. In this method, the hole is drilled, and a shock device is located a known distance away. A shock detector is located at some known depth in the hole and a shock applied. As with the cross-hole method, we can measure the time \( T_h \) for arrival of the shear wave and, by computing the diagonal side of the triangle, obtain the travel distance \( D_h \). The detector device is then placed at a greater depth and the test repeated, etc., until a reasonably average value of \( V_s \) is obtained. Hoar and Stokoe...
Figure 20-8 Two recommended in situ methods for obtaining shear modulus $G'$. 

(1978) discuss in some detail precautionary measures to take in making either of these two tests so that the results can justify the test effort.

**SEISMIC CONE METHOD.** This method directly measures the shear wave velocity by incorporating a small velocity seismometer (an electronic pickup device) inside the cone penetrometer. Essentially, the test proceeds by pushing the seismic cone to some depth $z$ and then applying a shock at the ground surface, using a hammer and striking plate or similar. This method seems to have been developed ca. 1986 [see Robertson et al. (1986)]. This device has the advantage of not requiring a borehole. It requires that the site soil be suitable for a CPT (fine-grained, with little to no gravel).
SPECTRAL ANALYSIS OF SURFACE WAVES (SASW) METHOD. This is a modification of the seismic surface method based on the dispersive characteristics of Rayleigh waves in layered media. It involves applying a vibration to the soil surface, measuring the wave speed between two electronic pickup devices a known distance apart, and then interpreting the data. The test procedure and early development is described in substantial detail by Nazarian and Desai (1993); the theory and use by Yuan and Nazarian (1993). An in-depth test program based on the SASW by Lefebvre et al. (1994) found that empirical correlations based on laboratory tests and using void ratio, OCR, and the mean effective stress \( \sigma_o \) as equation parameters may substantially underpredict the dynamic shear modulus \( G' \).

The SASW has particular value in not requiring a borehole or a large amount of field equipment.

SHEAR WAVE–\( G' \) CORRELATIONS. Schmertmann (1978a) suggests that \( V_s \) may be related to the SPT \( N \) value or to the CPT \( q_c \). From a plot of a large number of \( N \) values at a test site in sand, he suggested that \( V_s \approx 15N_{60} \)—for that site. From this it appears that

\[
V_s = 10 \text{ to } 20N_{60} \quad \text{(m/s)}
\]

(20-17)

with the range to account for increasing density, fine or coarse sand and other variables. Use Table 20-5 as an additional guide.

Seed et al. (1986) and later Jamiolkowski et al. (1988) suggested that the shear velocity is approximately

\[
V_s = C_1N_{60}^{0.17}z^{0.2}F_1F_2 \quad \text{(m/s)}
\]

(20-17a)

where

- \( C_1 = \) empirical constant; Seed et al. (1986) suggested 69; Jamiolkowski et al. (1988) suggested 53.5
- \( z = \) depth in soil where blow count \( N_{60} \) is taken, m
- \( F_1 = \) age factor:
  - = 1 for Holocene age (alluvial deposits)
  - = 1.3 for Pleistocene age (diluvial deposits)
- \( F_2 = \) soil factor as follows:

<table>
<thead>
<tr>
<th></th>
<th>Clay</th>
<th>Fine sand</th>
<th>Med sand</th>
<th>Coarse sand</th>
<th>Sand &amp; Gravel</th>
<th>Gravel</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_2 = )</td>
<td>1.0</td>
<td>1.09</td>
<td>1.07</td>
<td>1.11</td>
<td>1.15</td>
<td>1.45</td>
</tr>
</tbody>
</table>

Yoshida et al. (1988) give an equation for \( V_s \) as follows:

\[
V_s = C_1(\gamma z)^{0.14}N_{60}^{0.25} \quad \text{m/s}
\]

(20-17b)

where

- \( \gamma z = \) average overburden pressure in depth \( z \) of interest, kPa
- \( C_1 = \) coefficient depending on soil type as follows:

<table>
<thead>
<tr>
<th>Soil</th>
<th>Fine sand</th>
<th>25% Gravel</th>
<th>50% Gravel</th>
<th>All soils</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 = )</td>
<td>49</td>
<td>56</td>
<td>60</td>
<td>55</td>
</tr>
</tbody>
</table>
The $V_s$ estimate from Eqs. (20-17) are then used together with an estimated (or measured) value of soil density $\rho$ to back-compute $G'$ using Eq. (20-15). With some attention to details the computed $G'$ will probably not be in error more than $\pm 25$ percent—but it can be as much as 100 percent.

Mayne and Rix (1995) suggest a correlation using either a seismic cone or a piezocone with $q_c$ corrected for pore pressure to $q_j$ in the following form:

$$G' = \frac{99.5 \times p_a^{0.31} \times q_j^{0.69}}{e^n} \text{ (kPa)} \quad (20-17c)$$

Here, $p_a = \text{atmospheric pressure, kPa.}$ In most cases using $p_a = 100 \text{ kPa (vs. actual value of about 101.4 kPa)}$ is sufficiently precise; $q_j$ and the in situ void ratio $e$ have been previously defined. The $n$-exponent for $e$ has a value ranging from 1.13 to 1.3. For $p_a = 100 \text{ kPa, } q_j = 180 \text{ kPa, } e^n = (wG_s)^n = 1.08^{1.13}$, Eq. (20-17b) gives $G' = 13683 \text{ kPa (13.7 MPa)}$.

Alternatively, you may convert $q_c$ from the CPT to an equivalent SPT $N$ using Eq. (3-20); adjust this $N$ to $N_{60}$ and use Eqs. (20-17) to obtain $V_s$ then use Eq. (20-15) to compute $G'$.

Poisson’s ratio is more troublesome, however, since a difference between $\mu = 0.3$ and 0.4 can result in about 16 percent error in computing the soil spring.

### 20-5.3 Soil or Material Damping Ratio $D_i$

Soil damping, defined here as the ratio of Eq. (20-3), i.e., actual damping, $c_i$/critical soil damping, $c_{ci}$,

$$D_i = \frac{c_i}{c_{ci}} \times 100 \text{ (%)}$$

is usually estimated in the range of 0 to about 0.10 (0 to 10 percent). This damping ratio range has been suggested by Whitman and Richart (1967), who compiled values from a number of sources available at that time including Barkan (1962).

The recent work of Stewart and Campanella (1993) reasonably validates the earlier range of values. However, they also suggested that although the damping ratio is frequency- (as well as material-) dependent, it can be estimated in about the following range:

<table>
<thead>
<tr>
<th>Soil Type</th>
<th>S &amp; C (1993)</th>
<th>Summary of Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clay</td>
<td>1.0 to 5</td>
<td>1.7 to 7</td>
</tr>
<tr>
<td>Silt</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>Alluvium</td>
<td>3.5 to 12</td>
<td></td>
</tr>
<tr>
<td>Sand</td>
<td>0.5 to 2</td>
<td>1.7 to 6</td>
</tr>
</tbody>
</table>

The use of $D_i$, % is consistent with the S & C (1993) reference, but for use in such as Eq. (20-4), $D_i$ is a decimal, i.e., $1.0/100 = 0.01, 5/100 = 0.05$, etc. The values in the table above are suggested for the vertical-mode damping ratio $D_2$. Values will seldom be the same for sliding ($D_x$ or $D_y$) and rotational ($D_\theta$) modes.

### 20-6 UNBALANCED MACHINE FORCES

The unbalanced forces from the machinery, engines, or motors and their location with respect to some reference point from the machine base are required. The manufacturer must supply
this information for machinery and motors. The project engineer would have to obtain it in some manner if the vibrations are from wind gusts and such.

To illustrate the concept of an engine producing both primary and secondary forces we will briefly examine the single-cylinder engine idealized in Fig. 20-9. We define \( z_p \) = downward displacement of the piston from zero (when \( \omega t = 0 \)) at top dead center; maximum \( z_p \) occurs at \( \omega t = \pi \) rad counterclockwise. At any time \( t \) we have for \( \omega t \) as shown

\[
z_p = r(1 - \cos \omega t) + L(1 - \cos \alpha)
\]

but \( \alpha = f(\omega t) \) since \( y_c \) is common to both \( r \) and \( L \) so that

\[
\sin \alpha = \frac{r}{L} \sin \omega t
\]

Using a number of trigonometric relationships [see Den Hartog (1952)], we finally obtain

\[
\begin{align*}
z_p &= \left( r + \frac{r^2}{4L} \right) - r \left( \cos \omega t + \frac{r}{4L} \cos 2\omega t \right) \\
\dot{z}_p &= r\omega \left( \sin \omega t + \frac{r}{2L} \sin 2\omega t \right) \\
\ddot{z}_p &= r\omega^2 \left( \cos \omega t + \frac{r}{L} \cos 2\omega t \right)
\end{align*}
\]

Figure 20-9 Moving parts of a single-cylinder engine producing unbalanced frequency-dependent forces.
A similar exercise can be done for the crank to obtain

\[
\begin{align*}
  y_c &= -r \sin \omega t \\
  \dot{y}_c &= -r \omega \cos \omega t \\
  \ddot{y}_c &= r \omega^2 \sin \omega t \\
  z_c &= r(1 - \cos \omega t) \\
  \dot{z}_c &= r \omega \sin \omega t \\
  \ddot{z}_c &= r \omega^2 \cos \omega t
\end{align*}
\]

(20.19)

Designating the mass of the piston plus a part of the connecting rod as the vertical reciprocating mass \( m_{\text{rec}} \) concentrated at point \( C \) and that of the crank plus the remainder of the connecting rod as the rotating mass \( m_{\text{rot}} \) concentrated at \( D \), we obtain the unbalanced forces as

Vertical:

\[
F_z = m_{\text{rec}} \ddot{z}_p + m_{\text{rot}} \ddot{y}_c
\]

\[
F_z = (m_{\text{rec}} + m_{\text{rot}})r \omega^2 \cos \omega t + m_{\text{rec}} \frac{r^2 \omega^2}{L^2} \cos 2\omega t
\]

(20-20)

Horizontal:

\[
F_y = m_{\text{rot}} \ddot{y}_c = m_{\text{rot}} r \omega^2 \sin \omega t
\]

(20-21)

From these forces we have in Eq. (20-20) two parts:

A primary force:

\[
= (m_{\text{rec}} + m_{\text{rot}})r \omega^2 \cos \omega t
\]

A secondary force:

\[
= m_{\text{rec}} \frac{r^2 \omega^2}{L^2} \cos 2\omega t
\]

These are vertical primary and secondary forces and are a maximum at \( \omega t = 2\omega t = 0 \) and multiples of \( \pi \) so that the cosine term = 1 with the same sign. Note that these forces are frequency-dependent, so the forces are larger at, say, 3000 r/min (rpm) than at 2000 rpm.

Equation (20-21) gives a horizontal primary force; there is no secondary force because there is only one term. This force is a maximum at \( \omega t = \pi/4, 5\pi/4, \text{ etc.} \), and will be at some distance \( \ddot{y} \) above the center of the base-ground interface and will therefore produce a rocking moment about the \( x \) axis (which is perpendicular to the plane of the paper and passes through point \( O \) of Fig. 20-9). In this case the horizontal force produces both a sliding mode and a rocking mode. As we will see in the next section these two modes are generally interdependent or coupled.

Most motors have more than one cylinder, and manufacturers attempt to keep the unbalanced forces small (use small \( r \) and masses; have one crank rotate counterclockwise while another is rotating clockwise, etc.). Although it is possible to minimize the unbalanced forces and resulting rocking moments they are never completely eliminated.

Computational procedures can be used to obtain the unbalanced forces but as this simple example illustrates the work is formidable (for example, how does one allocate \( L \) between \( m_{\text{rot}} \) and \( m_{\text{rec}} \)?). For this reason equipment manufacturers use electronic data acquisition equipment such as displacement transducers and accelerometers located at strategic points on the machinery to measure displacements and accelerations at those points for several operational frequencies (or rpm’s). These data can be used to back-compute the forces since the total machine mass can be readily obtained by weighing. Using these methods, we can directly obtain the unbalanced forces without using the mass of the several component parts.

These data should be requested from the manufacturer in order to design the equipment base for any vibration control. We should also note that in case the base does not function as intended (vibrations too large or machinery becomes damaged) it is usual to put displacement transducers and accelerometers on the installation to ascertain whether the foundation was improperly designed or whether the manufacturer furnished incorrect machinery data.
20-7 DYNAMIC BASE EXAMPLE

Now that we have identified the soil properties and machine forces and other data needed to solve Eq. (20-4a) or (20-4b) we can use this information for the following example.

Example 20-1. Use the computer program FADDYNFL on your computer diskette and data set EXAM201.DTA and obtain the six displacements for the base as shown in Fig. E20-1a. Note that rocking modes will be about the center of area both in plan and elevation ($B, L, T_b/2$). The following data are given:

Soil: $G' = 239 400$ kPa

Damping factor: $D_i = 0.05$ (estimated—see Sec 20-5.3; same for all modes)

$\gamma_s = 19.65$ kN/m$^3$

$\mu = 0.333$ (estimated)

Machine: rpm = 900 (operating speed)

For purposes of illustration we will only use the primary forces.

- $F_{ox} = 45$ kN = $F_{oy}$ (horizontal for sliding)
- $F_{oz} = 90$ kN (vertical)
- $M_{ox} = 20.3$ kN m (about $x$ axis)
- $M_{oy} = 27.1$ kN m (about $y$ axis)
- $M_{oz} = 33.9$ kN m (about $z$ axis)

Solution. Compute the remaining parameters. Note that we will only make a solution for the operating speed of 900 rpm for cases 2 through 7 (Example 20-1a–f on your diskette file EXAM201.DTA). The special case of Example 20-1a is provided to illustrate the case of frequency-dependent forces. In this case we have a frequency-dependent vertical force $F_o = 90$ kN at 900 rpm. In these examples we are not inputting any secondary forces or secondary moments—although they usually exist.
EXAMPLE 20-1A- A VERTICAL MODE  FZ = 90 KN

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = VERT
FORCE TYPE (FO = 1; M*E = 2) = 1  IMET (SI > 0) = 1

BASE DATA:
DIMENSIONS:  B = 3.660  L = 7.320 M
INERTIA:  IX = 29.9  IY = 119.6  JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA  XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY, RHO = 2.00000  KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = .000  KN-SEC**2/M
MASS OF BLOCK + MACHINE = 19.99000  KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = 90.000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = .000 .000 KN-M

\[
\text{STATIC SPRING} = 43049E+07 \text{ KN-M} \\
\text{NATURAL FREQ} \ WN = 464.064 \text{ RAD/SEC} \\
\text{CRITICAL DAMPING} \ CC = 18553E+05 \text{ KN-SEC/M} \\
\text{MASS USED} = 19.990 \text{ KN-SEC**2/M} \\
\text{DR (C/CC)} = 2.0 \text{SQRT(SPRING*MASS)/CC} \\
\]

RPM  SPRING  DAMPING  FREQUENCY  W/WN  C/CC  YP/YS  YP,MM
900.  .39960E+07  34044.  94.2  .203  .963  .966  .20189E-01

EXAMPLE 20-1A VERTICAL MODE  FZ = 90 KN AT RPM = 900

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = VERT
FORCE TYPE (FO = 1; M*E = 2) = 2  IMET (SI > 0) = 1

BASE DATA:
DIMENSIONS:  B = 3.660  L = 7.320 M
INERTIA:  IX = 29.9  IY = 119.6  JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA  XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY, RHO = 2.00000  KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 800. 1000. 100.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = .000  KN-SEC**2/M
MASS OF BLOCK + MACHINE = 19.99000  KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = 90.000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = .000 .000 KN-M

OPERATING MACHINE SPEED, RPM0 = 900. RPM

\[
\text{STATIC SPRING} = 43049E+07 \text{ KN-M} \\
\text{NATURAL FREQ} \ WN = 464.064 \text{ RAD/SEC} \\
\text{CRITICAL DAMPING} \ CC = 18553E+05 \text{ KN-SEC/M} \\
\text{MASS USED} = 19.990 \text{ KN-SEC**2/M} \\
\text{DR (C/CC)} = 2.0 \text{SQRT(SPRING*MASS)/CC} \\
\]

RPM  SPRING  DAMPING  FREQUENCY  W/WN  C/CC  YP/YS  YP,MM
800.  .40415E+07  34606.  83.8  .181  .969  .972  .16057E-01
900.  .39960E+07  34044.  94.2  .203  .963  .966  .20189E-01
1000. .39482E+07  33591. 104.7 .226 .958 .959 .24749E-01

Figure E20-1a
in most rotating equipment. If you have any, you would input them when asked by the program when the data file is first being built.

When $F_0$ is of the form $F = F_0 \sin \omega t$, it is only necessary to look at the maximum $F_0$ that occurs at $\omega t = \pi/2$ and the program option control parameter IFORC = 1 (it is 2 for Example 20-1a).

In most real cases we would probably set the range of rpm from about 100 or 200 up to about 1200, which includes the operating speed. This is usually necessary since no machine suddenly starts spinning at 900 rpm with a constant force $F_0$; it is possible an rpm rate between 0 and 900 produces a larger displacement than one at 900 rpm (i.e., resonance with damping occurs). Note how the data were set up for Example 20-1a. One can write

$$ F_o = 2m \bar{y} \omega^2 $$

but this expression is equivalent to the following:

$$ F_{o,i} = F_{o,\text{max}} \left( \frac{\text{rpm}_i}{\text{rpm}_o} \right)^2 $$

and the operating rpm are required input for this case (as is shown on the output sheet).

**Foundation parameters.**

Ratio $L/B = 2L/2B = 7.32/3.66 = 2.00$

(falls on all curves and equations for easy reader checking without interpolation)

$$ B/L = 1.83/3.66 = 0.5 $$

Base area $A = 2B \times 2L = 3.66 \times 7.32 = 26.8 \text{ m}^2$

$$ J_a = \frac{A}{4L^2} = \frac{26.8}{4 \times 3.66^2} = 0.5 \quad \text{(as on Fig. 20-6)} $$

$$ I_x = \frac{bh^3}{12} = \frac{7.32 \times 3.66^3}{12} = 29.91 \text{ m}^4 $$

$$ I_y = \frac{hh^3}{12} = \frac{3.66 \times 7.32^3}{12} = 119.6 \text{ m}^4 $$

$$ J = I_x + I_y = 29.91 + 119.6 = 149.51 \text{ m}^4 $$

We will use a concrete base ($\gamma_c = 23.6 \text{ kN/m}^3$) with a thickness $T_b = 0.31 \text{ m}$. Then

Base mass $m_b = V_b \gamma_c / g = 3.66 \times 7.32 \times 0.31 \times 23.6/9.807$

$$ = 19.99 \text{ kN} \cdot \text{s}^2/\text{m} $$

**Moments.** We must compute the rotational mass moments of inertia using equations from Table 20-1. About the $x$ axis use dimensions perpendicular to axis; since these are through the geometric (and in this case the mass) center, all rocking will be with respect to the center of mass:

$$ I_{\theta_x} = \frac{m}{12} (a^2 + B^2) = \frac{19.99}{12} (0.31^2 + 3.66^2) = 22.47 \text{ kN} \cdot \text{m}^3 \cdot \text{s}^2 $$

$$ I_{\theta_y} = \frac{m}{12} (a^2 + L^2) = \frac{19.99}{12} (0.31^2 + 7.32^2) = 89.42 \text{ kN} \cdot \text{m}^3 \cdot \text{s}^2 $$

$$ I_{\theta_z} = \frac{m}{12} (B^2 + L^2) = \frac{19.99}{12} (3.66^2 + 7.32^2) = 111.57 \text{ kN} \cdot \text{m}^3 \cdot \text{s}^2 $$

We will use $m$ and the just-computed $I_{\theta_i}$ values in the mass term of Eq. (20-4a).
EXAMPLE 20-1B SLIDING MODE—PARALLEL TO X-AXIS IDIRS = 1

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = SLID
FORCE TYPE (FO = 1; M*E = 2) = 1
IMET (SI > 0) = 1

SLIDING PARALLEL TO LENGTH DIMENSION

BASE DATA:
DIMENSIONS: B = 3.660 L = 7.320 M
INERTIA: IX = 29.9 IY = 119.6 JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY ,RHO = 2.00000 KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = .000 KN-SEC**2-M
MASS OF BLOCK + MACHINE = 19.99000 KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = 45.000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = .000 .000 KN-M

STATIC SPRING = .36291E+07 KN-M
NATURAL FREQ WN = 426.079 RAD/SEC
CRITICAL DAMPING CC = .17035E+05 KN-SEC/M
MASS USED = 19.990 KN-SEC**2/M
DR (C/CC) = 2.*SQRT(SPRING*MASS)/CC

900. .33323E+07 20116. 94.2 .221 .958 .960 .11908E-01

Figure E20-1b

Soil parameters.

\[ \rho = \frac{\gamma_s}{g} = \frac{19.65}{9.807} = 2.00 \text{ kN} \cdot \text{s}^2/\text{m}^4 \]

The shear velocity [using Eq. (20-15)] is

\[ V_s = \sqrt{\frac{G'}{\rho}} = \sqrt{\frac{239400}{2.0}} = 346 \text{ m/sec} \]

Computations. The preceding computed items are required for input to create the data file EXAM201.DTA on your diskette, which is executed to produce Fig. E20-1b for the six different d.o.f.

The program computes the following frequency parameters but they are also computed here so you can see how the computations are made:

\[ f = \frac{\text{rpm}}{60} = \frac{900}{60} = 15 \text{ Hz} \]

\[ \omega = 2\pi f = 2\pi \times 15 = 94.3 \text{ rad/s} \]

\[ a_o = \frac{\omega B}{V_s} = \frac{4.3 \times 1.83}{346} = 0.4985 \] (Note: \( B = B/2 = 3.66/2 = 1.83 \text{ m} \))

Let us check selected values shown on the output sheet for EXAMPLE 20-1A.
EXAMPLE 20-1C  SLIDING MODE--PARALLEL TO Y-AXIS  IDIRS = 2

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = SLID
FORCE TYPE (FO = 1; M*E = 2) = 1  IMET (SI > 0) = 1

SLIDING PARALLEL TO WIDTH DIMENSION

BASE DATA:
DIMENSIONS:  B = 3.660  L = 7.320 M
INERTIA:  IX = 29.9  IY = 119.6  JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA  XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY ,RHO = 2.00000 KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = 19.99000 KN-SEC**2/M
MASS OF BLOCK + MACHINE = 19.99000 KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = .000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = .000 .000 KN-M

STATIC SPRING = .36291E+07 KN-M
NATURAL FREQ WN = 426.079 RAD/SEC
CRITICAL DAMPING CC = .17035E+05 KN-SEC/M
MASS USED = 19.990 KN-SEC**2/M
DR (C/CC) = 2.*SQRT(SPRING*MASS)/CC

RPM  SPRING  DAMPING  FREQ W  W/WN  C/CC  YP/YS  YP,MM
900. .35999E+07  22831.  94.2  .221  .996  .954  .11830E-01

EXAMPLE 20-1D  ROCKING MODE--ABOUT Y-AXIS  IDIRR = 1

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = ROCK
FORCE TYPE (FO = 1; M*E = 2) = 1  IMET (SI > 0) = 1

ROCKING RESISTED BY LENGTH DIMENSION

BASE DATA:
DIMENSIONS:  B = 3.660  L = 7.320 M
INERTIA:  IX = 29.9  IY = 119.6  JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA  XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY ,RHO = 2.00000 KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = 89.420 KN-SEC**2/M
MASS OF BLOCK + MACHINE = 19.99000 KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = .000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = .000 .000 KN-M

STATIC SPRING = .41352E+08 KN-M/RAD
NATURAL FREQ WN = 680.035 RAD/SEC
CRITICAL DAMPING CC = .12162E+06 KN-SEC/M
MASS USED = 89.420 KN-SEC**2/M
DR (C/CC) = 2.*SQRT(SPRING*MASS)/CC

RPM  SPRING  DAMPING  FREQ W  W/WN  C/CC  YP/YS  YP,RADS
900. .33709E+08  82096.  94.2  .139  .903  .988  .64744E-06
EXAMPLE 20-1E ROCKING MODE—ABOUT X-AXIS IDIRR = 2

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = ROCK
FORCE TYPE (FO = 1; M*E = 2) = 1

ROCKING RESISTED BY WIDTH DIMENSION

BASE DATA:
DIMENSIONS: B = 3.660 L = 7.320 M
INERTIA: IX = 29.9 IY = 119.6 JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY ,RHO = 2.00000 KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = 22.470 KN-SEC**2-M
MASS OF BLOCK + MACHINE = 19.99000 KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = .000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = 20.300 .000 KN-M

STATIC SPRING = 0.10341E+08 KN-M/RAD
NATURAL FREQ WN = 678.378 RAD/SEC
CRITICAL DAMPING CC = .30486E+05 KN-SEC/M

MASS USED = 22.470 KN-SEC**2/M
DR (C/CC) = 2. *SQRT(SPRING*MASS)/CC

RPM SPRING DAMPING FREQ W W/WN C/CC YP/YS YP, RADS
900. .95866E+07 16556. 94.2 .139 .963 .984 .19312E-05

EXAMPLE 20-1F TORSION MODE MODE—ABOUT Z-AXIS

DISK DATA FILE USED FOR THIS EXECUTION: EXAM201.DTA

VIBRATION MODE = TORS
FORCE TYPE (FO = 1; M*E = 2) = 1

BASE DATA:
DIMENSIONS: B = 3.660 L = 7.320 M
INERTIA: IX = 29.9 IY = 119.6 JT = 149.5 M**4
ACTUAL BASE AREA = 26.80 SQ M

SOIL DATA:
GS = 239400.0 KPA XMU = .330
SHEAR WAVE VELOCITY, VS = 346.0 M/S
SOIL DENSITY ,RHO = 2.00000 KN-SEC**2/M**4
SOIL MATERIAL DAMPING, BETA = .05

STARTING, ENDING AND RPM INCREMENT = 900. 900. 0.

ROTATIONAL MASS MOMENT OF INERTIA OF BASE, IPSI = 111.570 KN-SEC**2-M
MASS OF BLOCK + MACHINE = 19.99000 KN-SEC**2/M

INPUT PRIMARY AND SECONDARY FORCES, FORCP, FORCS = .000 .000 KN
INPUT PRIMARY AND SECONDARY MOMENTS, MOPR, MOSEC = 33.900 .000 KN-M

STATIC SPRING = 0.39003E+08 KN-M/RAD
NATURAL FREQ WN = 591.259 RAD/SEC
CRITICAL DAMPING CC = .13193E+06 KN-SEC/M

MASS USED = 111.570 KN-SEC**2/M
DR (C/CC) = 2. *SQRT(SPRING*MASS)/CC

RPM SPRING DAMPING FREQ W W/WN C/CC YP/YS YP, RADS
900. .35276E+08 69545. 94.2 .159 .951 .980 .85156E-06
Using Table 20-2, we obtain

\[ K_z = s_z \frac{2LG'}{1 - \mu} \]

and from Table 20-3

\[ S_z = 0.73 + 1.54 J_0^{0.75} \]

Solving, we find \( S_z = 0.73 + 1.54 \times 0.5^{0.75} = 1.646 \). We were given that also \( \mu = 0.333 \); \( G' = 0.23964E+6 \); and \( 2L = 7.32 \) m. By substitution,

\[ K_z = 1.646 \times \frac{7.32 \times 239400}{1 - 0.333} = 0.432445E+7 \text{ (output = 0.43049E+7)} \]

Large numbers will produce minor internal computer rounding errors, and with so many values having been estimated double precision is too much computational accuracy.

From Eq. (20-1) we compute

\[ \omega_n = \sqrt{\frac{K_z}{m}} = \sqrt{\frac{0.43245E+7}{19.99}} = 465 \text{ (output = 464.06)} \text{ rad/s} \]

We can also compute

\[ c_c = 2 \sqrt{K_z m} = 2 \sqrt{0.43245E+7 \times 19.99} = 0.18595E+5 \text{ (output = 0.18553E+5)} \]

From Eq. (20-2) we obtain the damping ratio \( D_r = c/c_c (\text{= computer variable DR}) \) using the computer-generated \( K_z \) for 900 rpm, giving

\[ DR = \frac{c}{c_c} = \frac{2 \times 0.39960E+7 \times 19.99}{0.18553E+5} = 0.963 \]

and

\[ \frac{\omega_d}{\omega_n} = \frac{94.2}{464.06} = 0.203 \]

\[ Y_s = \frac{P_v}{K_{z,s}} = \frac{90.0}{0.43049E+7} = 0.20906E-4 \text{ m} \]

\[ = 0.020906 \text{ mm} \]

Dynamic displacement \( Y_p \) must be computed using Eq. (20-4b); dynamic springs, using \( K_z = \eta_i K_i \); and damping constants, using \( \bar{c}_i = \lambda_i c_i \) [see Eqs. (20-9) and (20-10)]. These dynamic values are then used in Eq. (20-4b) to find \( z_i/(F_i/K_z) = V = Y_p/Y_s \) and \( Y_p = VY_s \), with both values shown on the output sheet.

If you check Fig. E20-1b for the output labeled Example 20-1A you will find that the several “constants” and the line of data for 900 rpm are exactly the same as in Example 20-1A-A even though the program computed first for 800 rpm. This small check demonstrates that the program is working correctly. Also since 1000 rpm gives a larger vertical displacement, it would appear the resonance frequency is above 900 rpm.

// //

20-8 COUPLED VIBRATIONS

Figure 20-10a is a machine on a base with the center of mass (or center of gravity = cg) as shown. At the crankshaft a distance of \( z_o \) above the base we have a horizontal force \( F_y = \)
The equivalent of \( a \) and using superposition of effects.

Figure 20-10 Coupled sliding and rocking.

\( F_o \cos \omega t \). It is evident that this force produces both translation and rocking about the \( x \) axis through the c.g.m. From use of the transfer formula at the cg we have

\[
F_y = F_o \cos \omega t \\
M = (z_o - h_o)F_y
\]

The base is usually used as a reference (or the top of the base or the top of the pedestal) to locate the center of mass of the component parts and locate the unbalanced machine forces.

With the cg located and the forces \( F_y \) and \( M \) acting, we can replace the system with the block base of Fig. 20-10b. From the figure geometry we can write by inspection the base movement (needed since this is what develops base-to-soil resistance) \( y_b \) from the translation at the cg of \( y_g \) as

\[
y_b = y_g - h_o \theta = \text{net base translation} \\
y_b = \dot{y}_b - h_o \dot{\theta}
\]  

(a)

The net base translation \( y_b \) produces a sliding resistance \( P_y \) as

\[
P_y = c_y \dot{y}_b + K_y y_b
\]  

(b)

The base rotational resistance (and using single subscripts to simplify the equations but noting \( \theta \) should be interpreted as \( \theta_y \), for example, \( I_\theta \) is actually \( I_{\theta_y} \)) is as for uncoupled cases

\[
R_\theta = c_\theta \dot{\theta} + K_\theta \theta
\]  

(c)
Summing horizontal forces through the cg, we have

\[ m\ddot{y}_g + P_y = F_y \]  

(d)

Summing moments about the cg and noting the base resistance \( P_y \) produces an additional moment to consider, we obtain

\[ I_\theta \theta + R_\theta - h_o P_y = M \]  

(e)

Now substituting for \( P_y, R_\theta, y_b, \) and \( \dot{y}_b \), we obtain

\[ m\ddot{y}_g + c_y \dot{y}_g + K_y y_g - h_o(K_y \theta + c_y \dot{\theta}) = F_o \cos \omega t \]  

(20-22)

\[ I_\theta \theta + (c_\theta + h_o^2 c_y)\dot{\theta} + (K_\theta + h_o^2 K_y) \theta - h_o(c_y \dot{y}_g + K_y y_g) = M \]  

(20-23)

Coupling can now be identified from all terms containing \( h_o \). If \( h_o = 0 \) these two equations would reduce to the basic (or uncoupled) form of Eq. (f) of Sec. 20-2. Also note that a vertical force anywhere on the base is not coupled to rocking even if it produces a moment from eccentricity with respect to the cg. If the presence of \( h_o \) produces coupling, the effects evidently reduce with \( h_o \) and was the reason for analyzing Example 20-1 using a 0.31 m thick base giving \( h_o \approx 0.155 \) m.

Also note that the moment and \( F_y \) are always in phase (act so results are cumulative). In this case the center of rotation would always be below the cg. If they are out of phase the center of rotation will lie somewhere along a vertical line through the cg and may be above this point depending on the relative magnitude of the translation and moment forces.

With rocking taking place about the cg you will have to use the transfer formula (shown on Fig. 20-1) when computing \( I_{\theta i} \) as necessary. For example, \( I_{\theta y} \) and \( I_{\theta x} \) of Fig. 20-10b will require use of the transfer formula component \( m d^2 \), and it is possible that \( I_{\theta z} \) will require its use as well. The axis subscript is about that axis, not parallel with it.

With reference to Eqs. (20-22) and (20-23) it is convenient to write the displacements in complex form \( (e = 2.71828 \ldots) \) as follows:

<table>
<thead>
<tr>
<th>Force</th>
<th>Rotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_g = (X_1 + iX_2)e^{i\omega t} )</td>
<td>( \theta = (X_3 + iX_4)e^{i\omega t} )</td>
</tr>
<tr>
<td>( \dot{y}_g = \omega(iX_1 - X_2)e^{i\omega t} )</td>
<td>( \dot{\theta} = \omega(iX_3 - X_4)e^{i\omega t} )</td>
</tr>
<tr>
<td>( \ddot{y}_g = -\omega^2(X_1 + iX_2)e^{i\omega t} )</td>
<td>( \ddot{\theta} = -\omega^2(X_3 + iX_4)e^{i\omega t} )</td>
</tr>
</tbody>
</table>

The process of substituting these displacement functions into Eqs. (20-22) and (20-23), simplifying, and collecting real and imaginary terms for Eq. (20-22) gives two equations in the four values of \( X_i \). Similarly, the real and imaginary terms of Eq. (20-23) give two equations in the four values of \( X_i \). These equations are given as follows:

\[
\begin{align*}
(K_y - m \omega^2)X_1 - c_y \omega X_2 - h_o K_y X_3 + h_o c_y \omega X_4 &= F_y \\
c_y \omega X_1 + (K_y - m \omega^2)X_2 - h_o c_y \omega X_3 - h_o K_y X_4 &= 0 \\
-h_o K_y X_1 + h_o c_y \omega X_2 + (h_o^2 K_y + K_\theta - I_\theta \omega^2)X_3 &= 0 \\
h_o c_y \omega X_1 - h_o K_y X_2 + (h_o^2 c_y \omega + c_\theta \omega)X_3 &= M \\
-h_o c_y \omega X_1 - h_o K_y X_2 + (h_o^2 c_y \omega + c_\theta \omega)X_3 + (h_o^2 K_y + K_\theta - I_\theta \omega^2)X_4 &= 0
\end{align*}
\]

(20-24)
These equations can be programmed to give the unknowns $X_i$, which are then used to obtain the displacements (noting the complex definition of displacements) as

$$y_g = \sqrt{X_1^2 + X_2^2} \quad \theta = \sqrt{X_3^2 + X_4^2}$$  \hspace{1cm} (20-25)

Note $X_1, X_2 =$ translations of m, ft, etc. and $X_3, X_4 =$ rotations in radians. Since the springs and damping constants are frequency-dependent it is usually necessary to cycle the problem using the range of values of $\omega$ from 0 to somewhat above the operating frequency $\omega_o$ (or rpm) for the “worst” case.

Secondary forces that are out of phase will require a second computer analysis with the rotations and displacements summed with the primary values and giving careful attention to signs. In-phase secondary forces (or select in-phase values) can simply be added to the primary forces for direct analysis.

A Computer Program

The computer program used in Example 20-1 was modified (see B-29) with some effort for allowing a coupling analysis.

20-9 EMBEDMENT EFFECTS ON DYNAMIC BASE RESPONSE

The previous methods of analysis considered the dynamic base on the ground surface. Most bases supporting machinery will be embedded some depth into the ground so as to be founded on more competent soil below the zone of seasonal volume change.

It is generally accepted from both a theoretical analysis and field measurements that placing the base into the ground affects the system response to excitation forces. It appears that embedment tends to increase the resonant frequency and may decrease the amplitude.

Several methods to account for vertical vibration exist, including those of Novak and Beredugo (1972), Dobry and Gazetas (1985), and as attributed to Whitman by Arya et al. (1979). Those of Novak and Beredugo and in Arya et al. are for round bases and will not be used here since rectangular base response is substantially different.

The Arya et al. (1979) reference is the only one the author located purporting to allow for rocking and sliding as well as vertical excitation. It is suggested, however, that rocking and sliding spring adjustments for depth should be used cautiously—if at all—for these reasons:

1. Rocking of the base into the side soil may produce a gap over time.
2. Sliding of the base into the side soil may produce gaps over time.
3. The space around the base would have to be carefully backfilled and compacted to provide any appreciable side resistance unless the excavation was excavated and the base poured without using concrete forms.
4. It is not uncommon, where wooden concrete forms are used, to leave them in place.
5. A slight adjustment for depth is automatically accounted for since the effective normal stress at a depth is larger [see Eqs. (20-12) through (20-13)] so that $G'$ is larger. This in turn increases the computed soil springs.

The method given by Dobry and Gazetas (1985) is suggested, however, for the vertical vibration mode spring, as it is both rational and applicable to rectangular- (and other-) shaped bases. Referring to Fig. 20-11, we may define the vertical dynamic spring as the product of

$$K'_z = K_z \times \kappa_{tre} \times \kappa_{wall}$$  \hspace{1cm} (20-26)
where \( K_z \) = static spring computed using the formula given in Table 20-2
\[ \kappa_{\text{tre}} = \text{factor} > 1 \text{ from a curve-fitting scheme from the base being at the bottom of a trench (the excavation), given as} \]
\[ \kappa_{\text{tre}} = 1 + \frac{D}{21B} \left( 1 + \frac{4}{3} J_a \right) \]  
\( \kappa_{\text{wall}} = \text{factor} > 1 \text{ from contact of base sides against soil—either backfill or original ground given as} \]
\[ \kappa_{\text{wall}} = 1 + 0.19 \left( \frac{A_s}{A} \right)^{0.67} \]

with \( A_s = \text{area of sides of base in contact with side soil and gives for a rectangular base of } 2B \times 2L \times T_b \)
\[ A_s = 2T_b(2B + 2L) \quad A = 2B \times 2L \]

This is the more theoretical form given by the reference and is recommended as it allows adjustment in the side contact area \( A_s \). For example, we might compute \( A_s \) and decide, based on a site study, to use only 0.25, 0.50, or some other fraction rather than the full value. For damping it is suggested to use the computed damping + addition of side damping as
\[ c'_z = c_z + \rho V_s A_s \]
Example 20-2. Assume the base of Example 20-1 is 1 m in the ground. What are the revised values of the static spring and damping coefficient $c_z'$ [which are then used to compute the dynamic value(s)]?

Solution. From the computer printout obtain the surface static spring as $K_z = .43049E+7$. Also $2B = 3.66$ m, $2L = 7.32$ m giving $J_a = 0.5$ (computed in example) and $A = 26.8$ m$^2$. $A_s = 2 \times 0.31 \times (3.66 + 7.32) = 6.8$ m$^2$. We will assume $A_s$ is 50 percent effective so use $A_s = 3.4$ m$^2$.

Substituting into Eq. (20-27), we obtain for $D = 1$ m and $B = 3.66$ m

$$\kappa_{tre} = 1 + \frac{1.0}{21(3.66)} \left(1 + \frac{4}{3}(0.5)\right) = 1.02$$

Substituting into Eq. (20-28) gives

$$\kappa_{wall} = 1 + 0.19 \left(\frac{3.4}{26.8}\right)^{0.67} = 1.05$$

from which the static spring adjusted for embedment is

$$K_z' = K_z\kappa_{tre}\kappa_{wall} = .43049E + 7(1.02)(1.05) = .46105E+7 \text{ kN/m}$$

The damping constant, being directly additive, gives

$$c_z' = c_z + \rho V_s A_s \quad \text{(where $A_s = 3.4$ m$^2$ as previously used)}$$

$$= .18553E+5 + 2 \times 346 \times 3.4 = .20906E+5$$

From Fig. 20-11 we note that the base sliding or rocking against the side soil could be similarly accommodated for damping as for the vertical mode. For spring adjustments we could do the following:

For sliding. This is equivalent to vertical vibration rotated 90°, so we might compute a horizontal spring using the vertical spring equations and place it (or a fraction) in parallel with the horizontal spring.

For rocking. This is equivalent to base rocking, so compute the side equivalent rocking value ($2B = base$ thickness rocking against side soil) and put this spring in parallel with the base rocking spring.

Springs in parallel are directly additive as

$$K_{tot} = K_1 + K_2 \quad (20-30)$$

Springs in series are

$$\frac{1}{K_{tot}} = \frac{1}{K_1} + \frac{1}{K_2} \quad (20-30a)$$

20-10 GENERAL CONSIDERATIONS IN DESIGNING DYNAMIC BASES

Experience has provided some guidelines for the analysis of foundation blocks to control vibrations. Other guides may be obtained from carefully analyzing Eq. (20-4a) or by making a series of parametric studies using the provided computer program. Some particular
considerations are as follows:

1. If a dynamic analysis predicts a resonance condition at the operating frequency $f_o$ you must increase or decrease the mass or alter the spring constant. Even if the resonance amplitude is acceptable we do not want to have $f_o = f_r$. It is usually suggested to keep $f_o$ at least $\pm 20$ percent from $f_r$.

2. Try to adjust the base so the center of gravity of equipment and block are coincident. Doing this provides reasonably uniform soil pressure and static settlement. Proportion the base dimension for about half the allowable static soil bearing-capacity pressure. The static + dynamic pressure should not be much over 75 percent of the allowable static pressure.

4. Use as wide a base as possible to resist rocking. Try to use a width that is greater than or equal to $z_o$ to $1.15z_o$ of Fig. 20-10a. Rocking about the narrow dimension will very likely produce vibration amplitudes that are too large. Also the edge pressures may be excessive, so the base eventually tilts.

5. Use a base thickness of at least 0.6 m to produce a "rigid" foundation, in line with the general theory used to develop Eq. (20-4a).

6. Use a machinery/block ($W_n/W_b$) mass ratio of 2 to 3 for centrifugal machinery and 3 for reciprocating equipment.

7. Try to provide a 300-mm clearance all around the machinery frame for any maintenance or other requirements.

It is seldom necessary to use high-strength concrete for vibrating bases since mass is usually more critical than strength; however, $f'_c < 21$ MPa is not recommended.

When a foundation is designed and put into service and problems develop, a question arises of what remedial action to take. Often a first step is to check if increasing the mass will solve the problem. A temporary mass increase can be made by use of sandbags placed on the block (symmetrically to maintain uniform soil pressure). Other alternatives consist in stiffening the base soil by drilling holes through the base (if not too thick) and injecting grout into the underlying soil in a zone up to about $3B$ in depth.

In many cases the problem can be solved by a combination of increasing the mass and the base area. This can be fairly easy to accomplish by simply pouring a perimeter enlargement that is well-bonded (using dowels) to the original base—often without having to take the machine out of service. Contrary to some opinion, concrete will harden while being vibrated (at low amplitudes)—and usually will have some strength gain from the greater resulting density and slightly lowered $w/c$ ratio.

**20-11 PILE-SUPPORTED DYNAMIC FOUNDATIONS**

When the soil is loose or soft, or when it is necessary to alter the foundation frequency, piles may be used. Intuitively, one sees that piles provide a greater apparent soil stiffness; and for the same supported mass $m$ it is evident from

$$\omega_n = \sqrt{\frac{K}{m}}$$

that an increase in $K$ also increases the natural frequency $\omega_n$ of the foundation block.
The piles provide additional spring and damping contributions to the system, so some means is necessary to incorporate the significant properties of the two materials into equivalent springs and damping factors. When we do this we can then use Eq. (20-4a) to obtain the solution (or the coupling concepts) for that vibration mode.

There are few theories and even fewer reported data from field performance studies on full-scale dynamically loaded bases supported by pile foundations. For this reason the theories are substantially uncertain; however, rational estimates are better than simply guessing at the response.

It is generally accepted that using piles will:

1. Decrease geometric (or radiation) damping
2. Increase the resonant frequency \( f_r \) and may also increase \( f_n \)
3. Influence the amplitude near resonance
4. When laterally loaded, produce dynamic responses that are uncertain to estimate

The principal effort in dynamic pile analyses has been undertaken by and under the direction of the late Professor M. Novak at the University of Western Ontario, Canada. The basic theory is given by Novak (1974) and Novak and Howell (1977) for torsion. The dynamic pile equations of Novak (1974) are of the following general form using Novak’s notation and noting \( i = \sqrt{-1} \):

**Horizontal and rocking:**

\[
G'(S_{u,1} + iS_{u,2})u(z, t)dz = F(t)
\]

**Vertical:**

\[
G'(S_{w,1} + iS_{w,2})w(z, t)dz = F(t)
\]

The parameters \( S_{i,j} \) depend on Poisson’s ratio \( \mu \) and \( x_o = a_o\sqrt{q} = (r_o\omega \sqrt{q})/V_s \). Terms are defined in the following list if not identified here. The term \( q \) is given as

\[
q = \frac{1 - 2\mu}{2 - 2\mu}
\]

From using \( i = \sqrt{-1} \) we can see the \( S_{u,j} \) factors are complex and in the original derivation include Hankel functions of the second kind of orders 0, 1, and 2 based on \( a_o \) and \( x_o \).

The \( S_{w,j} \) factors are also complex and include Bessel functions of order 0 and 1 based on \( a_o \) and \( x_o \). It is convenient to program the Bessel and Hankel function computations as subroutines to obtain the \( S_{i,j} \) functions without having to use charts, tables, or curve-fitting schemes. This step is done in computer program B-30.

The following list of variables are also significant problem parameters:

- \( E_p \) = modulus of elasticity of pile
- \( G' \) = shear modulus of soil (and depends on \( \mu \))
- \( \gamma_p, \gamma_s \) = unit weights of pile material and soil, respectively
- \( V_p, V_s \) = shear wave velocities in pile and soil respectively [for the pile compute \( V_p = \sqrt{E_p/\rho} \); for the soil use Eq. (20-15)]
- \( L_p/r_o \) = ratio of pile length \( L_p \)/effective radius of pile \( r_o \)
- \( r_o \) = effective radius of pile = radius of round pile and the equivalent for a square or rectangular pile computed as \( r_o = \sqrt{\text{area}/\pi} \)
\[ a_o = \text{dimensionless frequency factor previously used but here defined as} \]
\[ a_o = \omega r_0 \sqrt{q/V_s}; \quad q = \text{Poisson ratio value previously defined}; \quad \omega \text{ is same as used in Eq. (20-4)} \]

One must use consistent units, and with \( a_o \) as a problem parameter it is evident the pile springs and damping constants will be frequency-dependent since \( a_o \) is used to obtain the \( S_{ij} \) factors.

The general solution is only practical by using a computer program to develop the necessary constants for use in the stiffness and damping constants. Novak (1974) provides a number of curves and a table of some values, but invariably a practical problem requires interpolation or falls out of the table range. The references give the necessary information so that one can produce a computer program, but it will have to be written in a computer programming language, which allows manipulation of complex variables.

Solutions are provided for all six degrees of freedom of the base with proper interpretation and for piles with the head fixed in the base and the lower end either pinned or fixed. It appears that for the pile lengths (in terms of the \( L_p/r_o \) ratio) likely to be used the fixed lower end case will occur for nearly all cases. The theoretical solutions for the fixed lower end and the pinned lower end converge at about \( L_p/r_o = 25 \) to 30. The solution produces factors \( f_{ii} \) that are multipliers to obtain the actual spring and damping constants. Generally these \( f_{ii} \) constants depend on the following:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Amount of dependency</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_s/V_p )</td>
<td>Considerable as illustrated in Table 20-6</td>
</tr>
<tr>
<td>( L_p/r_o )</td>
<td>Not much for ( L_p/r_o &gt; 25 )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Not much, e.g., for ( V_s/V_p = 0.030 ) and ( L_p/r_o &gt; 25 ), |</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( f_{18,1} )</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0373</td>
</tr>
<tr>
<td>0.33</td>
<td>0.0373</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0373</td>
</tr>
<tr>
<td>( a_o )</td>
<td>Substantial—particularly above 0.50</td>
</tr>
</tbody>
</table>

Table 20-7 lists the spring and damping constants computed using the \( f_{ii} \) constants given in Table 20-6 for a typical concrete pile.

When the spring and damping constants are computed for a single pile it is necessary somehow to concentrate the several piles to an equivalent total or global spring and damping coefficient that, together with the block mass \( m \), are used in Eq. (20-4a) to compute displacement amplitudes and other data. There are conflicting opinions on how to make the summing process. Most persons agree that if the pile spacing ratio \( s/D \) is greater than 5 or 6 one can make a summation by simply adding the individual pile contributions (where the piles are all similar and there are \( n \) piles the global spring = \( n \times K_{\text{pile}} \) and global damping = \( n \times c_{\text{pile}} \)). When the \( s/D \) ratio is less, there is opinion that corner piles contribute more than side piles and side piles contribute more than interior piles. A method suggested by Poulos (1979) has been noted by Novak (1974) and suggested by Arya et al. (1979). Others having used the Poulos (1979) method have found it does not predict
TABLE 20-6
Novak’s $f_{ij}$ values for an intermediate value of $\mu = 0.33$ for a concrete pile with $\rho_s/\rho_p = 0.7$

Values from author’s computer program based on Novak (1974) and Novak and Howell (1977). Values $f_{12,i}$ are for torsion and use author’s identification.

Fixed parameters: $L/r_o = 30$, $a_o = 0.3$ and for torsion $\beta = 0.10$, $\mu = 0.33$.

<table>
<thead>
<tr>
<th>$V_i/V_c$</th>
<th>$f_{18,1}$</th>
<th>$f_{7,1}$</th>
<th>$f_{9,1}$</th>
<th>$f_{11,1}$</th>
<th>$f_{12,1}$</th>
<th>$f_{18,2}$</th>
<th>$f_{7,2}$</th>
<th>$f_{9,2}$</th>
<th>$f_{11,2}$</th>
<th>$f_{12,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.034</td>
<td>0.199</td>
<td>-0.019</td>
<td>0.004</td>
<td>0.045</td>
<td>0.002</td>
<td>0.136</td>
<td>-0.028</td>
<td>0.008</td>
<td>0.002</td>
</tr>
<tr>
<td>0.02</td>
<td>0.035</td>
<td>0.282</td>
<td>-0.038</td>
<td>0.010</td>
<td>0.072</td>
<td>0.007</td>
<td>0.198</td>
<td>-0.056</td>
<td>0.023</td>
<td>0.007</td>
</tr>
<tr>
<td>0.03</td>
<td>0.037</td>
<td>0.345</td>
<td>-0.057</td>
<td>0.018</td>
<td>0.105</td>
<td>0.016</td>
<td>0.245</td>
<td>-0.084</td>
<td>0.043</td>
<td>0.011</td>
</tr>
<tr>
<td>0.04</td>
<td>0.040</td>
<td>0.398</td>
<td>-0.076</td>
<td>0.027</td>
<td>0.139</td>
<td>0.027</td>
<td>0.283</td>
<td>-0.112</td>
<td>0.066</td>
<td>0.015</td>
</tr>
<tr>
<td>0.05</td>
<td>0.044</td>
<td>0.445</td>
<td>-0.095</td>
<td>0.038</td>
<td>0.174</td>
<td>0.041</td>
<td>0.314</td>
<td>-0.141</td>
<td>0.092</td>
<td>0.019</td>
</tr>
<tr>
<td>0.06</td>
<td>0.049</td>
<td>0.448</td>
<td>-0.114</td>
<td>0.050</td>
<td>0.208</td>
<td>0.055</td>
<td>0.346</td>
<td>-0.169</td>
<td>0.122</td>
<td>0.022</td>
</tr>
</tbody>
</table>

Displacement amplitudes very well. The method does, however, consider interior piles to contribute less resistance than exterior and corner piles. Since the Poulos method does not predict very well and it is fairly computationally intensive, the author suggests either doing nothing but sum values or considering the following approach if $s/D$ is less than about 3.5:

1. When displacement piles are driven the soil densifies in the vicinity of the pile. The densification is more concentrated at the interior of a pile group than around the exterior piles. This suggests that we should use a base factor $G'$ for the soil (prior to the pile insertion.

TABLE 20-7
Pile spring and damping constants [Novak (1974), Novak and Howell (1977)]

<table>
<thead>
<tr>
<th>Mode</th>
<th>Spring $K_1$</th>
<th>Damping $c_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>$K_z = \frac{EA}{r_o} f_{18,1}$</td>
<td>$c_z = \frac{EA}{V_s} f_{18,2}$</td>
</tr>
<tr>
<td>Horizontal</td>
<td>$K_h = \frac{EI}{r_o^3} f_{11,1}$</td>
<td>$c_h = \frac{EI}{r_o^2 V_s} f_{11,2}$</td>
</tr>
<tr>
<td>Rocking</td>
<td>$K_\theta = \frac{EI}{r_o} f_{7,1}$</td>
<td>$c_\theta = \frac{EI}{V_s} f_{7,2}$</td>
</tr>
<tr>
<td>Cross-stiffness/damping</td>
<td>$K_{x\theta} = \frac{EI}{r_o} f_{9,1}$</td>
<td>$c_{x\theta} = \frac{EI}{r_o V_s} f_{9,2}$</td>
</tr>
<tr>
<td>Torsion</td>
<td>$K_t = \frac{G'J}{r_o} f_{12,1}$</td>
<td>$c_t = \frac{G'J}{V_s} f_{12,2}$</td>
</tr>
</tbody>
</table>

Use consistent units for all
where $E$ = modulus of elasticity of pile
$A$ = cross-section as area of pile
$G'$ = shear modulus of pile
$I$ = moment of inertia of pile about axis to resist displacement
$J$ = torsion (or polar) moment of inertia of pile
process) and increase it some amount for side piles (perhaps use a factor of 1.1 to 1.25). Interior piles might be increased by a factor of 1.25 to 1.5. Call this factor A.

2. Solve a typical interior pile of the group using \( G'' = G'/A \), a side pile using the intermediate \( G'' \), and the corner piles using \( G' \). Inspection of Table 20-7 indicates this action will give reduced springs and damping constants for the interior compared to the sides and corner piles.

3. Now make a summation by adding all the interior springs + all the side springs + all the corner springs to obtain the global spring. Make a similar sum for the damping.

4. Use this global spring and damping value with the block mass \( m \) in Eq. (20-4a) to obtain data for that frequency \( \omega \).

Piles also have internal damping \( \beta_d \). As a first approximation we may estimate the damping ratio \( D \) on the order of 0.05 to 0.10 and use Eqs. (20-9) and (20-10) to adjust the spring and damping coefficients. A global adjustment is about the best the problem data can generally justify; however, you may make individual pile adjustments where reliable problem parameters are used.

**Example 20-3.** Compute the several single-pile spring and damping constants for the pile-supported block of Fig. E20-3. Use the vertical spring and damping values to compute the displacement in the vertical mode using Eq. (20-4a).

You are given these data:

**Piles:** precast concrete 300 × 300 mm square

- \( L_p = 9.1 \text{ m} \) (spacing \( s \) for \( s/D \) shown in Fig. E20-3)
- \( E_p = 27800 \text{ MPa} \)
- \( \gamma_p = 23.6 \text{ kN/m}^3 \)
- \( 
\mu_p = 0.15 \)

**Soil:** \( G' = 17700 \text{ kPa} \)

- \( \mu_s = 0.33 \)
- \( \gamma_s = 16.5 \text{ kN/m}^3 \)

**Other:** \( \omega = 179.2 \text{ rad/sec} \) (for current rpm)

**Solution.**

\[
\frac{r_o}{r_o} = \sqrt{\frac{A_p}{\pi}} = \sqrt{\frac{0.3 \times 0.3}{\pi}} = 0.169 \text{ m}
\]

\[
L_p/r_o = 9.1/0.169 = 54 > 30 \quad \text{(O.K. to use Table 20-6)}
\]

\[
V_s = \sqrt{\frac{G'/\rho_s}{\gamma_s}} = \sqrt{17700 \times 9.807/16.5} = 103 \text{ m/sec}
\]

\[
V_p = \sqrt{\frac{E_p/\rho_p}{\gamma_p}} = \sqrt{\frac{27.8 \times 6 	imes 9.807}{23.6}} = 3400 \text{ m/sec}
\]

\[
V_s/V_p = 103/3400 = 0.0303 \quad \text{(use 0.030 for table)}
\]

\[
\rho_s/\rho_p = \frac{\gamma_s}{\gamma_p} = 16.50/23.60 = 0.70
\]

\[
L_p = bh^3/12 = \frac{0.30^4}{12} = 0.6750 \times 3 \text{ m}^4
\]

For torsion constant \( J \) use an equivalent round pile based on \( r_o \), or

\[
J = \frac{\pi r_o^4}{2} = \frac{\pi \times 0.169^4}{2} = 0.1281 \text{E}-2 \text{ m}^4
\]

The dimensionless frequency factor \( a_o \) is computed as

\[
a_o = \frac{\omega r_o}{V_s} = \frac{179.2 \times 0.169}{103} = 0.29
\]
With these several data values computed, we can compute the several spring and damping constants using equations given in Table 20-7 with $f_{i,i}$ values from Table 20-6 ($E$, $A$, $I$, $J$ are pile values):

**Vertical.**

\[ K_z = \frac{EA}{r_0} f_{18,1} = \frac{27800 \times 0.3^2}{0.169} \times 0.037 = 547.8 \text{ MN/m} \]

\[ c_z = \frac{EA}{V_z} f_{18,2} = \frac{27800 \times 0.09}{103} \times 0.016 = 0.389 \text{ MN \cdot s/m} \]

**Horizontal.**

\[ K_h = \frac{EI}{r_0^3} f_{11,1} = \frac{17800 \times 0.6750E-3}{0.169^3} \times 0.018 = 44.8 \text{ MN/m} \]

\[ c_h = \frac{EI}{r_0^2V_z} f_{11,2} = \frac{12,015}{0.169^2 \times 103} \times 0.043 = 0.1756 \text{ MN \cdot s/m} \]
With these data and the large $s/D = 1.8/0.3 = 6$, the vertical spring and damping constants will be summed to obtain a global value for the nine piles as

$$K_Z = 9 \times 547.8 = 4930.2 \text{ MN/m}$$

$$c_z = 9 \times 0.389 = 3.501 \text{ MN \cdot s/m}$$

We can compute the block mass from the weight of block and machinery shown in Fig. E20-3 (in MN) to obtain

$$m = 4.12/9.807 = 0.4201 \text{ MN \cdot s^2/m}$$

and, using Eq. (20-4a),

$$z = \frac{F_0}{\sqrt{K_Z - (m_0^2c_0^2 + (c_z^2 \omega)^2}$$

and making group substitutions for $K_Z$, $c_z$ and $\omega = 179.2$ we obtain

$$z = \frac{0.296}{\sqrt{[4930.2 - (0.4201 \times 179.2)^2 + (3.501 \times 179.2)^2]}$$

$$= 0.3058 \text{ m} \rightarrow 0.306 \text{ mm}$$

**Comments.**

1. The first term under the square root is negative, so it appears that the vertical displacement can be reduced most economically by either increasing $\omega$ or the damping $c_z$. Reducing the vertical force would also reduce the displacement, but this is probably not possible.

2. The soil velocity $V_s$ should be reduced, but this approach is also not possible. Increasing the soil density $\rho_s$ usually increases $G_p$, so soil improvement does not appear to be a solution.

3. Adding piles does not appear to be a good solution, but increasing the base thickness to increase $m$ may be of some aid. Increasing pile size to $600 \times 600$ mm would reduce the $s/B$ to 3 and would not be of much help—even if it were possible to reduce pile length (so $L_p/r_o \approx 31$ or 32).

4. We do not know the static displacement; however, we may obtain coefficients at $a_o \rightarrow 0$ that would approximate “static” values for computing the natural system frequency and critical damping if that is desired.
GENERAL COMMENTS ON USING PILES.

1. Probably the best piles to use are concrete piles or pipe piles filled with concrete. Where wood piles are available they might be used to some advantage. HP piles are not a good choice for vibration control.

2. Use as large a pile spacing as possible—preferably $s/D > 5$ where $D =$ pile diameter or width.

3. Use low pile stresses. A rule of thumb is to limit static stresses to not more than one-half the allowable design stress for the pile material. The pile stresses in Example 20-3 are quite low at $4.12/(9 \times 0.09) = 5.09 \text{ MN}$.

4. Pile cap (or block) mass should be about 1.5 to 2.5 $\times$ mass of centrifugal machines and 2.5 to 4 $\times$ mass of reciprocating machines.

5. Arrange the centroid of the pile group to coincide with the centroid of the block mass as closely as practicable.

6. Consider batter piles with large horizontal dynamic forces. Here we could compute the axial spring of the batter pile and use the horizontal component together with the horizontal springs of the vertical piles in the group.

7. Be sure the cap is well anchored to the piles. Use shear connectors in combination with at least 300 mm of pile embedment.

8. The soil properties—particularly $G'$—after driving the piles will be substantially different from those obtained initially. Unless you can somehow determine the parameters after the piles have been installed for use in the equations given here, great refinement in spring and damping coefficients for use in Eq. (20-4a) is not necessary, and the equations and methodology given are satisfactory. Note, too, that it would be difficult to determine the parameters after driving by the down-hole or cross-hole method if the shear waves travel through both pile and soil to the detection unit.

PROBLEMS

20-1. Use your computer program FADDYNF1 and compute a value of $F_z$ (refer to Example 20-1) that would increase the given displacement by a factor of 8. To do this, make a copy of data set EXAM201.DTA and then separate the several different vibration modes into separate disk files. Revise the set labeled EXAMPLE 20-1A-A and make several copies of the file with different forces $F_z$ and make a plot of $z$ versus $F_z$ to find the resonant value.

20-2. Using your computer program, make a parametric study of the effect of $G'$ on the vertical mode of Example 20-1. That is, make a plot of $G'$ versus $z$ for 50, 75, and 150 percent of the given $G' = 239400 \text{ kPa}$.

20-3. Use your computer program as in Problem 20-2 but for the sliding mode parallel to the $x$ axis.

20-4. Use your computer program as in Problem 20-2 but for the rocking mode about the $z$ axis.

20-5. A single-cylinder engine weighs 24.15 kN. The unbalanced vertical forces are these: primary $= 18.50$ kN, secondary $= 9.75$ kN, at the operating speed of 1600 rpm. The soil is a very sandy clay with $q_u = 250 \text{ kPa}$. Find the amplitude of vibration for the system using a concrete foundation block $1.2 \times 2.4 \times 1.0$ m thick. Find the displacements in the range of rpm from 0 to 1800. (Assume this is a $m_c\ddot{y}+\omega^2$ type with the above vertical forces occurring at 1200 rpm.) Use increments of 100 rpm. If you find the resonance frequency is in the operating range, make a second run starting at 100 rpm before resonance to 100 rpm beyond, using increments of 25 rpm.
20-6. Estimate the revised lateral spring and damping for Example 20-2.

20-7. Compute the estimated horizontal displacement of Example 20-3 for a dynamic lateral force of 50 kN acting 1.5 m above the block base at ground line. Should coupling be considered in this case?

20-8. Referring to Example 20-3, we know that $V_s/V_p = 0.06$. Back-compute the corresponding soil $G'$, recompute the vertical soil springs $K_z$ and damping $c_z$, and compute the resulting vertical displacement $z$. Can you draw any conclusions about the effect of $G'$ on this class of problems?

20-9. Redo Example 20-3 for the displacement mode assigned if the frequency is either 149.2 or 209.2 rad/sec (also as assigned). Compare the spring value to that in the example, which uses 179.2 rad/sec. Does frequency $\omega$ appear to have a significant effect on the displacements? *Hint:* Compare 1/square root term computed using the example springs and $\omega$ and your springs and $\omega$.

20-10. Write a short computer program and verify Figs. 20-4.